TRIPLE-ADAPTIVE INERTIAL SUBGRADIENT EXTRAGRADIENT ALGORITHMS FOR BILEVEL SPLIT VARIATIONAL INEQUALITY WITH FIXED POINTS CONSTRAINT

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In this paper, we introduce triple-adaptive inertial subgradient extragradient rule for solving a bilevel split pseudomonotone variational inequality problem (BSPVIP) with the common fixed point problem (CFPP) constraint of finitely many nonexpansive mappings in real Hilbert spaces, where the BSPVIP involves the fixed point problem (FPP) of a demimetric mapping. The rule exploits the strong monotonicity of one operator at the upper-level problem and the pseudomonotonicity of another mapping at the lower level. The strong convergence result for the proposed algorithm is established under some suitable assumptions. Our results improve and extend some recent results.

Keywords: adaptive inertial subgradient extragradient algorithm, bilevel split pseudomonotone variational inequality, demimetric mapping, fixed point.

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1. **Introduction**

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\emptyset \neq C \subset \mathcal{H}$ be a convex and closed set. Let $A : C \to \mathcal{H}$ be a mapping. Consider the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0,\ \forall x \in C.$$

The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. The VIP has been investigated extensively, see [3, 16, 24, 31–34]. It is well known that one of the most popular approaches for settling the VIP is the extragradient method invented by Korpelevich [18] in 1976, that is, for any initial $p_0 \in C$, the sequence $\{p_n\}$ is fabricated below

$$\begin{cases} q_n = P_C(p_n - \ell A p_n), \\ p_{n+1} = P_C(p_n - \ell A q_n), \forall n \geq 0. \end{cases}$$

The literature on the VIP is numerous and Korpelevich's extragradient method has received extensive attention given by many scholars, who ameliorated it in various aspects; see e.g., [4–9, 11, 12, 19, 23, 25, 27, 29] and references therein.

In 2018, Thong and Hieu [23] put forward the inertial subgradient extragradient method, that is, for any initial $p_1, p_0 \in \mathcal{H}$, the sequence $\{p_n\}$ is generated by

$$\begin{cases} w_n = p_n + \alpha_n(p_n - p_{n-1}), \\ y_n = P_C(w_n - \ell A w_n), \\ C_n = \{p \in \mathcal{H} : \langle w_n - \ell A w_n - y_n, y_n - p\rangle \geq 0\}, \\ p_{n+1} = P_{C_n}(w_n - \ell A y_n), \forall n \geq 1. \end{cases}$$

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Under suitable conditions, they proved the weak convergence of $\{p_n\}$ to an element of $\mathrm{VI}(C, A)$. Very recently, Ceng et al. [8] introduced the following modified inertial subgradient extragradient method for solving the VIP with pseudomonotone and Lipschitz continuous mapping $A : \mathcal{H} \to \mathcal{H}$ and the common fixed point problem (CFPP) of finitely many nonexpansive self-mappings $\{S_i\}_{i=1}^N$ on $\mathcal{H}$.

Algorithm 1.1 ([8])**.** *Let $\lambda_1 > 0$, $\alpha > 0$, $\mu \in (0,1)$ and $x_1, x_0 \in \mathcal{H}$ be arbitrary. Calculate x_{n+1} as follows: Step 1. Given the iterates x_n and x_{n-1} $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where*

$$\bar{\alpha}_n = \begin{cases} \min\{\alpha, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, \text{otherwise.} \end{cases}$$

Step 2. Compute $w_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$. Step 3. Construct the half-space $C_n := \{y \in \mathcal{H} : \langle w_n - \lambda_n A w_n - y_n, y_n - y\rangle \geq 0\}$, and compute $z_n = P_{C_n}(w_n - \lambda_n A y_n)$. Step 4. Calculate $x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \beta_n \rho F) z_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n\rangle}, \lambda_n\}, & \text{if } \langle A w_n - A y_n, z_n - y_n\rangle > 0, \\ \lambda_n, \text{otherwise.} \end{cases}$$

Set $n := n+1$ and return to Step 1.

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ denote a bounded linear operator and $A, F : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings. Recall that the bilevel split variational inequality problem (BSVIP) ([2]) is specified below:

$$\text{Seek } z^* \in \Omega \text{ such that } \langle F z^*, z - z^*\rangle \geq 0, \ \forall z \in \Omega, \tag{1}$$

where $\Omega := \{z \in \mathrm{VI}(C, A) : Tz \in \mathrm{VI}(Q, B)\}$ is the solution set of the split variational inequality problem (SVIP), which was introduced by Censor et al. [10] and formulated as:

$$\text{Find } x^* \in C \text{ such that } \langle A x^*, x - x^*\rangle \geq 0, \ \forall x \in C, \tag{2}$$

and

$$y^* = T x^* \in Q \text{ such that } \langle B y^*, y - y^*\rangle \geq 0, \ \forall y \in Q. \tag{3}$$

Censor et al. proposed and analyzed the following iterative method for approximating the solution of (2)-(3), i.e., for any initial $x_1 \in \mathcal{H}_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I - \lambda A)(x_n + \gamma T^*(P_Q(I - \lambda B) - I) T x_n), \quad \forall n \geq 1, \tag{4}$$

where A and B are inverse-strongly monotone. Consequently, the split problems have been studied extensively, see [13, 14, 20–22, 28, 30]. We note that the VIP can be expressed as the FPP: $Sz = P_Q(z - \mu B z)$, $\mu > 0$, with $\mathrm{VI}(Q, B) = \mathrm{Fix}(S)$, where $\mathrm{Fix}(S)$ denotes the fixed point set of the operator S. Consequently, we can reformulate the BSVIP in (1) below: Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be quasimonotone and L-Lipschitz continuous, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be κ-Lipschitzian and η-strongly monotone, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be non-zero bounded linear operator, and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be τ-demimetric mapping with $\tau \in (-\infty, 1)$. Then,

$$\text{Seek } z^* \in \Omega \text{ such that } \langle F z^*, z - z^*\rangle \geq 0, \quad \forall z \in \Omega, \tag{5}$$

where $\Omega := \{z \in \mathrm{VI}(C, A) : Tz \in \mathrm{Fix}(S)\}$. In this case, such a problem is referred to as a bilevel split quasimonotone variational inequality problem (BSQVIP). Very recently, Abuchu et al. [1] proposed a modified relaxed inertial subgradient extragradient iterative algorithm for solving the BSQVIP (5). Under suitable assumptions, it was proven in [18] that the sequence generated by the proposed algorithm converges strongly to a unique solution of the BSQVIP (5). Inspired by the previous research works, we introduce and study a bilevel split pseudomonotone variational inequality problem (BSPVIP) with the common fixed point problem (CFPP) constraint:

$$\text{Seek } z^* \in \Xi \text{ such that } \langle(\rho F - f)z^*, p - z^*\rangle \geq 0, \quad \forall p \in \Xi, \tag{6}$$

where $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction, $A : \mathcal{H}_1 \to \mathcal{H}_1$ is pseudomonotone and L-Lipschitz continuous and $\{S_i\}_{i=1}^N$ is finitely many nonexpansive self-mappings on $\mathcal{H}_1$ such that $\Xi := \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$.

In this paper, we propose triple-adaptive inertial subgradient extragradient algorithm for settling the above BSPVIP with the CFPP constraint in real Hilbert spaces, where the BSPVIP involves the FPP of demimetric mapping S. The rule exploits the strong monotonicity of the operator F at the upper-level problem and the pseudomonotonicity of the mapping A at the lower level. The strong convergence result for the proposed algorithm is established under mild assumptions. Our results improve and extend the corresponding ones in [1, 8].

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A mapping $S : C \to \mathcal{H}$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. Given a sequence $\{x_n\} \subset \mathcal{H}$, we denote by $x_n \to x$ (resp., $x_n \rightharpoonup x$) the strong (resp., weak) convergence of $\{x_n\}$ to x. For each $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$.

Lemma 2.1 ([15])**.** *The following hold:*
(i) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in \mathcal{H}$;
(ii) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leq 0$, $\forall x \in \mathcal{H}, y \in C$;
(iii) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$, $\forall x \in \mathcal{H}, y \in C$;
(iv) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, $\forall x, y \in \mathcal{H}$;
(v) $\|sx + (1 - s)y\|^2 = s\|x\|^2 + (1 - s)\|y\|^2 - s(1 - s)\|x - y\|^2$, $\forall x, y \in \mathcal{H}$, $s \in [0, 1]$.

Recall also that $S : C \to \mathcal{H}$ is called ([17])
(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t. $\|Sx - Sy\| \leq L\|x - y\|$, $\forall x, y \in C$;
(ii) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Sx - Sy, x - y\rangle \geq \alpha\|x - y\|^2$, $\forall x, y \in C$;
(iii) monotone if $\langle Sx - Sy, x - y\rangle \geq 0$, $\forall x, y \in C$;
(iv) pseudomonotone if $\langle Sx, y - x\rangle \geq 0 \Rightarrow \langle Sy, y - x\rangle \geq 0$, $\forall x, y \in C$;
(v) quasimonotone if $\langle Sx, y - x\rangle > 0 \Rightarrow \langle Sy, y - x\rangle \geq 0$, $\forall x, y \in C$;
(vi) τ-demicontractive if $\exists \tau \in (0, 1)$ s.t. $\|Sx - p\|^2 \leq \|x - p\|^2 + \tau\|x - Sx\|^2$, $\forall x \in C$, $p \in \mathrm{Fix}(S) \neq \emptyset$;
(vii) τ-demimetric if $\exists \tau \in (-\infty, 1)$ s.t. $\langle x - Sx, x - p\rangle \geq \frac{1-\tau}{2}\|x - Sx\|^2$, $\forall x \in C$, $p \in \mathrm{Fix}(S) \neq \emptyset$;
(viii) sequentially weakly continuous if $\forall \{x_n\} \subset C$, the relation holds: $x_n \rightharpoonup x \Rightarrow Sx_n \rightharpoonup Sx$.

Lemma 2.2 ([26])**.** *Let $\lambda \in (0, 1]$, $S : C \to \mathcal{H}$ be a nonexpansive mapping, and the mapping $S^\lambda : C \to \mathcal{H}$ be defined by $S^\lambda x := Sx - \lambda\rho F(Sx)$, $\forall x \in C$, where $F : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitzian and η-strongly monotone. Then S^λ is a contraction provided $0 < \rho < \frac{2\eta}{\kappa^2}$, i.e., $\|S^\lambda x - S^\lambda y\| \leq (1 - \lambda\zeta)\|x - y\|$, $\forall x, y \in C$, where $\zeta = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0, 1]$.*

Lemma 2.3. *Assume that $A : C \to \mathcal{H}$ is pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, v - u\rangle \geq 0$, $\forall v \in C$, if and only if $\langle Av, v - u\rangle \geq 0$, $\forall v \in C$.*

Lemma 2.4 ([26])**.** *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\gamma_n$ $\forall n \geq 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^\infty \lambda_n = \infty$, and (ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=1}^\infty |\lambda_n\gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5 ([15]). *Let $S : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I - S)x_n \to 0$, then $(I - S)x = 0$, where I is the identity mapping of $\mathcal{H}$.*

Lemma 2.6 ([19]). *Let $\{\mathbf{\Gamma}_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\mathbf{\Gamma}_{m_k}\} \subset \{\mathbf{\Gamma}_m\}$ s.t. $\mathbf{\Gamma}_{m_k} < \mathbf{\Gamma}_{m_k+1}$, $\forall k \geq 1$. Let the sequence $\{\phi(m)\}_{m\geq m_0}$ of integers be formulated $\phi(m) = \max\{k \leq m : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}$, with integer $m_0 \geq 1$ satisfying $\{k \leq m_0 : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\} \neq \emptyset$. Then the following hold:*

(i) $\phi(m_0) \leq \phi(m_0 + 1) \leq \cdots$ *and* $\phi(m) \to \infty$*;*

(ii) $\mathbf{\Gamma}_{\phi(m)} \leq \mathbf{\Gamma}_{\phi(m)+1}$ *and* $\mathbf{\Gamma}_m \leq \mathbf{\Gamma}_{\phi(m)+1}$, $\forall m \geq m_0$.

3. Convergence criteria

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ both are real Hilbert spaces and the feasible set C is nonempty, closed and convex in $\mathcal{H}_1$. For the convergence analysis of our proposed algorithm for treating the BSPVIP (1.6) with the CFPP constraint, we assume always that the following hold:

- $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a non-zero bounded linear operator with the adjoint T^*, and $S : \mathcal{H}_2 \to \mathcal{H}_2$ is a τ-demimetric mapping such that $I - S$ is demiclosed at zero, where $\tau \in (-\infty, 1)$;
- $A : \mathcal{H}_1 \to \mathcal{H}_1$ is pseudomonotone and L-Lipschitz continuous mapping satisfying the condition: $\|Au\| \leq \liminf_{n\to\infty} \|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup u$;
- $\{S_i\}_{i=1}^N$ is finitely many nonexpansive self-mappings on $\mathcal{H}_1$ such that $\Xi := \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$, with $\Omega := \{z \in \mathrm{VI}(C, A) : Tz \in \mathrm{Fix}(S)\}$;
- $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction with constant $\delta \in [0, 1)$, and $F : \mathcal{H}_1 \to \mathcal{H}_1$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$;
- $\{\beta_n\}, \{\gamma_n\}, \{\varepsilon_n\}$ are positive sequences such that $\beta_n + \gamma_n < 1$, $\sum_{n=1}^\infty \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $\varepsilon_n = o(\beta_n)$.

In addition, we write $S_n := S_{n \bmod N}$ for integer $n \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, that is, whenever $n = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, one has that $S_n = S_N$ if $q = 0$ and $S_n = S_q$ if $0 < q < N$.

Algorithm 3.1. *Let $\lambda_1 > 0$, $\epsilon > 0$, $\sigma \geq 0$, $\mu \in (0, 1)$, $\alpha \in [0, 1]$ and $x_0, x_1 \in \mathcal{H}_1$ be arbitrary. Calculate x_{n+1} as follows: Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where*

$$\bar{\alpha}_n = \begin{cases} \min\{\alpha, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{7}$$

Step 2. Compute $w_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$.

Step 3. Construct the half-space $C_n := \{y \in \mathcal{H}_1 : \langle w_n - \lambda_n A w_n - y_n, y_n - y\rangle \geq 0\}$, and compute $v_n = P_{C_n}(w_n - \lambda_n A y_n)$ and $z_n = v_n - \sigma_n T^*(I - S)Tv_n$.

Step 4. Calculate $x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \beta_n \rho F)z_n$, update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|v_n - y_n\|^2}{2\langle Aw_n - Ay_n, v_n - y_n\rangle}, \lambda_n\}, & \text{if } \langle Aw_n - Ay_n, v_n - y_n\rangle > 0, \\ \lambda_n, \text{otherwise,} \end{cases} \tag{8}$$

and for any fixed $\epsilon > 0$, σ_n is chosen to be the bounded sequence satisfying

$$0 < \epsilon \leq \sigma_n \leq \frac{(1 - \tau)\|Tv_n - STv_n\|^2}{\|T^*(Tv_n - STv_n)\|^2} - \epsilon, \quad \text{if } Tv_n \neq STv_n, \tag{9}$$

otherwise set $\sigma_n = \sigma \geq 0$.

Set $n := n + 1$ and return to Step 1.

Remark 3.1. *It is easy to see that, from* (7) *we get* $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. *Indeed, we have* $\alpha_n\|x_n - x_{n-1}\| \leq \varepsilon_n$, $\forall n \geq 1$, *which together with* $\lim_{n\to\infty}\frac{\varepsilon_n}{\beta_n} = 0$ *implies that* $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\beta_n} \to 0$ *as* $n \to \infty$.

Lemma 3.1. *Let* $\{\lambda_n\}$ *be generated by* (8). *Then* $\{\lambda_n\}$ *is a nonincreasing sequence with* $\lambda_n \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$ $\forall n \geq 1$, *and* $\lim_{n\to\infty}\lambda_n \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$.

Proof. First, from (8) it is clear that $\lambda_n \geq \lambda_{n+1}$ $\forall n \geq 1$. Also, observe that

$$\left.\begin{array}{l}\frac{1}{2}(\|w_n - y_n\|^2 + \|v_n - y_n\|^2) \geq \|w_n - y_n\|\|v_n - y_n\| \\ \langle Aw_n - Ay_n, v_n - y_n\rangle \leq L\|w_n - y_n\|\|v_n - y_n\|\end{array}\right\} \Rightarrow \lambda_{n+1} \geq \min\{\lambda_n, \frac{\mu}{L}\}.$$

□

Lemma 3.2. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 3.1. Then, the stepsize* σ_n *formulated in* (9) *is well-defined.*

Proof. It suffices to show that $\|T^*(Tv_n - STv_n)\|^2 \neq 0$. Take a $p \in \Xi$ arbitrarily. Since S is τ-demimetric mapping, we obtain

$$\begin{aligned}\|v_n - p\|\|T^*(Tv_n - STv_n)\| &\geq \langle v_n - p, T^*(Tv_n - STv_n)\rangle \\ &= \langle Tv_n - Tp, Tv_n - STv_n\rangle \\ &\geq \frac{1-\tau}{2}\|Tv_n - STv_n\|^2.\end{aligned} \tag{10}$$

If $Tv_n \neq STv_n$, then $\|Tv_n - STv_n\|^2 > 0$. Thus, $\|T^*(Tv_n - STv_n)\|^2 > 0$. □

Lemma 3.3. *Let* $\{w_n\}, \{y_n\}, \{v_n\}$ *be the sequences generated by Algorithm 3.1. Then*

$$\|v_n - p\|^2 \leq \|w_n - p\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n - y_n\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|v_n - y_n\|^2, \ \forall p \in \Xi.$$

Proof. First, by the definition of $\{\lambda_n\}$ we claim that

$$2\langle Aw_n - Ay_n, v_n - y_n\rangle \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|^2 + \frac{\mu}{\lambda_{n+1}}\|v_n - y_n\|^2, \ \forall n \geq 1. \tag{11}$$

Indeed, if $\langle Aw_n - Ay_n, v_n - y_n\rangle \leq 0$, then inequality (11) holds. Otherwise, from (8) we get (11). Also, observe that for each $p \in \Xi \subset C \subset C_n$,

$$\begin{aligned}\|v_n - p\|^2 = \|P_{C_n}(w_n - \lambda_n Ay_n) - P_{C_n}p\|^2 &\leq \langle v_n - p, w_n - \lambda_n Ay_n - p\rangle \\ &= \frac{1}{2}\|v_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|v_n - w_n\|^2 - \langle v_n - p, \lambda_n Ay_n\rangle,\end{aligned}$$

which hence yields

$$\|v_n - p\|^2 \leq \|w_n - p\|^2 - \|v_n - w_n\|^2 - 2\langle v_n - p, \lambda_n Ay_n\rangle. \tag{12}$$

From $p \in \mathrm{VI}(C, A)$, we get $\langle Ap, x - p\rangle \geq 0$ $\forall x \in C$. By the pseudomonotonicity of A on C we have $\langle Ax, x - p\rangle \geq 0$ $\forall x \in C$. Putting $x := y_n \in C$ we get $\langle Ay_n, p - y_n\rangle \leq 0$. Thus,

$$\langle Ay_n, p - v_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - v_n\rangle \leq \langle Ay_n, y_n - v_n\rangle. \tag{13}$$

Substituting (13) for (12), we obtain

$$\|v_n - p\|^2 \leq \|w_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle. \tag{14}$$

Since $v_n = P_{C_n}(w_n - \lambda_n Ay_n)$, we have that $v_n \in C_n$ and hence

$$\begin{aligned}2\langle w_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle &= 2\langle w_n - \lambda_n Aw_n - y_n, v_n - y_n\rangle + 2\lambda_n\langle Aw_n - Ay_n, v_n - y_n\rangle \\ &\leq 2\lambda_n\langle Aw_n - Ay_n, v_n - y_n\rangle,\end{aligned}$$

which together with (11), implies that

$$2\langle w_n - \lambda_n A y_n - y_n, v_n - y_n\rangle \le \mu \frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 + \mu \frac{\lambda_n}{\lambda_{n+1}}\|v_n - y_n\|^2. \tag{15}$$

Therefore, substituting (15) for (14), we obtain the desired result. □

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.*

Proof. First of all, we show that $P_\Xi(f + I - \rho F)$ is a contraction. Indeed, for any $x, y \in \mathcal{H}_1$, by Lemma 2.2, we have

$$\begin{aligned}\|P_\Xi(f + I - \rho F)x - P_\Xi(f + I - \rho F)y\| &\le \|f(x) - f(y)\| + \|(I - \rho F)x - (I - \rho F)y\| \\ &\le \delta\|x - y\| + (1 - \zeta)\|x - y\| = [1 - (\zeta - \delta)]\|x - y\|,\end{aligned}$$

which implies that $P_\Xi(f + I - \rho F)$ is a contraction. Hence, $P_\Xi(f + I - \rho F)$ has a unique fixed point. Say $z^* \in \mathcal{H}_1$, that is, $z^* = P_\Xi(f + I - \rho F)z^*$. Thus, there exists the unique solution $z^* \in \Xi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega$ to the VIP

$$\langle(\rho F - f)z^*, p - z^*\rangle \ge 0, \quad \forall p \in \Xi. \tag{16}$$

This also means that there exists the unique solution $z^* \in \Xi$ to the BSPVIP (6) with the CFPP constraint.

Now, by the definition of w_n in Algorithm 3.1, we have

$$\|w_n - z^*\| = \|S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1}) - z^*\| \le \|x_n - z^*\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.$$

From Remark 3.1, we know that $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. This means that $\{\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|\}$ is bounded. Thus, $\exists M_1 > 0$ s.t. $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1,\ \forall n \ge 1$. Hence,

$$\|w_n - z^*\| \le \|x_n - z^*\| + \beta_n M_1, \quad \forall n \ge 1. \tag{17}$$

From Step 3 of Algorithm 3.1, using the definition of z_n, we get

$$\|z_n - z^*\|^2 = \|v_n - z^*\|^2 - 2\sigma_n\langle T(v_n - z^*), (I - S)Tv_n\rangle + \sigma_n^2\|T^*(I - S)Tv_n\|^2. \tag{18}$$

Since the operator S is τ-demimetric, from (18) we get

$$\begin{aligned}\|z_n - z^*\|^2 &\le \|v_n - z^*\|^2 - \sigma_n(1 - \tau)\|(I - S)Tv_n\|^2 + \sigma_n^2\|T^*(I - S)Tv_n\|^2 \\ &= \|v_n - z^*\|^2 + \sigma_n[\sigma_n\|T^*(I - S)Tv_n\|^2 - (1 - \tau)\|(I - S)Tv_n\|^2].\end{aligned} \tag{19}$$

But from the stepsize σ_n in (9), we get

$$\sigma_n + \epsilon \le \frac{(1 - \tau)\|Tv_n - STv_n\|^2}{\|T^*(I - S)Tv_n\|^2},$$

if and only if

$$\sigma_n(\sigma_n\|T^*(I - S)Tv_n\|^2 - (1 - \tau)\|Tv_n - STv_n\|^2) \le -\sigma_n\epsilon\|T^*(I - S)Tv_n\|^2. \tag{20}$$

Using $0 < \epsilon \le \sigma_n$ in (9), we have that $-\epsilon^2 \ge -\sigma_n\epsilon$ and hance

$$-\sigma_n\epsilon\|T^*(I - S)Tv_n\|^2 \le -\epsilon^2\|T^*(I - S)Tv_n\|^2. \tag{21}$$

Combining (19),(20) and (21), we obtain

$$\|z_n - z^*\|^2 \le \|v_n - z^*\|^2 - \epsilon^2\|T^*(I - S)Tv_n\|^2 \le \|v_n - z^*\|^2. \tag{22}$$

In addition, by Lemma 3.1, we have $\lim_{n\to\infty} \lambda_n \ge \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$, which hence leads to $\lim_{n\to\infty}(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0$. Without loss of generality, we may assume that

$1-\mu\frac{\lambda_n}{\lambda_{n+1}}>0\ \forall n\geq 1$. Thus, by Lemma 3.3, we get

$$\begin{aligned}\|v_n-z^*\|^2 &\leq \|w_n-z^*\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|v_n-y_n\|^2\\ &\leq \|w_n-z^*\|^2.\end{aligned}\tag{23}$$

Combining (17),(22) and (23), we obtain

$$\|z_n-z^*\|\leq\|v_n-z^*\|\leq\|w_n-z^*\|\leq\|x_n-z^*\|+\beta_nM_1,\quad \forall n\geq 1.\tag{24}$$

Since $\beta_n+\gamma_n<1,\ \forall n\geq 1$, we get $\frac{\beta_n}{1-\gamma_n}<1\ \forall n\geq 1$. So, from Lemma 2.2 and (24) it follows that

$$\begin{aligned}\|x_{n+1}-z^*\| &= \|\beta_nf(x_n)+\gamma_nx_n+((1-\gamma_n)I-\beta_n\rho F)z_n-z^*\|\\ &\leq \beta_n\|f(x_n)-z^*\|+\gamma_n\|x_n-z^*\|\\ &\quad+(1-\beta_n-\gamma_n)\|(\frac{1-\gamma_n}{1-\beta_n-\gamma_n}I-\frac{\beta_n}{1-\beta_n-\gamma_n}\rho F)z_n-z^*\|\\ &\leq \beta_n(\|f(x_n)-f(z^*)\|+\|f(z^*)-z^*\|)+\gamma_n\|x_n-z^*\|\\ &\leq \beta_n(\delta\|x_n-z^*\|+\|f(z^*)-z^*\|)+\gamma_n\|x_n-z^*\|\\ &\quad+(1-\gamma_n-\beta_n\zeta)(\|x_n-z^*\|+\beta_nM_1)+\beta_n\|(I-\rho F)z^*\|\\ &\leq \max\{\|x_n-z^*\|,\frac{M_1+\|f(z^*)-z^*\|+\|(I-\rho F)z^*\|}{\zeta-\delta}\}.\end{aligned}$$

By induction, we obtain $\|x_n-z^*\|\leq\max\{\|x_1-z^*\|,\frac{M_1+\|f(z^*)-z^*\|+\|(I-\rho F)z^*\|}{\zeta-\delta}\},\ \forall n\geq 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{v_n\},\{y_n\},\{z_n\},\{Fz_n\},\{S_nx_n\}$. □

Lemma 3.5. *Let $\{v_n\},\{w_n\},\{x_n\},\{y_n\},\{z_n\}$ be the sequences generated by Algorithm 3.1. Suppose that $x_n-x_{n+1}\to 0$, $w_n-x_n\to 0$, $w_n-y_n\to 0$ and $v_n-z_n\to 0$. Then $\omega_w(\{x_n\})\subset\Xi$, with $\omega_w(\{x_n\})=\{z\in\mathcal{H}_1 : x_{n_k}\rightharpoonup z$ for some $\{x_{n_k}\}\subset\{x_n\}\}$.*

Proof. Take an arbitrary fixed $z\in\omega_w(\{x_n\})$. Then, $\exists\{x_{n_k}\}\subset\{x_n\}$ s.t. $x_{n_k}\rightharpoonup z\in\mathcal{H}_1$. Thanks to $w_n-x_n\to 0$, we know that $\exists\{w_{n_k}\}\subset\{w_n\}$ s.t. $w_{n_k}\rightharpoonup z\in\mathcal{H}_1$. In what follows, we claim that $z\in\Xi$. In fact, from Algorithm 3.1, we get $w_n-x_n=S_nx_n-x_n+\alpha_n(S_nx_n-S_nx_{n-1}),\ \forall n\geq 1$, and hence

$$\begin{aligned}\|S_nx_n-x_n\| &\leq \|w_n-x_n\|+\alpha_n\|S_nx_n-S_nx_{n-1}\|\\ &\leq \|w_n-x_n\|+\beta_n\cdot\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|.\end{aligned}$$

Using Remark 3.1 and the assumption $w_n-x_n\to 0$, we have

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0.\tag{25}$$

Also, from $y_n=P_C(w_n-\lambda_nAw_n)$, we have $\langle w_n-\lambda_nAw_n-y_n,y_n-y\rangle\geq 0,\ \forall y\in C$, and hence

$$\frac{1}{\lambda_n}\langle w_n-y_n,y-y_n\rangle+\langle Aw_n,y_n-w_n\rangle\leq\langle Aw_n,y-w_n\rangle,\quad\forall y\in C.\tag{26}$$

According to the Lipschitz continuity of A, $\{Aw_{n_k}\}$ is bounded. Note that $\lambda_n\geq\min\{\lambda_1,\frac{\mu}{L}\}$. So, from (26) we get $\liminf_{k\to\infty}\langle Aw_{n_k},y-w_{n_k}\rangle\geq 0,\ \forall y\in C$. Meantime, observe that $\langle Ay_n,y-y_n\rangle=\langle Ay_n-Aw_n,y-w_n\rangle+\langle Aw_n,y-w_n\rangle+\langle Ay_n,w_n-y_n\rangle$. Since $w_n-y_n\to 0$, from L-Lipschitz continuity of A we obtain $Aw_n-Ay_n\to 0$, which together with (26) arrives at $\liminf_{k\to\infty}\langle Ay_{n_k},y-y_{n_k}\rangle\geq 0,\ \forall y\in C$. Next we show that $\lim_{n\to\infty}\|x_n-S_lx_n\|=0$ for $l=1,...,N$. Indeed, note that for $i=1,...,N$,

$$\begin{aligned}\|x_n-S_{n+i}x_n\| &\leq \|x_n-x_{n+i}\|+\|x_{n+i}-S_{n+i}x_{n+i}\|+\|S_{n+i}x_{n+i}-S_{n+i}x_n\|\\ &\leq 2\|x_n-x_{n+i}\|+\|x_{n+i}-S_{n+i}x_{n+i}\|.\end{aligned}$$

Hence from (25) and the assumption $x_n - x_{n+1} \to 0$ we get $\lim_{n\to\infty} \|x_n - S_{n+i}x_n\| = 0$ for $i = 1, ..., N$. This immediately implies that

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0, \quad \text{for } l = 1, ..., N. \tag{27}$$

We now take a sequence $\{\varsigma_k\} \subset (0,1)$ satisfying $\varsigma_k \downarrow 0$ as $k \to \infty$. For all $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\langle Ay_{n_j}, y - y_{n_j}\rangle + \varsigma_k \geq 0, \quad \forall j \geq m_k. \tag{28}$$

Since $\{\varsigma_k\}$ is decreasing, it is clear that $\{m_k\}$ is increasing.

Again from the assumption on A, we know that $\liminf_{k\to\infty} \|Ay_{n_k}\| \geq \|Az\|$. If $Az = 0$, then z is a solution, i.e., $z \in \mathrm{VI}(C, A)$. Let $Az \neq 0$. Then we have $0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|$. Without loss of generality, we may assume that $Ay_{n_k} \neq 0$, $\forall k \geq 1$. Noticing that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $Ay_{n_k} \neq 0$ $\forall k \geq 1$, we set $u_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we get $\langle Ay_{m_k}, u_{m_k}\rangle = 1$, $\forall k \geq 1$. So, from (28) we get $\langle Ay_{m_k}, y + \varsigma_k u_{m_k} - y_{m_k}\rangle \geq 0$, $\forall k \geq 1$. Again from the pseudomonotonicity of A we have $\langle A(y + \varsigma_k u_{m_k}), y + \varsigma_k u_{m_k} - y_{m_k}\rangle \geq 0$, $\forall k \geq 1$. This immediately yields

$$\langle Ay, y - y_{m_k}\rangle \geq \langle Ay - A(y + \varsigma_k u_{m_k}), y + \varsigma_k u_{m_k} - y_{m_k}\rangle - \varsigma_k \langle Ay, u_{m_k}\rangle, \quad \forall k \geq 1. \tag{29}$$

We claim that $\lim_{k\to\infty} \varsigma_k u_{m_k} = 0$. Indeed, from $x_{n_k} \rightharpoonup z$ and $x_n - y_n \to 0$ (due to $w_n - x_n \to 0$ and $w_n - y_n \to 0$), we obtain $y_{n_k} \rightharpoonup z$. So, $\{y_n\} \subset C$ guarantees $z \in C$. Note that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\varsigma_k \downarrow 0$ as $k \to \infty$. So it follows that $0 \leq \limsup_{k\to\infty} \|\varsigma_k u_{m_k}\| = \limsup_{k\to\infty} \frac{\varsigma_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty} \varsigma_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0$. Hence we get $\varsigma_k u_{m_k} \to 0$.

Next we show that $z \in \varXi$. Indeed, using (27) we have $x_{n_k} - S_l x_{n_k} \to 0$ for $l = 1, ..., N$. Note that Lemma 2.5 guarantees the demiclosedness of $I - S_l$ at zero for $l = 1, ..., N$. Thus, from $x_{n_k} \rightharpoonup z$, we get $z \in \mathrm{Fix}(S_l)$. Since l is an arbitrary element in the finite set $\{1, ..., N\}$, it follows that $z \in \cap_{i=1}^N \mathrm{Fix}(S_i)$. Also, letting $k \to \infty$, we deduce that the right-hand side of (29) tends to zero by the uniform continuity of A, the boundedness of $\{w_{m_k}\}, \{u_{m_k}\}$ and the limit $\lim_{k\to\infty} \varsigma_k u_{m_k} = 0$. Thus, we get $\langle Ay, y - z\rangle = \liminf_{k\to\infty} \langle Ay, y - y_{m_k}\rangle \geq 0$, $\forall y \in C$. By Lemma 2.3 we have $z \in \mathrm{VI}(C, A)$. Furthermore, we claim $Tz \in \mathrm{Fix}(S)$. In fact, noticing $z_n = v_n - \sigma_n T^*(I - S)Tv_n$, from $0 < \epsilon \leq \sigma_n$ and $v_n - z_n \to 0$, we get

$$\epsilon \|T^*(I - S)Tv_n\| \leq \sigma_n \|T^*(I - S)Tv_n\| = \|v_n - z_n\| \to 0,$$

which together with the τ-demimetricness of S, leads to

$$\begin{aligned} \frac{1-\tau}{2} \|(I - S)Tv_n\|^2 &\leq \langle (I - S)Tv_n, T(v_n - z^*)\rangle \\ &\leq \|T^*(I - S)Tv_n\| \|v_n - z^*\| \to 0 \quad (n \to \infty). \end{aligned} \tag{30}$$

Noticing $x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \beta_n \rho F)z_n$, we have

$$\begin{aligned} (1 - \gamma_n)\|z_n - x_n\| &= \|x_{n+1} - x_n - \beta_n (f(x_n) - \rho F z_n)\| \\ &\leq \|x_{n+1} - x_n\| + \beta_n (\|f(x_n)\| + \|\rho F z_n\|). \end{aligned}$$

Since $0 < \liminf_{n\to\infty}(1 - \gamma_n)$, $x_n - x_{n+1} \to 0$ and $\beta_n \to 0$, from the boundedness of $\{x_n\}$ and $\{z_n\}$, we get $\lim_{n\to\infty} \|z_n - x_n\| = 0$, which hence yields

$$\|v_n - x_n\| \leq \|v_n - z_n\| + \|z_n - x_n\| \to 0 \quad (n \to \infty).$$

From $x_{n_k} \rightharpoonup z$, we get $v_{n_k} \rightharpoonup z$. Since T is bounded linear operator, it is easy to see that T is weakly continuous on $\mathcal{H}_1$. So it follows that $Tv_{n_k} \rightharpoonup Tz$. By the assumption on S, we know that $I - S$ is demiclosed at zero. Hence, from (30) we derive $Tz \in \mathrm{Fix}(S)$. Therefore, $z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \varOmega = \varXi$. This completes the proof. □

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ converges strongly to the unique solution $z^* \in \varXi$ of the BSPVIP (6) with the CFPP constraint.*

Proof. First of all, in terms of Lemma 3.4 we obtain that $\{x_n\}$ is bounded. From its proof we know that there exists the unique solution $z^* \in \Xi$ of the BSPVIP (6) with the CFPP constraint, that is, the VIP (16) has the unique solution $z^* \in \Xi$. In order to show the conclusion of the theorem, we divide the rest of the proof into several steps.

Step 1. We claim that

$$\begin{aligned}&(1-\beta_n\zeta-\gamma_n)[(1-\mu\frac{\lambda_n}{\lambda_{n+1}})(\|w_n-y_n\|^2+\|v_n-y_n\|^2)+\epsilon^2\|T^*(I-S)Tv_n\|^2]\\&\le\|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2+\beta_nM_4,\end{aligned}$$

for some $M_4>0$. Indeed, observe that

$$\begin{aligned}x_{n+1}-z^*&=\beta_n(f(x_n)-z^*)+\gamma_n(x_n-z^*)+(1-\beta_n-\gamma_n)\{\frac{1-\gamma_n}{1-\beta_n-\gamma_n}[(I-\frac{\beta_n}{1-\gamma_n}\rho F)z_n\\&\quad-(I-\frac{\beta_n}{1-\gamma_n}\rho F)z^*]+\frac{\beta_n}{1-\beta_n-\gamma_n}(I-\rho F)z^*\}\\&=\beta_n(f(x_n)-f(z^*))+\gamma_n(x_n-z^*)+(1-\gamma_n)[(I-\frac{\beta_n}{1-\gamma_n}\rho F)z_n\\&\quad-(I-\frac{\beta_n}{1-\gamma_n}\rho F)z^*]+\beta_n(f-\rho F)z^*.\end{aligned}$$

Then by Lemma 2.2 and the convexity of the function $h(s)=s^2,\ \forall s\in\mathbf{R}$, we get

$$\begin{aligned}\|x_{n+1}-z^*\|^2&\le\|\beta_n(f(x_n)-f(z^*))+\gamma_n(x_n-z^*)+(1-\gamma_n)[(I-\frac{\beta_n}{1-\gamma_n}\rho F)z_n\\&\quad-(I-\frac{\beta_n}{1-\gamma_n}\rho F)z^*]\|^2+2\beta_n\langle(f-\rho F)z^*,x_{n+1}-z^*\rangle\\&\le\beta_n\delta\|x_n-z^*\|^2+\gamma_n\|x_n-z^*\|^2+(1-\beta_n\zeta-\gamma_n)\|z_n-z^*\|^2\\&\quad+2\beta_n\langle(f-\rho F)z^*,x_{n+1}-z^*\rangle\\&\le\beta_n\delta\|x_n-z^*\|^2+\gamma_n\|x_n-z^*\|^2+(1-\beta_n\zeta-\gamma_n)\|z_n-z^*\|^2+\beta_nM_2\end{aligned}\tag{31}$$

where $\sup_{n\ge1}2\|(f-\rho F)z^*\|\|x_n-z^*\|\le M_2$ for some $M_2>0$. Substituting (22) for (31), by Lemma 3.3 we get

$$\begin{aligned}\|x_{n+1}-z^*\|^2&\le\beta_n\delta\|x_n-z^*\|^2+\gamma_n\|x_n-z^*\|^2+(1-\beta_n\zeta-\gamma_n)[\|v_n-z^*\|^2\\&\quad-\epsilon^2\|T^*(I-S)Tv_n\|^2]+\beta_nM_2\\&\le\beta_n\delta\|x_n-z^*\|^2+\gamma_n\|x_n-z^*\|^2+(1-\beta_n\zeta-\gamma_n)[\|w_n-z^*\|^2+\beta_nM_2\\&\quad-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})(\|w_n-y_n\|^2+\|v_n-y_n\|^2)-\epsilon^2\|T^*(I-S)Tv_n\|^2].\end{aligned}\tag{32}$$

Also, from (24) we have

$$\begin{aligned}\|w_n-z^*\|^2&\le(\|x_n-z^*\|+\beta_nM_1)^2\\&=\|x_n-z^*\|^2+\beta_n(2M_1\|x_n-z^*\|+\beta_nM_1^2)\\&\le\|x_n-z^*\|^2+\beta_nM_3,\end{aligned}\tag{33}$$

where $\sup_{n\geq 1}(2M_1\|x_n - z^*\| + \beta_n M_1^2) \leq M_3$ for some $M_3 > 0$. Combining (32) and (33), we obtain

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\leq \beta_n\delta\|x_n - z^*\|^2 + \gamma_n\|x_n - z^*\|^2 + (1 - \beta_n\zeta - \gamma_n)[\|x_n - z^*\|^2 + \beta_n M_3] \\ &\quad - (1 - \beta_n\zeta - \gamma_n)[(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|w_n - y_n\|^2 + \|v_n - y_n\|^2) \\ &\quad + \epsilon^2\|T^*(I - S)Tv_n\|^2] + \beta_n M_2 \\ &\leq \|x_n - z^*\|^2 - (1 - \beta_n\zeta - \gamma_n)[(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|w_n - y_n\|^2 + \|v_n - y_n\|^2) \\ &\quad + \epsilon^2\|T^*(I - S)Tv_n\|^2] + \beta_n M_4,\end{aligned}$$

where $M_4 := M_2 + M_3$. This immediately implies that

$$\begin{aligned}&(1 - \beta_n\zeta - \gamma_n)[(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|w_n - y_n\|^2 + \|v_n - y_n\|^2) + \epsilon^2\|T^*(I - S)Tv_n\|^2] \\ &\leq \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \beta_n M_4.\end{aligned} \tag{34}$$

Step 2. We claim that

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\leq [1 - \beta_n(\zeta - \delta)]\|x_n - z^*\|^2 \\ &\quad + \beta_n(\zeta - \delta)[\frac{2}{\zeta - \delta}\langle(f - \rho F)z^*, x_{n+1} - z^*\rangle + \frac{3M}{\zeta - \delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n - x_{n-1}\|]\end{aligned}$$

for some $M > 0$. Indeed, we have

$$\begin{aligned}\|w_n - z^*\|^2 &\leq [\|x_n - z^*\| + \alpha_n\|x_n - x_{n-1}\|]^2 \\ &\leq \|x_n - z^*\|^2 + \alpha_n\|x_n - x_{n-1}\|[2\|x_n - z^*\| + \alpha_n\|x_n - x_{n-1}\|].\end{aligned} \tag{35}$$

Combining (24), (31) and (35), we have

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\leq \beta_n\delta\|x_n - z^*\|^2 + \gamma_n\|x_n - z^*\|^2 + (1 - \beta_n\zeta - \gamma_n)\|z_n - z^*\|^2 \\ &\quad + 2\beta_n\langle(f - \rho F)z^*, x_{n+1} - z^*\rangle \\ &\leq \beta_n\delta\|x_n - z^*\|^2 + \gamma_n\|x_n - z^*\|^2 + (1 - \beta_n\zeta - \gamma_n)\|w_n - z^*\|^2 \\ &\quad + 2\beta_n\langle(f - \rho F)z^*, x_{n+1} - z^*\rangle \\ &\leq [1 - \beta_n(\zeta - \delta)]\|x_n - z^*\|^2 + \beta_n(\zeta - \delta)\cdot[\frac{2\langle(f - \rho F)z^*, x_{n+1} - z^*\rangle}{\zeta - \delta} \\ &\quad + \frac{3M}{\zeta - \delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n - x_{n-1}\|],\end{aligned} \tag{36}$$

where $\sup_{n\geq 1}\{\|x_n - z^*\|, \alpha_n\|x_n - x_{n-1}\|\} \leq M$ for some $M > 0$.

Step 3. We claim that $\{x_n\}$ converges strongly to the unique solution $z^* \in \Xi$ to the VIP (16). Indeed, putting $\mathbf{\Gamma}_n = \|x_n - z^*\|^2$, we show the convergence of $\{\mathbf{\Gamma}_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0 \geq 1$ such that $\{\mathbf{\Gamma}_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\mathbf{\Gamma}_n = d < +\infty$ and $\lim_{n\to\infty}(\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1}) = 0$. From (34) we obtain

$$\begin{aligned}&(1 - \beta_n\zeta - \gamma_n)[(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|w_n - y_n\|^2 + \|v_n - y_n\|^2) + \epsilon^2\|T^*(I - S)Tv_n\|^2] \\ &\leq \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \beta_n M_4 = \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + \beta_n M_4.\end{aligned}$$

Since $\lim_{n\to\infty}(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0$, $\liminf_{n\to\infty}(1 - \gamma_n) > 0$, $\beta_n \to 0$ and $\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} \to 0$, one has

$$\lim_{n\to\infty}\|w_n - y_n\| = \lim_{n\to\infty}\|v_n - y_n\| = \lim_{n\to\infty}\|T^*(I - S)Tv_n\| = 0. \tag{37}$$

Noticing $z_n = v_n - \sigma_n T^*(I-S)Tv_n$ and the boundedness of $\{\sigma_n\}$, from (37) we get

$$\|v_n - z_n\| = \sigma_n \|T^*(I-S)Tv_n\| \to 0 \quad (n \to \infty). \tag{38}$$

and hence

$$\|w_n - z_n\| \le \|w_n - y_n\| + \|y_n - v_n\| + \|v_n - z_n\| \to 0 \quad (n \to \infty). \tag{39}$$

Moreover, noticing $x_{n+1} - z^* = \gamma_n(x_n - z^*) + (1-\gamma_n)(z_n - z^*) + \beta_n(f(x_n) - \rho F z_n)$, we obtain from (24) that

$$\begin{aligned}\|x_{n+1} - z^*\|^2 \le\ & \gamma_n\|x_n - z^*\|^2 + (1-\gamma_n)\|z_n - z^*\|^2 - \gamma_n(1-\gamma_n)\|x_n - z_n\|^2 \\ & + 2\|\beta_n(f(x_n) - \rho F z_n)\|\|x_{n+1} - z^*\| \\ \le\ & \gamma_n\|x_n - z^*\|^2 + (1-\gamma_n)\|z_n - z^*\|^2 - \gamma_n(1-\gamma_n)\|x_n - z_n\|^2 \\ & + 2\beta_n(\|f(x_n)\| + \|\rho F z_n\|)\|x_{n+1} - x^*\| \\ \le\ & \|x_n - z^*\|^2 + \beta_n M_1[2\|x_n - z^*\| + \beta_n M_1] \\ & - \gamma_n(1-\gamma_n)\|x_n - z_n\|^2 + 2\beta_n(\|f(x_n)\| + \|\rho F z_n\|)\|x_{n+1} - z^*\|,\end{aligned}$$

which immediately arrives at

$$\begin{aligned}& \gamma_n(1-\gamma_n)\|x_n - z_n\|^2 \le \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 \\ & + \beta_n M_1[2\|x_n - z^*\| + \beta_n M_1] + 2\beta_n(\|f(x_n)\| + \|\rho F z_n\|)\|x_{n+1} - z^*\| \\ & \le \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + \beta_n M_1[2\mathbf{\Gamma}_n^{\frac{1}{2}} + \beta_n M_1] + 2\beta_n(\|f(x_n)\| + \|\rho F z_n\|)\mathbf{\Gamma}_{n+1}^{\frac{1}{2}}.\end{aligned}$$

Since $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$, $\beta_n \to 0$, $\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} \to 0$ and $\lim_{n\to\infty}\mathbf{\Gamma}_n = d < +\infty$, from the boundedness of $\{x_n\}, \{z_n\}$, we infer that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

So it follows from (39) that

$$\|w_n - x_n\| \le \|w_n - z_n\| + \|z_n - x_n\| \to 0 \quad (n \to \infty). \tag{40}$$

Also, from Algorithm 3.1 we obtain that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\beta_n f(x_n) + (1-\gamma_n)(z_n - x_n) - \beta_n\rho F z_n\| \\ &\le (1-\gamma_n)\|z_n - x_n\| + \beta_n\|f(x_n) - \rho F z_n\| \\ &\le \|z_n - x_n\| + \beta_n(\|f(x_n)\| + \|\rho F z_n\|) \to 0 \quad (n \to \infty).\end{aligned} \tag{41}$$

In addition, from the boundedness of $\{x_n\}$ it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f - \rho F)z^*, x_n - z^*\rangle = \lim_{k\to\infty}\langle (f - \rho F)z^*, x_{n_k} - z^*\rangle. \tag{42}$$

Since $\mathcal{H}_1$ is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup \widetilde{z}$. Thus, from (42) one gets

$$\limsup_{n\to\infty}\langle (f - \rho F)z^*, x_n - z^*\rangle = \lim_{k\to\infty}\langle (f - \rho F)z^*, x_{n_k} - z^*\rangle = \langle (f - \rho F)z^*, \widetilde{z} - z^*\rangle. \tag{43}$$

Since $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - y_n \to 0$ and $v_n - z_n \to 0$, by Lemma 3.5 we deduce that $\widetilde{z} \in \omega_w(\{x_n\}) \subset \varXi$. Hence from (16) and (43) one gets

$$\limsup_{n\to\infty}\langle (f - \rho F)z^*, x_n - z^*\rangle = \langle (f - \rho F)z^*, \widetilde{z} - z^*\rangle \le 0, \tag{44}$$

which together with (41), leads to

$$\begin{aligned}& \limsup_{n\to\infty}\langle (f - \rho F)z^*, x_{n+1} - z^*\rangle \\ & \le \limsup_{n\to\infty}[\|(f - \rho F)z^*\|\|x_{n+1} - x_n\| + \langle (f - \rho F)z^*, x_n - z^*\rangle] \le 0.\end{aligned} \tag{45}$$

Note that $\{\beta_n(\zeta-\delta)\}\subset[0,1]$, $\sum_{n=1}^\infty \beta_n(\zeta-\delta)=\infty$, and

$$\limsup_{n\to\infty}\Big[\frac{2\langle (f-\rho F)z^*, x_{n+1}-z^*\rangle}{\zeta-\delta}+\frac{3M}{\zeta-\delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\Big]\le 0.$$

Consequently, applying Lemma 2.4 to (36), one has $\lim_{n\to\infty}\|x_n-z^*\|^2=0$.

Case 2. Suppose that $\exists\{\mathbf{\Gamma}_{n_k}\}\subset\{\mathbf{\Gamma}_n\}$ s.t. $\mathbf{\Gamma}_{n_k}<\mathbf{\Gamma}_{n_k+1}$, $\forall k\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\phi:\mathcal{N}\to\mathcal{N}$ by $\phi(n):=\max\{k\le n:\mathbf{\Gamma}_k<\mathbf{\Gamma}_{k+1}\}$. By Lemma 2.6, we get $\mathbf{\Gamma}_{\phi(n)}\le\mathbf{\Gamma}_{\phi(n)+1}$ and $\mathbf{\Gamma}_n\le\mathbf{\Gamma}_{\phi(n)+1}$. From (34) we have

$$\begin{aligned}&(1-\beta_{\phi(n)}\zeta-\gamma_{\phi(n)})\Big[(1-\mu\frac{\lambda_{\phi(n)}}{\lambda_{\phi(n)+1}})(\|w_{\phi(n)}-y_{\phi(n)}\|^2+\|v_{\phi(n)}-y_{\phi(n)}\|^2)\\&+\epsilon^2\|T^*(I-S)Tv_{\phi(n)}\|^2\Big]\\&\le\|x_{\phi(n)}-z^*\|^2-\|x_{\phi(n)+1}-z^*\|^2+\beta_{\phi(n)}M_4=\mathbf{\Gamma}_{\phi(n)}-\mathbf{\Gamma}_{\phi(n)+1}+\beta_{\phi(n)}M_4,\end{aligned}\tag{46}$$

which immediately yields

$$\lim_{n\to\infty}\|w_{\phi(n)}-y_{\phi(n)}\|=\lim_{n\to\infty}\|v_{\phi(n)}-y_{\phi(n)}\|=\lim_{n\to\infty}\|T^*(I-S)Tv_{\phi(n)}\|=0.$$

Using the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|v_{\phi(n)}-z_{\phi(n)}\|=\lim_{n\to\infty}\|w_{\phi(n)}-x_{\phi(n)}\|=\lim_{n\to\infty}\|x_{\phi(n)+1}-x_{\phi(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle (f-\rho F)z^*, x_{\phi(n)+1}-z^*\rangle\le 0.\tag{47}$$

On the other hand, from (36) we obtain

$$\begin{aligned}\beta_{\phi(n)}(\zeta-\delta)\mathbf{\Gamma}_{\phi(n)}&\le\mathbf{\Gamma}_{\phi(n)}-\mathbf{\Gamma}_{\phi(n)+1}+\beta_{\phi(n)}(\zeta-\delta)\Big[\frac{2\langle (f-\rho F)z^*, x_{\phi(n)+1}-z^*\rangle}{\zeta-\delta}\\&+\frac{3M}{\zeta-\delta}\cdot\frac{\alpha_{\phi(n)}}{\beta_{\phi(n)}}\cdot\|x_{\phi(n)}-x_{\phi(n)-1}\|\Big]\\&\le\beta_{\phi(n)}(\zeta-\delta)\Big[\frac{2\langle (f-\rho F)z^*, x_{\phi(n)+1}-z^*\rangle}{\zeta-\delta}+\frac{3M}{\zeta-\delta}\cdot\frac{\alpha_{\phi(n)}}{\beta_{\phi(n)}}\cdot\|x_{\phi(n)}-x_{\phi(n)-1}\|\Big],\end{aligned}$$

which hence arrives at

$$\limsup_{n\to\infty}\mathbf{\Gamma}_{\phi(n)}\le\limsup_{n\to\infty}\Big[\frac{2\langle (f-\rho F)z^*, x_{\phi(n)+1}-z^*\rangle}{\zeta-\delta}+\frac{3M}{\zeta-\delta}\cdot\frac{\alpha_{\phi(n)}}{\beta_{\phi(n)}}\cdot\|x_{\phi(n)}-x_{\phi(n)-1}\|\Big]\le 0.$$

Thus, $\lim_{n\to\infty}\|x_{\phi(n)}-z^*\|^2=0$. Owing to $\mathbf{\Gamma}_n\le\mathbf{\Gamma}_{\phi(n)+1}$, we get

$$\|x_n-z^*\|^2\le\|x_{\phi(n)}-z^*\|^2+2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-z^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2\to 0.$$

That is, $x_n\to z^*$ as $n\to\infty$. This completes the proof. □

4. **Concluding remarks**

In this paper, we study a bilevel split pseudomonotone variational inequality problem (BSPVIP) with the common fixed point problem (CFPP) constraint of finitely many nonexpansive mappings in real Hilbert spaces. We introduce a triple-adaptive inertial subgradient extragradient algorithm [Algorithm 3.1] for solving BSPVIP with the CFPP constraint (6), where the BSPVIP involves the FPP of demimetric mapping S. The algorithm exploits the strong monotonicity of the operator F at the upper-level problem and the pseudomonotonicity of the mapping A at the lower level. We prove the strong convergence theorem [Theorem 3.1] under mild assumptions. Our results improve and extend the corresponding ones in [1, 8].

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