REMARKS ON THE AUTOCOMMUTATOR SUBGROUP AND ABSOLUTE CENTER OF A GROUP

Azam Pourmirzaei[1], Rasoul Hatamian[2], Mitra Hassanzadeh[3]

Let $L(G)$ and $K(G)$ denote the absolute center and the autocommutator subgroup of a group G, respectively. In this paper, we prove some new results regarding the relation between $G/L(G)$ and $K(G)$. Also, we consider some generalizations of the absolute center and obtain similar results.

Keywords: Absolute center, Autocommutator subgroup, Locally finite groups, Locally nilpotent groups.

MSC2020: 20E34, 20E36, 20F28, 20D15.

1. Introduction

A very famous theorem of Schur asserts that in a group G, if $G/Z(G)$ is finite, then G' is finite. Many questions arise from this theorem. The most natural one may be about the validity of the converse of this theorem. Derek Holt considered a group G with generators x_i, y_i, $i > 0$, and z, subject to the relations $x_i^p = y_i^p = z^p = 1, [x_i, x_j] = [y_i, y_j] = 1$, and $[x_i, y_i] = z, [x_i, y_j] = 1, i = j$, and $[z, x_i] = [z, y_i] = 1$, for all i. He showed that $G' = Z(G)$ is finite, but $G/Z(G)$ is infinite.(See [9]). By considering this infinite extra special p-group the answer is obviously negative. Hekster [7] proved that a partially converse is valid if the group is finitely generated. There are many results in literature, related to Schur's theorem. For instance, Mann [11] showed that if $G/Z(G)$ is locally finite of exponent n, then G' is locally finite and its exponent is bounded by a function of n. Hilton [8] proved that in a nilpotent group G, $G/Z(G)$ is a p-group if and only if G' is p-group, where $p > 0$ is a prime number.

Hegarty [6] took a different approach regarding this topic. His work started by introducing two new subgroups called the autocommmutator subgroup and absolute center of a group. In the following we recall these definitions. Before reviewing the following definition note that we denote by $Aut(G)$, the group of automorphisms of a group G.

Definition 1.1. *Let G be a group. The set*

$$L(G) = \{g \in G : \alpha(g) = g, \quad for\ all\ \alpha \in Aut(G)\}$$

is called the absolute center of G.

Clearly $L(G)$ is a central subgroup of G.

[1] Department of Pure Mathematics, Hakim Sabzevari University, Sabzevar, Iran. e-mail: a.pormirzaei@hsu.ac.ir

[2] Department of Basic Sciences, School of Mathematical Sciences, P. O. Box 19395-3697, Payame Noor University, Tehran, Iran. e-mail: hatamianr@pnu.ac.ir

[3] Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran. e-mail: mtr.hassanzadeh@gmail.com

Definition 1.2. *For a group G, the subgroup generated by all $[g,\alpha] = g^{-1}\alpha(g)$, $g \in G$ and $\alpha \in Aut(G)$, is called the autocommutator subgroup of G. This subgroup is denoted by $K(G)$.*

Hegarty [6] proved that if $G/L(G)$ is finite, then $K(G)$ is finite. In [5], Hegarty give an example of a group with finite autocommutator subgroup and infinite absolute central factor. This means that the converse of the mentioned theorem is not valid. Now one may ask whether there are conditions on G to validate a converse. One of these conditions is stated by Hegarty [5] as "Aut(G) is finite".
In the present work, in Theorem 2.2 we prove that if G is a group with $G/L(G)$ finitely generated and $K(G)$ finite, then $G/L(G)$ is finite. Also, for capable groups, in Corollary 2.1 we show that if $Z(G)/L(G)$ is finitely generated and $K(G)$ is finite, then $G/L(G)$ is finite. Furthermore, we argue on some other properties which transfer from $G/L(G)$ to $K(G)$ or vice versa in a group G. For instance,

(a) In theorem 2.3, we prove that if $G/L(G)$ is locally finite, then $K(G)$ is locally finite.
(b) In Theorem 2.4, we show that if $K(G)$ is locally finite and $G/L(G)$ is a torsion group, then $G/L(G)$ is locally finite.
(c) In Theorem 2.5, we prove that the order of $K(G)$ is bounded above by $f(n) = n^{\frac{1}{2}log_2 n + [log_2 n]}$.
(d) In Theorem 2.6, we show that if G is a finite group and $G/L(G)$ is a p-group,then $K(G)$ is also a p-group.
(e) In Theorem 2.7, we show that if G is a locally nilpotent torsion group, then $G/L(G)$ is a p-group if and only if $K(G)$ is a p-group.
(f) In Theorem 2.10, we prove that for a nilpotent group G of class c, if $G/L(G)$ is a p-group of exponent m, then $K(G)$ is a p-group of exponent dividing m^c.
(g) In Theorem 2.11, we prove that for a nilpotent group G of class c, if $K(G)$ is a p-group of exponent m, then $G/L(G)$ is a p-group of exponent dividing m^c.

Eventually, we focus on class preserving automorphisms and central automorphisms and we introduce two central subgroups related to them. In this direction we obtain some results which are achieved fairly straightforward but interesting (See Theorem 2.13, Theorem 2.14 and Theorem 2.15).

2. Autocommutator subgroup

Definition 2.1. *Let G be a group and set $\gamma_1(G) = G$. Suppose that $\gamma_i(G)$ is defined inductively for $i \geq 1$. By setting $\gamma_{i+1}(G) = [\gamma_i(G), G]$ a series of subgroups is obtained as follows*

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$$

which is called the lower central series of G.

Note that $\gamma_n(G)/\gamma_{n+1}(G)$ lies in the center of $G/\gamma_{n+1}(G)$ and that each $\gamma_n(G)$ is a fully-invariant subgroup of G. There is an ascending series that is dual to the lower central series. This is the *upper central series*

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$$

which is defined by

$$\frac{Z_{n+1}(G)}{Z_n(G)} = Z\left(\frac{G}{Z_n(G)}\right).$$

Each $Z_n(G)$ is characteristic but not necessarily fully-invariant in G. This series need not reach G.

Now, we recall the main theorem from [4] which is severally used throughout the paper.

Theorem 2.1. *If G is a group, $\gamma_{n+1}(G)$ is finite and $G/Z_n(G)$ is finitely generated, then*

$$\left|\frac{G}{Z_n(G)}\right| \leq |\gamma_{n+1}(G)|^{d(G/Zn(G))^n},$$

where $d(X)$ is the minimal number of generators of the group X.

Theorem 2.2. *For an arbitrary group G, if $G/L(G)$ is finitely generated and $K(G)$ is finite then $G/L(G)$ is finite.*

Proof. First we note that $G/Z(G)$ is a factor group of $G/L(G)$. Now the assumption "$G/L(G)$ is finitely generated" yields that $G/Z(G)$ is finitely generated. Since G' is a subgroup of $K(G)$, it is finite. Use Theorem 2.1 in case $n = 1$ to have $G/Z(G)$ is finite. The following isomorphism

$$\frac{G}{Z(G)} \cong \frac{G/L(G)}{Z(G)/L(G)}$$

implies that $Z(G)/L(G)$ is a finitely generated abelian group. Therefore

$$\frac{Z(G)}{L(G)} \cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \ldots \oplus \mathbb{Z}_{m_t} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$$

in which for every $1 \leq i \leq t$, $m_i > 1$.
Let r denotes the Betti number of $Z(G)/L(G)$. We now prove that $r = 0$; then $Z(G)/L(G)$ is finite and since $G/Z(G)$ is finite as shown above, the assertion follows. Suppose that $r > 0$. Then $Z(G)/L(G)$ has an element of infinite order say $gL(G)$. By definition of the absolute center and the choice of $gL(G)$, for each $n \in \mathbb{N}$, there exists an automorphism α_n such that $\alpha_n(g^n) \neq g^n$. Let s be the order of $K(G)$ and set $m = (s+1)!$. Obviously, for every $t \mid m$, we have $\alpha_m(g^t) \neq g^t$ i.e. $[g^t, \alpha_m] \neq 1$. So Definition 1.2 and a simple argument show that $[g^i, \alpha_m] \neq [g^j, \alpha_m]$, for every $1 \leq i \neq j \leq (s+1)$. This yields at least $(s+1)$ pairwise distinct generators for $K(G)$ which is a contradiction. □

A group G is called *capable* if it is the group of inner automorphisms of some group. Podoski and Szegedy proved in [12] that if H is a capable group and $|H'| = n$, then $[H : Z(H)] \leq n^{2\log_2 n}$. Using this theorem we have the following corollary.

Corollary 2.1. *Let G be a capable group such that $Z(G)/L(G)$ is finitely generated and $K(G)$ is finite. Then $G/L(G)$ is finite.*

Proof. Since G is capable and $G' \leq K(G)$ is finite, Podoski's result shows that $G/Z(G)$ is finite. Therefore $G/L(G)$ is finitely generated. Now the assertion follows from the previous theorem. □

If χ is a property of groups, a group G is called a locally χ -group if each finite subset of G is contained in a χ-subgroup of G. For a group G, if $G/L(G)$ is locally finite, then $K(G)$ is locally finite. This fact was proved by Dietrich and Moravec in [2]. (They also find a bound for the order of $K(G)$ corresponding to the order of $G/L(G)$). The following theorem can be investigated when we use a theorem of Mann [11,Theorem1] and elementary group theory techniques.

Theorem 2.3. *Let G be a group such that $G/L(G)$ is locally finite. Then $K(G)$ is locally finite.*

Proof. Since $G/L(G)$ is locally finite, it follows that $G/Z(G)$ is locally finite. Hence by [11,Theorem 1] G' is locally finite. To prove the statement it is enough to show that $K(G)/G'$ is locally finite.

First, we show that $K(G)/G'$ is a torsion group. Let $[x,\alpha]G'$ be an arbitrary generator of $K(G)/G'$, where $x \in G$ and $\alpha \in Aut(G)$. Since $G/L(G)$ is locally finite, there exists a natural number n such that $x^n \in L(G)$. Since $K(G)/G'$ is abelian, we have

$$[x,\alpha]^n G' = x^{-n}\alpha(x^n)G' = x^{-n}x^n G' = G'.$$

Therefore every finitely generated subgroup of $K(G)/G'$ is an abelian torsion group which immediately achieve the goal. □

Our next aim is to find a sufficient condition such that the converse of the above theorem holds. In order to do so, we recall two definitions and a lemma from [7].

Definition 2.2. *A group S is called a stem group if $Z(S) \leq S'$.*

Definition 2.3. *Let G and H be two groups. An isoclinism from G to H is a pair of isomorphisms (α,β) with $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ such that the following diagram is commutative:*

$$\begin{array}{ccccc} \frac{G}{Z(G)} & \times & \frac{G}{Z(G)} & \xrightarrow{\gamma(2,G)} & G' \\ & \alpha^2 \downarrow & & & \beta\downarrow \\ \frac{H}{Z(H)} & \times & \frac{H}{Z(H)} & \xrightarrow{\gamma(2,H)} & H'. \end{array}$$

Whenever the groups G and H are isoclinic, we write $G \sim H$.

Lemma 2.1 ([7], Lemma 6.1)**.** *For every group G, there exists a stem group S such that $G \sim S$.*

We recall that a torsion group G is a group whose elements have finite order.

Theorem 2.4. *Let G be a group such that $K(G)$ is locally finite and $G/L(G)$ is a torsion group. Then $G/L(G)$ is locally finite.*

Proof. Assume that $H/L(G)$ is a finitely generated subgroup of $G/L(G)$. One may observe that $H/Z(H)$ is a homomorphic image of $H/L(G)$ and so it is torsion and finitely generated. From Lemma 2.1 it follows that there exists a stem group S isoclinic to H. Thus $S/Z(S)$ is also a finitely generated torsion group and since $Z(S) = Z(S) \cap S' \leq \phi(S)$, where $\phi(S)$ denotes the Frattini subgroup of S, S is finitely generated (Note that the Frattini subgroup of a group is the set of non-generators of the group.). Therefore S/S' is a finite abelian group which yields S' and thus H' are finitely generated. From Theorem 2.1 it follows that $H/Z(H)$ is finite and therefore $Z(H)/L(G)$ is a subgroup of finite index of the finitely generated group $H/L(G)$. This follows that $Z(H)/L(G)$ is finitely generated. Go back to the assumption "$G/L(G)$ is locally finite" to have the result. □

The following proposition is very interesting despite its easy proof.

Proposition 2.1. *If $G/L(G)$ is locally nilpotent, then G is locally nilpotent and also $K(G)$ is locally nilpotent.*

Proof. Let H be a finitely generated subgroup of G. Clearly $HZ(G)/Z(G)$ is a finitely generated subgroup of the locally nilpotent group $G/Z(G)$ which implies that it is nilpotent. Therefore $H/Z(H)$ and so H is nilpotent. □

The following theorem is proved by using basic results of group theory. First we recall that for a torsion group G, if the order of its elements is bounded, the exponent of G which is denoted by $exp(G)$, is the least common multiple of all the orders of its elements.

Theorem 2.5. *For a group G, if $G/L(G)$ is of order n, then $K(G)$ is finite and the order of $K(G)$ is bounded by $f(n) = n^{\frac{1}{2}\log_2 n + [\log_2 n]}$.*

Proof. It is clear that the order of $G/Z(G)$ is at most n. Hence with respect to the bound obtained by Wiegold [15], we have $|G'| \leq n^{\frac{1}{2}\log_2 n}$. From the proof of Theorem 2.3 it follows that $K(G)/G'$ is a torsion abelian group of exponent at most n. Thus if $G/L(G) = \langle g_1L(G), g_2L(G), \dots g_kL(G)\rangle$ where k is the minimal number of generators of $G/L(G)$ then we claim that $K(G)/G'$ is also generated with at most k elements. For this, consider $[g,\alpha]G' \in K(G)/G'$ such that $g \in G$ and $\alpha \in Aut(G)$. One can see that $g = g_1^{m_1} g_2^{m_2} \dots g_k^{m_k} l$, where $l \in L(G)$ and for every $1 \leq i \leq k$, $m_i \geq 0$. Since $K(G)/G'$ is abelian, we have $[g,\alpha]G' = [g_1,\alpha]^{m_1}[g_2,\alpha]^{m_2} \dots [g_k,\alpha]^{m_k}G'$. Now we conclude that $K(G)/G'$ is a finite group and an easy argument shows that $k \leq [\log_2 n]$. Therefore

$$\left|\frac{K(G)}{G'}\right| \leq \left|\prod_{j=1}^{k} \mathbb{Z}_n\right| = n^k \leq n^{[\log_2 n]}.$$

The result is now clear. □

Theorem 2.6. *Let G be a finite group. If $G/L(G)$ is a p-group for some prime number p, then $K(G)$ is also a p-group.*

Proof. As $L(G)$ is a finite abelian group, by concentrating on the presentation of finitely generated abelian groups, one can write $L(G) = A \times B$, where A is a π-group and B is a π'-group when $\pi = \{p\}$. Moreover

$$[G:B] = [G:L(G)][L(G):B] = p^\gamma,$$

for some natural number γ. This means that B is a Hall π'-group. By the Schur-Zassenhouse Theorem [13], there is a subgroup H of G such that $G = HB$ and $H \cap B = 1$. Clearly H is a p-group since $|H| = [G:B]$. Suppose that $[g,\alpha]$, $g \in G, \alpha \in Aut(G)$, be an arbitrary generator of $K(G)$. There are elements $h \in H$ and $b \in B$ in which $g = hb$. Thus

$$[g,\alpha] = [hb,\alpha] = (hb)^{-1}\alpha(hb) = h^{-1}\alpha(h).$$

The last equality holds since $b \in L(G) \leq Z(G)$. Let $\alpha(h) = h'b'$ for some $h' \in H$ and $b' \in B$ and $|H| = p^\beta$, $\beta \in \mathbb{N}$. Then

$$1 = (\alpha(h))^{p^\beta} = (h'b')^{p^\beta} = h'^{p^\beta} b'^{p^\beta} = b'^{p^\beta}.$$

But $(|b'|, p^\beta) = 1$, hence $b' = 1$. Thus $\alpha(h) \in H$ and $K(G) \leq H$. This shows that $[g,\alpha]$ is a p-element. □

In the following we show that in locally nilpotent, torsion groups G, the property 'being p-group' of $K(G)$ carries over to $G/L(G)$ and vice versa.

Theorem 2.7. *Let G be a locally nilpotent torsion group. Then $K(G)$ is a p-group if and only if $G/L(G)$ is a p-group.*

Proof. First suppose that the autocommutator subgroup of G is a p-group. Assume by contradiction that $G/L(G)$ is not a p-group. Then it has an element, say $xL(G)$, of order q^n for some prime number $q \neq p$ and $n \geq 1$. Set $H = \langle x, L(G)\rangle$. It is clear that $H/L(G)$ is a finite q-group. Since G is a torsion group, it follows that cyclic group $\langle x\rangle$ is finite. So $\langle x\rangle = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \dots \oplus \mathbb{Z}_{p_t^{\alpha_t}}$, for some prime numbers p_i such that $1 \leq i \leq t$. Put $\mathbb{Z}_{p_i^{\alpha_i}} = \langle x_i\rangle$, for each $1 \leq i \leq t$. Hence $x = x_1x_2...x_t$ where for each i, $1 \leq i \leq t$ and x_i is the generator of p_i-Sylow subgroup of $\langle x\rangle$. Obviously, for some i, $p_i = q$. Assume $p_1 = q$. From the fact $q \mid |xL(G)|$ and x_i 's are members of distinct Sylow subgroups, we find out x_1 does not belong to $L(G)$. Thus there is an automorphism β of G such that $\beta(x_1) \neq x_1$. Consider the subgroup $M/L(G) = \langle x_1L(G), \beta(x_1)L(G)\rangle$ of $G/L(G)$. Thus since G is locally nilpotent, it follows that $M/L(G)$ is a nilpotent group and since it is finitely generated and torsion, it is finite. Therefore it is the product of its Sylow subgroups. Consequently $q \mid \left|x_1^{-1}\beta(x_1)L(G)\right|$

which is a contradiction.
Now suppose that $K(G)$ is not a p-group whereas $G/L(G)$ is a p-group. In this case since G is a torsion group, there exists a q-element, in $K(G)$ where $q \neq p$ is a prime number. For an arbitrary element $x \in K(G)$, we have $x = \prod_{i=1}^{m} g_i^{-1}\alpha_i(g_i)$ in which $g_i \in G$ and $\alpha_i \in Aut(G)$ for every $1 \leq i \leq m$. Consider the finitely generated subgroup

$$H = \langle g_1, g_2, ..., g_m, \alpha_1(g_1), \alpha_2(g_2), ..., \alpha_m(g_m)\rangle.$$

Clearly H is a finitely generated torsion nilpotent group and so it is finite. Let $H = \prod_{j=1}^{k} P_j \times Q$, where P_j is a p_j-Sylow subgroups and Q is a q-Sylow subgroups of H. Thus for each i, $g_i = x_{i_1}x_{i_2}...x_{i_k}y_i$ in which $x_{i_j} \in P_j$ and $y_i \in Q$. Note that the elements of distinct Sylow subgroups commutates and so one can readily deduce the following inequality.

$$g_i^{-1}\alpha_i(g_i) = x_{i_1}^{-1}\alpha_i(x_{i_1})x_{i_2}^{-1}\alpha_i(x_{i_2})...x_{i_k}^{-1}\alpha_i(x_{i_k})y_i^{-1}\alpha_i(y_i).$$

Since $x \in Q$, for some i we have $y_i^{-1}\alpha_i(y_i) \neq 1$ and thus y_i is not contained in $L(G)$. This implies that $q \mid |y_iL(G)|$ which is a contradiction. □

Hilton [8] Proved two remarkable theorems about the relation between the central factor group and the derived subgroup in a nilpotent group in the aspects of 'exponent' and 'being p-group'. In the following, invoking them, we prove similar statements for the absolute center and the autocommutator subgroup. We recall that for a nilpotent group G, the length of a shortest central series of G is the nilpotent class of G.

Theorem 2.8 ([8], Theorem 1.4)**.** *Let G be a nilpotent group of class c. If $G/Z(G)$ is a p-group of exponent m, then G' is a p-group of exponent dividing m^{c-1}.*

Theorem 2.9 ([8], Theorem 1.3)**.** *In a nilpotent group G, if G' is a p-group of exponent m, then $G/Z(G)$ is a p-group of exponent dividing m^{c-1}, where c is the nilpotency class of G.*

Theorem 2.10. *Let G be a nilpotent group of class c. If $G/L(G)$ is a p-group of exponent m, then $K(G)$ is also a p-group of exponent dividing m^c.*

Proof. From Theorem 2.7, $K(G)$ is a p-group. As $exp\,(G/L(G)) = m$ and

$$\frac{G}{Z(G)} \cong \frac{G/L(G)}{Z(G)/L(G)},$$

it follows that the exponent of $G/Z(G)$ divides m. Denote the latter number by n. By Theorem 2.8, $exp(G')|n^{c-1}$. On the other hand $K(G)/G'$ is an abelian group. Now it is enough to argue on its generators to bound its exponent. Let $g \in G$ and $\alpha \in Aut(G)$. $([g, \alpha]G')^m = g^{-m}\alpha(g^m)G' = G'$. Consequently, $exp\,(K(G)/G') \mid m$. Now the result is clear. □

Theorem 2.11. *Suppose that G is a nilpotent group of class c. If the subgroup $K(G)$ is a p-group and $exp\,(K(G)) = m$, then $G/L(G)$ is a p-group. Furthermore $exp\,(G/L(G)) \mid m^c$.*

Proof. The first assertion follows from Theorem 2.7. For the second part, assume that $g \in G$ and $g^{m^c} \notin L(G)$. By definition of the absolute center there is an automorphism α such that $\alpha\left(g^{m^c}\right) \neq g^{m^c}$ i.e. $[g^{m^c}, \alpha] \neq 1$.
Clearly G' is a subgroup of $K(G)$ which yields that it is a p-group of exponent dividing m. From Theorem 2.9 we deduce that $g^{m^{c-1}} \in Z(G)$. Therefore $[g^{m^c}, \alpha] = [g^{m^{c-1}}, \alpha]^m = 1$. The latter equality which is obtained from the fact $exp\,(K(G)) = m$ leads to a contradiction and hence $exp(G/L(G)) \mid m^c$. □

In the following, we consider two special classes of automorphisms and we introduce several subgroups related to them which are similar to $L(G)$ and $K(G)$. For this aim we review the definitions of class preserving automorphisms and central automorphisms from [14].

Definition 2.4. *A class preserving automorphism of a group G is an automorphism α of G such that for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1} x g_x$.*

It is easy to show that the set of all class preserving automorphisms of a group G is a normal subgroup of $Aut(G)$. This subgroup is denoted by $Aut_c(G)$. More precisely,

$$Aut_c(G) = \{\alpha \in Aut(G) : \forall x \in G, \exists g_x \in G; \alpha(x) = g_x^{-1} x g_x\}.$$

Note that $Aut_c(G)$ contains $Inn(G)$, where $Inn(G)$ denotes the set of inner automorphisms of G.

Definition 2.5. *An automorphism φ of a group G is called a central automorphism if for all elements $g \in G$, we have $g^{-1}\varphi(g) \in Z(G)$.*

It is obvious that the set of all central automorphisms of G is also a normal subgroup of $Aut(G)$. This group is denoted by $Aut_z(G)$. For a survey about these two subgroups of $Aut(G)$, the reader is referred to [1, 14, 16]. Now, we introduce two normal subgroups of G related to $Aut_c(G)$ and $Aut_z(G)$ respectively, as follows:

Definition 2.6. *Let G be a group. The set*

$$\{g \in G : [g, \alpha] = 1, \quad for \quad all \quad \alpha \in Aut_c(G)\}$$

is denoted by $L_c(G)$.

Definition 2.7. *Let G be a group. The set*

$$\{g \in G : [g, \alpha] = 1, \quad for \quad all \quad \alpha \in Aut_z(G)\}$$

is denoted by $L_z(G)$.

Note that $Aut_z(G)$ may not contain $Inn(G)$. However merely

$$Aut_z(G) = C_{Aut(G)}\left(Inn\left(G\right)\right)$$

and $Inn\,(G) \cap Aut_z\,(G) = Z\,(Inn\,(G))$. So $L_z(G)$ is not central but $L_z(G) \trianglelefteq G$. Clearly $L_c(G)$ is a central subgroup and so it is normal.
Now we can define $K_c(G)$ and $K_z(G)$ in the way of defining $K(G)$, i.e.

Definition 2.8. *For a group G, the subgroup*

$$\langle [g, \alpha] = g^{-1}\alpha(g) : \alpha \in Aut_c(G)\rangle$$

is denoted by $K_c(G)$.

Note that $K_c(G) = [G, Aut_c(G)]$ hence $G' = [G, Inn(G)] \leq K_c(G)$. On the other hand by the definition of a class preserving automorphism it is very simple to see $K_c(G) \leq G'$ and therefore $K_c(G) = G'$.

Definition 2.9. *For a group G, the subgroup*

$$\langle [g, \alpha] = g^{-1}\alpha(g) : \alpha \in Aut_z(G)\rangle$$

is denoted by $K_z(G)$.

In fact $K_z(G) = [G, Aut_z(G)]$.
Let φ be an automorphism of a group G and let $[G, \varphi] = \langle x^{-1}\varphi(x) : x \in G\rangle$ be the commutator subgroup of φ. Endimioni and Moravec [3] prove a converse of Schur's theorem for the commutator subgroup of G as follows:

Theorem 2.12. *Let φ be an automorphism of a group G such that the subgroup $[G, \varphi]$ is finite. Then the index of $C_G(\varphi)$ in G is finite.*

This theorem can help us to state the following.

Theorem 2.13. *Let $K_z(G)$ and $Aut_z(G)$ are finite groups. Then the index of $L_z(G)$ in G is finite.*

Proof. First note that for each $\alpha \in Aut_z(G)$, the subgroup $[G, \alpha]$ of $K_z(G)$ is finite. From Theorem 2.12 it follows that the index of $C_G(\alpha)$ in G is finite. By definition it is easy to see that $L_z(G) = \bigcap_{\alpha \in Aut_z(G)} C_G(\alpha)$ and the proof is done. □

Theorem 2.14. *For a group G, if G' and $Aut_c(G)$ are finite, then $G/L_c(G)$ is finite.*

Proof. Let $\alpha \in Aut_c(G)$. Then $[G : C_G(\alpha)]$ is finite, since G', is finite. But $L_c(G) = \bigcap_{\alpha \in Aut_c(G)} C_G(\alpha)$, and since this is a finite intersection, it follows that $[G : L_c(G)] = [G : \bigcap_{\alpha \in Aut_c(G)} C_G(\alpha)] \leq \prod_{\alpha \in Aut_c(G)} [G : C_G(\alpha)]$. Hence $G/L_c(G)$ is finite. □

Theorem 2.15. *Let G be a group such that G' is finite and $G/L_c(G)$ is finitely generated. Then $G/L_c(G)$ is finite.*

Proof. It is known that $L_c(G)$ is a central subgroup and so $G/Z(G)$ is finitely generated. Use Theorem 2.1 to observe that $G/Z(G)$ is finite. Consequently $Z(G)/L_c(G)$ is finite. Notice that the former is an abelian group and the proof is complete when we show that it is a torsion group. Consider an element $xL_c(G) \in Z(G)/L_c(G)$ of infinite order. Clearly for every natural number n, x^n is not a member of $L_c(G)$. Therefore there exists an element $g_x \in G$ such that $(x^n)^{g_x} \neq x^n$ but $x^n \in Z(G)$ and this is a contradiction. □

REFERENCES

[1] *W. Burnside*, On the automorphisms of a group, Proc. London Math. Soc.(2) **11** (1913), 40-42.

[2] *H. Dietrich, P. Moravec*, On the autocommutator subgroup and absolute center of a group, J. Alg. **341** (2011), 150-157.

[3] *G. Endimioni, P. Moravec*, On the centralizer and the commutator subgroup of an automorphism, Monatsh. Math. **167** (2011), 1-10.

[4] *R. Hatamian, M. Hassanzadeh, S. Kayvanfar*, A converse of Baer's theorem, Ricerche mat. **63** no. 1, (2014), 183-187.

[5] *P. Hegarty*, Autocommutator subgroups of finite groups, J. Alg. **190** (1997), 556-562.

[6] *P. Hegarty*, The absolute center of a group, J. Alg. **169** (1994), 929-935.

[7] *N. S. Hekster*, On the structure of n-isoclinism classes of groups, J. Pure Appl. Algebra, **40** (1) (1986), 63-85.

[8] *P. Hilton*, On a theorem of Schur, Int. J. Math. Sci., **28** (8) (2001), 455-460.

[9] *P. Hilton*, On variants of a theorem of Schur, Int. J. Math. Sci., **20** (2005), 3269-3276.

[10] *I. M. Isaacs*, Derived subgroups and centers of capable groups, Proc. Amer. Math. Soc. **129** (10) (2001), 2853-2859.

[11] *A. Mann*, The exponents of central factor and commutator groups, J. Group Theory **10** (2007), 435-436.

[12] *K. Podoski , B. Szegedy*, Bounds for the index of the centre in capable groups, Proc. Amer. Math. Soc. **133** (2005), 3441-3445.

[13] *D. J. S. Robinson*, A Course in the The Theory of Groups, Springer Verlage, New York, 1982.

[14] *C. H. Sah*, Automorphisms of finite groups, J. Algebra **10** (1968), 47-68.

[15] *J. Wiegold*, Multiplicators and groups with finite central factor-groups, Math. Z. **89** (1965), 345-347.

[16] *M.K. Yadav*, On automorphisms of some finite p-groups, Proc. Indian Acad. Sci. **118** (2008), 1-11.