MULTIVALUED (α) OPERATORS WITH GENERALIZED VARIATIONAL INEQUALITIES APPLIED TO MANAGEMENT OF MIGRATION EQUILIBRIUM

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In this paper, we derive some new existence results for the generalized variational inequalities by introducing multivalued (α) operators. The theory of variational inequalities has opened a tendency in modern mathematics. Variational inequalities allow the approach some problems more general than the classical ones and describe in convenient formulations the phenomena structure, including economic, geographic and demographic phenomena.

Keywords: multivalued (α) operators, generalized variational inequalities

1. Introduction

Variational inequalities (VI) have many applications in technical and natural sciences, particularly, in the plasticity theory, hydrodynamics, etc.

The variational inequality problem has been utilized to formulate and study the different problems in the different disciplines, ranging from market problems to the management of network equilibrium problems. The equilibrium is a central concept in numerous disciplines including economics, management and engineering. Date problems which have been formulated and studied as variational inequality problems include: traffic network equilibrium problems, financial equilibrium problems, migration equilibrium problems, as well as environmental network problems, and knowledge network problems etc.

Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems. Since equilibrium theory was the central theme to economics variational inequality theory provided a mechanism by which relationship between operations research and other disciplines could be established. For example, the migration phenomen. Human migration is a topic that has received attention not only from economists but also from sociologists, geographers and mathematicians.

It is generally accepted that migration is a shared responsibility of the countries of origin, but also of those of transit and destination. Starting with 2015, EU and Africa worked in a spirit of partnership to find common solutions to the

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challenges of mutual interest related to the migration phenomenon. We can consider that in the future the migration phenomenon will have a major impact on the economic and education areas. The “brain drain” will (continue to) be a pillar in the migration phenomen also. Taking this into consideration, we can consider that on the long term, the migration pattern will reach an equilibrium.

Assume a closed economy in which there are *n* locations, typically denoted by *i*, and *J* classes, typically denoted by *k*. Assume further that the attractiveness of any location i as perceived by class *k* is represented by a utility u_i^k.

Mathematically, a multiclass population vector $p^* \in K$ is said to be in equilibrium if for each class *k*; *k=1,...,J*;

$$u_i^k \begin{cases} = \lambda^k, \ if \ \ p_i^{k^*} > 0 \\ \le \lambda^k, \ if \ \ p_i^{k^*} = 0 \end{cases}.$$

Equilibrium conditions reveal that for a given class k only those locations i with maximal utility equal to an indicator λ^k will have a positive volume of the class.

It allows addressing broader issues then those classified in convenient formulation and description of the structure of phenomena. Over the years, solving variational inequalities have been used various methods, such as projection method and variants thereof, Wiener-Hopf equations, auxiliary principle technique, technical resolution and proximal equations. To resolve certain class of variational inequalities involving nonlinear functions indistinguishable using auxiliary principle technique introduced by Glowinski, Tremolieres and Lions [7].

Let $(X, \|\bullet\|)$ be a real Banach space with the topological dual $(X^*, \|\bullet\|_*)$, $<\cdot,\cdot>: X^* \times X \to \mathbb{R}$ the pairing of elements from X^* and X.

We denote by 2^{X^*} the totality of all nonempty subsets of X^* and consider the multivalued or set-valued mapping $T : K \subseteq X \to 2^{X^*}$. Let $D(T) = \{x \in X : T(x) \neq \varnothing\}$ be its *range* and $G(T) = R(T) \times D(T)$ be its *graph*. We be not distinguish between a set-valued mapping T and its graph $G(T)$. So that, T or $G(T)$ is *monotone* if

$$\langle f_1 - f_2, x_1 - x_2 \rangle \ge 0 \text{ for all } f_1 \in T(x_1) \quad \text{ and } f_2 \in T(x_2) \tag{1}$$

for all $(x_1, f_1), (x_2, f_2) \in G(T)$.

To prove the existence of a solution of the operator equation (inclusion) involving a monotone mapping $T(x) \ni f$ it is necessary to assume that T is maximal. The set-valued mapping T is *maximal monotone* if its graph $G(T)$ has no monotone extension in $X^* \times X$. The maximality ensures some required topological properties.

The variational inequalities can be regarded as generalizations of these equations.

In more general framework (we give up the finite dimension of X) let K be a nonempty closed convex set in X and T be multivalued mapping from X into X^*. Then, for a given $f \in X^*$, the problem of finding an elemnet $u \in K$ such that

$$\langle Tu - f, x-u \rangle \geq 0, \text{ for all } x \in K \quad (2)$$

is called a *variational inequality* (VI) *of the first kind*. More precisely, sometimes we denote it by $VI(T,K)$ and the set of solutions by $SOL(T,K)$.

Clearly, when $K = X$ or u is an interior point of K, then we range over a neighborhood of u and the variational inequality $VI(T,K)$ reduces to the equation $T(u) \ni f$.

2. Multivalued (α) operators and multivalued mappings

Let K be a nonempty closed subsetof a real Banach space X.

We consider the multivalued or set-valued mapping $T : K \subseteq X \to 2^{X^*}$, find $(x, f) \in K \times X^*$ such that $f \in T(x)$ and

$$\langle f - g, x-y \rangle \leq 0, \text{ for all } y \in K \quad (3)$$

is called a generalize variatonal inequalitity (GVI). More precisely, sometimes we denote it by $GVI(T,K)$ and the solutions by $SOL(T,K)$.

The symbols "$\to$", "$\rightharpoonup$" and "$\overset{*}{\rightleftharpoons}$" denote the norm-convergence, the weak convergence and the weak* convergence, respectively. Then T is *hemicontinuous* if the function $t \mapsto \langle T(tx + (1-t)y), x-y \rangle$ is continuous on $[0,1]$, and T is *demicontinuous if* $x_n \to x$ in X implies $Tx_n \overset{*}{\rightleftharpoons} Tx$ in X^*. Obviously, if T is demicountinuos then the restrictions of to any finite-dimensional subspaces or X are continuos.

The mapping $T: D(T) \subseteq X \to 2^{X^*}$ is *of class* (α), denoted $T \in (\alpha)$, if for any sequence $\{(x_n, x_n^*)\} \subset G(T)$ for which $(x_n, x_n^*) \rightharpoonup (x, x^*)$ in $X \times X^*$ and

$\limsup < x_n^*, x_n - x > \leq 0$,

it follows that the sequence $\{x_n\}$ converges strongly to x.

The multivalued mapping $T: D(T) \subseteq X \to 2^{X^*}$ is *a strongly* (α)-*monotonous*, if there exists a continuous strictly increasing function $c:[0,+\infty) \to [0,+\infty)$, such that $c(0)=0$ and

$\langle f-g, x-y \rangle \geq c(\|x-y\|)$, for all $x, y \in X$, $f \in T(x)$, $g \in T(y)$.

We say that T is *upper semicontinuous* if, for each open set $\omega \subset X^*$, the set $\{x \in K;\ T(x) \subset \omega\}$ is open in K. We recall that if the graph $G(T)$ is closed, then each T is closed.

Theorem 2.1.

Let $T: K \subseteq X \to X^*$, $T(x)$ is a nonempty, bounded, closed, and convex subset of X^***.** Suppose that T is upper semicontinuous from K into X^*. Then there exists a solution to the $GVI(T,K)$ (See [3]).

Now, we can establish the existence result:

Theorem 2.2.

Let K be a nonempty, convex, and weakly compact subset of the real reflexive Banach spaces X, and $T: K \to 2^{X^*}$ be an upper semicontinuous multifunction such that:

$T(x)$ is a nonempty, closed, and convex subset of X^*, for each $x \in K$;

T satisfies condition (α) and $T(K)$ is bounded.

Then, there exists a solution to the $GVI(T,K)$.

Proof

Let $\Gamma \subset X$ be a family, for each $M \in \Gamma$, by Theorem 2.1, there exists a point $(x_M, f_M) \in (K \cap M) \times X^*$ such that $f_M \in T(x_M)$ and

$\langle f_M, x_M - y \rangle \leq 0$, for all $y \in K \cap M$ (4)

For each $M \in \Gamma$, put $\Omega_M = \{(x_N, f_N): N \in \Gamma,\ M \subseteq N\}$. Since the family $\{\Omega_M\}$ has a finite intersection property and $K \times \overline{T(K)}^{\omega}$ is weakly compact in $X \times X^*$, it follows that $\bigcap_{M \in \Gamma} \overline{\Omega_M}^{\omega} \neq \emptyset$. Let $(x, f) \in \bigcap_{M \in \Gamma} \overline{\Omega_M}^{\omega}$, we claim that

(x, f) is a solution to the $GVI(T,K)$. Let $y \in K$, and let $M \in \Gamma$ such that $x, y \in M$. Since $(x,y) \in \overline{\Omega_M}^{\omega}$ and $X \times X^*$ is reflexive, there exist a sequence $\{(x_{M_n}, f_{M_n})\} \rightharpoonup (x, f)$ and $M \subseteq M_n$ for all $n \in \mathbb{N}$ (see [18]). From now, on we put $x_n = x_{M_n}$ and $f_n = f_{M_n}$. By (4), we have that $\langle f_n, x_n - x \rangle \le 0$, for all $n \in \mathbb{N}$.

It follows that $\limsup_{n\to\infty} \langle f_n, x_n - x \rangle \le 0$.

Since T has the *class* (α), this implies that the sequence $\{x_n\}$ has a subsequence norm converning *x*. We can suppose that the whole sequence $\{x_n\}$ is norm converging to x. By [18], the graph of *T* is closed, hence $f \in T(x)$. Again by (4), we have

$\langle f_n, x_n - y \rangle \le 0$, for all $n \in \mathbb{N}$.

Thus, we have

$$0 \ge \limsup_{n\to\infty} \langle f_n, x_n - y \rangle = \limsup_{n\to\infty} \langle f_n, x_n - x \rangle + \lim_{n\to\infty} \langle f_n, x - y \rangle = \langle f, x - y \rangle,$$

since $\lim_{n\to\infty} \langle f_n, x_n - x \rangle = 0$.

We remark the condition (4), emphasized by P. Cubiotti, J.-C. Yao [4] is weaker than hemicontinuity assuption and Theorem 2.2 extends the standard Stampacchia's existence result.

Theorem 2.3.

Let K be a nonempty, convex, and weakly compact subset of the real reflexive Banach spaces X, and $T : K \to 2^{X^*}$ be an upper semicontinuous multifunction such that:

$T(x)$ is a nonempty, closed, and convex subset of X^*, for each $x \in K$;

(ii) T satisfies condition (α);

The graph of T is closed in $K \times X^*$.

Further, assume that exists a nonempty bounded closed convex set $S \subseteq X^*$ such that $T(x) \cap S \ne \varnothing$ for all $x \in K$.

Then there exists a solution to the $GVI(T,K)$).

Now, we consider set-valued (multivalued) mappings $T : X \to 2^{X^*}$, $f \in X^*$ is a fixed element, and $K_f = \{ y \in \ | T(y) \ni f \}$. We study certain properties of the set K_f. Denote by $\mathrm{Conv}(X^*)$ the set of all convex closed subsets of X^* and let $\overline{B}_r = \{y \in X : \|y\| \le r\}$.

We introduce the upper and lower support functions for T by the formulas

$$\left[T(x), y\right]_+ = \sup_{x^* \in T(x)} \left\langle x^*, y \right\rangle \quad \text{and} \quad \left[T(x), y\right]_- = \inf_{x^* \in T(x)} \left\langle x^*, y \right\rangle,$$

with the upper norm on $\mathrm{Conv}(X^*)$ defined by

$$\|T(x)\|_+ = \sup_{x^* \in T(x)} \left\|x^*\right\|_{X^*}$$

Our operators could be non-convex and non-closed set-value, i.e., we distinguish $T(x)$ and $\overline{co}\,T(x)$ (the minimal closed convex set containing $T(x)$), and let $gr\ \overline{co}T = \left\{(x, g) \in D(T) \times X^* : g \in \overline{co}T(x)\right\}$.

In addition, the following relations hold:

$$\left[T(x), y\right]_+ = \left[coT(x), y\right]_+ = \left[\overline{co}T(x), y\right]_+, \ \forall x, y \in X;$$

$$\left[T(x), y\right]_- = \left[coT(x), y\right]_- = \left[\overline{co}T(x), y\right]_-, \ \forall x, y \in X;$$

$$\left[T(x), y_1 + y_2\right]_+ \ge \left[T(x), y_1\right]_+ + \left[T(x), y_2\right]_-, \ \forall x, y_1, y_2 \in X;$$

$$\left[T(x), y_1 + y_2\right]_- \le \left[T(x), y_1\right]_- + \left[T(x), y_2\right]_+, \ \forall x, y_1, y_2 \in X;$$

$$\left\|\overline{co}T(x)\right\|_+ = \left\|T(x)\right\|_+, \ \forall x \in X.$$

We know the following definitions:

1. A mapping $T : X \to Conv(X^*)$ is called *upper semicontinuous* at $x \in D(T)$ if for each neighborhood V of $T(x)$ in X^* there is a neighborhood U of x in X such that $T(U) \subset V$ and A is upper semicontinuous if it is upper semicontinuous at each point $x \in D(A)$. The upper semicontinuity plays an important role in the fixed-point theory for multivalued maps.
2. The mapping T is called *locally bounded* if for any $x \in \overline{D(T)}$ there are positive numbers ε and M such that

$$\|T(y)\|_+ \le M \ \text{ for } \ y \in X \ \text{ with } \ \|y - x\| < \varepsilon.$$

Our study of variational inequalities involving multivalued mappings is based on the following Brower fixed-point extension []:

Proposition 2.1

Let K *be a nonempty, convex compact set in a locally convex space* X *and* $T : K \to 2^K$ *a mapping such that the set* $T(x)$ *is nonempty and convex for all* $x \in K$*, and the preimages* $T^{-1}(y)$ *are relatively open with respect to* K *for all* $y \in K$*. Then* T *has a fixed point.*

We can now extend an existence result ([18], pp. 453) to variational inequalities with multivalued mappings, i.e., the following problem:
Find a pair $(x, g) \in K \times T(x)$ such that this satisfies the inequality

$$\langle g, y - x \rangle \geq 0 \text{ for all } y \in K \tag{5}$$

We give sufficient conditions for this problem to have solutions.

Theorem 2.4

Let $T : K \subset X \to 2^{X^*}$ be a multivalued mapping defined a nonempty subset $K \subset X$. If the following conditions are satisfied:

- the mapping T is locally bounded and upper semicontinuous;
- the set K is nonempty, convex, and compact.

Then the variational inequality (3.1) has a solution $(x, g) \in K \times T(x)$.

Proof

In the contrary case, to each $h \in T(x)$ there corresponds an element $z \in K$ such that

$$\langle h, z - x \rangle < 0 \tag{6}$$

Define the multivalued mapping $S : K \to 2^K$ by $S(x) := \{ z \in K \mid \langle h, z - x \rangle < 0 \}$.

Condition (6) implies that the set $S(x)$ is a nonempty for all $x \in K$. In addition, $S(x)$ is convex.

We denote that the $S^{-1}(x)$ is relatively open in K. First, specify $S^{-1}(z) := \{ x \in K \mid \langle h, z - x \rangle < 0 \}$.

Let $\{x_n\}$ be a sequence, with $x_n \to z$ and $h_n \in T(x_n)$, so that $\langle h_n, z - x_n \rangle \geq 0$ for all n. By the local boundedness, we also have $h_n \rightharpoonup g$ in X^*.

By the previous proposition, there exists a fixed point $x \in S(x)$. This leads to the contradiction $\langle h, x - x \rangle < 0$. Hence there is a $g \in T(x)$ and $x \in K$, satisfying (5).

3. Variational inequalities with multivalued mappings

Let $K \subset D(T)$ be a convex closed set. O. V Solonoukha [3] investigated the solvability for the multivariational inequality

$$\left[T(x), y-x\right]_{+} \geq \langle f, y-x\rangle,\ y \in K \tag{7}$$

Involving the multivalued mapping $T : K \to 2^{X^*}$, called briefly VIM.

We start giving an equivalence of $VIM(T,K)$ with a usual multivalued mapping in the form (5).

Theorem 3.1

Let x_0 be a solution of VIM (7) with $\overline{co}T(y)$ a bounded set. Assume that K in also compact set and T is locally bounded, upper semicontinuous and a generalized pseudomonotone mapping. The exists an element $g \in \overline{co}T(x_0)$ such that

$$\langle g, y-x_0\rangle \geq \langle f, y-x_0\rangle,\ (\forall)\ y \in K.$$

Proof

If the claim is not true, to each $g \in \overline{co}T(x_0)$ there corresponds an element $z \in K$ such that $\langle h, z-x_0\rangle \geq \langle h, z-x_0\rangle$. We define a similar multivalued mapping S and we follow the proof of Theorem 2.4.

This theorem allows us to approach the previous $VIM(T,K)$ by a simpler and regular form. In this setting, the mapping T is called *coercive* if

$$\frac{\left[T(x), x\right]_{-}}{\|x\|} \to \infty \text{ as } \|x\| \to \infty.$$

In the standard way [11], we can establish the following existence result:

Theorem 3.2.

Let $K \subset D(T)$ be a closed convex and compact set in a real reflexive Banach space and $T : X \to 2^{X^*}$ be a locally bounded, generalized pseudomonotone mapping.

Assume, further, that T is coercive. Then VIM (7) has a nonempty weakly compact set of solutions for any $f \in X^*$.

In the case $K = X$, if the mapping T satisfies the assumptions of theorem 3.2, then, for any $f \in X^*$, the operator inclusion $\overline{co}T(x) \ni f$ has at least one solution $x \in X$. In other words, $\overline{co}T$ is surjective, i.e., $R\left(\overline{co}T\right) = X^*$.

More generally, let $\varphi : D(\varphi) \to \mathbb{R}$ be a convex lower semicontinuous function with the domain $D(\varphi) = \{x \in X \mid \varphi(x) < \infty\}$. Consider the variational inequality of the second kind, that is, for a given $f \in X^*$, find $x \in D(\varphi)$ such that

$$\left[T(x), y - x\right]_+ + \varphi(y) - \varphi(x) \geq \langle f, y - x \rangle, \ (\forall)\ y \in D(\varphi) \tag{8}$$

The corresponding coerciveness condition has the form

$$\frac{\left[T(x), x\right]_- + \varphi(x)}{\|x\|} \to \infty \text{ as } \|x\| \to \infty \tag{9}$$

and we can prove a similar existence result:

Theorem 3.3

Let $\varphi : D(\varphi) \to \mathbb{R}$ be a convex lower semicontinuous function on a real reflexive Banach space X and $T : D(T) \to 2^{X^*}$ be a locally bounded generalized pseudomonotone mapping. Assume, further, that T satisfies the coerciveness condition (9). Then the VIM (8) has a nonempty weakly compact set of solutions for any $f \in X^*$.

4. Conclusions

It is well known that the equilibrium theory plays an important role in the study of variational inequality and its variant forms.

In this paper, we have introduced and studied a new class of variational inequalities, which is called multivalued extended general variational inequalities. Using the projection operator technique, it is shown that the multivalued extended general quasi-variational inequalities are equivalent to the fixed-point problems. This alternative equivalent fixed point formulation is used to discuss the existence of a solution of new class of variational inequalities.

Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems, for example, the migration phenomen. We can conclude that the equilibrium theory was the central theme to economics variational inequality theory, that provided a mechanism to bridge operations research to other disciplines. Hence the importance of a scientific, comprehensive approach to study the migration phenomenon, that is likely to have a significant impact in the future on the economic and education sectors.

R E F E R E N C E S

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