NUMERICAL INTEGRAL APPROACHES FOR BUCKLING ANALYSIS OF STRAIGHT BEAMS

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This paper presents two different approaches using an integral approximate method based on flexibility influence functions (Green's functions), concerning the critical buckling load calculation for straight Euler-Bernoulli beams. The integral formulation solves in fact the differential equation governing the bending behavior of a beam subjected to compression loads. The integrals are then computed by a summation using weighting numbers for a chosen number of collocation points on beam axis. Several examples concerning the pin-ended beams and clamped-free beams are analyzed. The first approach is formulated using an integral form of the corresponding differential equation in terms of bending deflection while in the second approach the differential equation is written in terms of bending slope. The numerical results show good agreement with the analytical one.

Keywords: Integral Method, Green's Functions, Beam Buckling, Collocation, Critical Loads.

1. Introduction

The integral approach based on the use of flexibility influence functions (Green's functions), as are they called in [1], was widely used in the structural and aeroelastic analysis for the fixed large aspect ratio cantilever wing problems in the works [2-4]. In [5] the differential equation governing the transverse bending vibration analysis for rotating beams was put in integral form using Green functions in order to obtain the natural frequencies, highlighting the stiffening effect due to the centrifugal forces. The approach was then extended to the coupled bending vibration analysis for pre twisted blades [6]. Then, in [7] was described the more general case of the coupled bending-bending-torsion vibration analysis for straight beams and blades. Other applications of the integral approach in static and dynamic response analysis of beams are described in [8]. In the case of the dynamic, stability or aeroelastic analysis this method leads to an eigenvalues and eigenvectors problem [9].

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This paper presents two standard cases of the buckling analysis for pin-ended and clamped-free beams using the integral forms of the corresponding differential equations and appropriate Green's functions. Two approaches are described: the first one obtains the integral form of the beam equations in terms of bending deflection, while the second one works with the beam bending slopes. For both approaches and both beam configurations, numerical applications are discussed allowing the comparison with known analytical results.

2. Integral and matrix forms for beam differential equations

The differential equation governing the bending behavior for a straight beam, of length L and subjected to a transverse distributed load force $p(x)$, can be written as:

$$[EI(x)w'']'' = p(x)\,. \tag{1}$$

This can be reformulated in the integral form, [1]:

$$w(x) = \int_0^L G_w(x,\xi)\,p(\xi)\,d\xi\,. \tag{2}$$

In this equation, the Green's function $G_w(x,\xi)$ are the bending deflections $w(x,\xi)$ measured at distances x due to unit forces applied at distance ξ (Fig. 1a).

From the Saint Venant torsional behavior of a straight beam of length L and subjected to the distributed torsion moment $m_t(x)$, the differential equation is:

$$[GJ(x)\phi']' + m_t(x) = 0\,. \tag{3}$$

It can take the integral form:

$$\phi(x) = \int_0^L G_t(x,\xi)\,m_t(\xi)\,d\xi\,, \tag{4}$$

using the Green's function $G_t(x,\xi)$ representing the twist deflection angles $\phi(x,\xi)$ at distances x due to unit torsion moments applied at distances ξ (Fig. 1b).

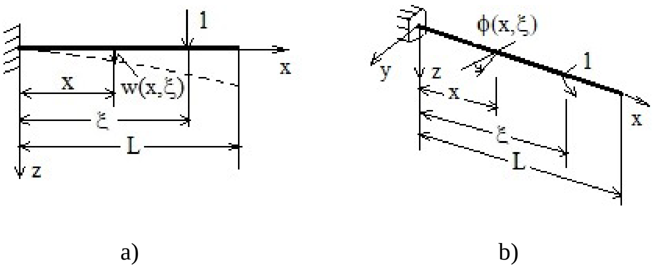

Fig. 1. Physical significance of Green's functions for bending and torsion

In these equations the material of the beam considered isotropic has the longitudinal elastic modulus E and the shear modulus G. The terms $I(x)$ and $J(x)$ represent the moment of inertia of the cross-section of the beam, respectively the torsional stiffness constant. Choosing n collocation points ξ_i with $f_i = f(\xi_i)$, the integral forms (2) and (4) can be approximated as:

$$\int_0^L f(\xi)d\xi = \sum_{i=1}^{n} f_i \cdot W_i \,, \tag{5}$$

where W_i are weighting numbers. In this paper the Simpson's method of integration was used for an even number $n = 2m$ of equally spaced collocation points:

$$\int_0^L f(\xi)d\xi = \frac{L}{n}\left[f(\xi_1) + 2\sum_{k=1}^{m-1} f(\xi_{2k}) + 4\sum_{k=1}^{m} f(\xi_{2k-1}) + f(\xi_n) \right]. \tag{6}$$

The equation (1) can take the matrix form:

$$\{w\} = [G_w][W]\{p\}. \tag{7}$$

In this relation:
$[G_w]$ is the (n,n) symmetric matrix of the Green's functions values,
$[W]$ is a (n,n) diagonal weighting matrix corresponding to the Simpson's method,
$\{w\}$ and $\{p\}$ are column vectors of the bending deflections and of the distributed transverse forces $p(\xi)$ in the chosen n collocation points respectively.
The equation (4) can also be written in matrix form:

$$\{\phi\} = [G_t][W]\{m_t\}, \tag{8}$$

where:
$[G_t]$ is the (n,n) symmetric matrix containing Green's functions values,
$\{\phi\}$ and $\{m_t\}$ are column vectors of the torsion deflections and of the distributed torsion moments $m_t(\xi)$ in the n collocation points respectively.

3. Buckling analysis of pin-ended straight beam

In the first case the buckling analysis of a pin-ended straight beam is carried out. The beam is loaded in compression by a force P. In this boundary conditions case, the Green's functions $G_w(x,\xi) = w(x,\xi)$ are shown in the figure below:

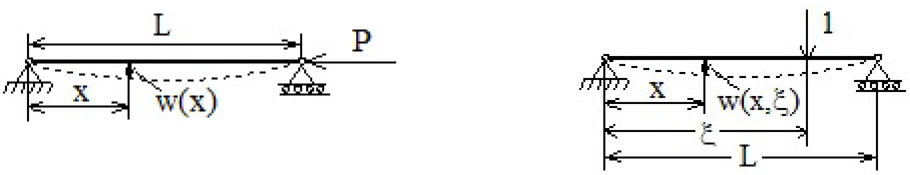

Fig. 2. Buckling of a pin-ended straight beam and the corresponding Green function

One starts from the equation governing the bending displacements written as:

$$EI(x)w'' = -Pw; \quad or\,[EI(x)w'']'' = -Pw''. \tag{9}$$

The matrix form of the last equations takes the form:

$$\{w\} = -P[G_w][W][D_2]\{w\} = -P[G]\{w\}. \qquad (10)$$

In the previous relation $[D_2]$ is a differentiating matrix used to obtain the second derivative of the bending deflection. This is calculated using a central difference operator. Equation (10) represents an eigenvalue problem having the dimension given by the number n of collocation points:

$$[[A_1] + P[I]]\{w\} = \{0\} \qquad (11)$$

where $[A_1] = inv[G]$. The eigenvalues of the matrix $[A_1]$ are the critical buckling loads ($\lambda = - P_c$). The collocation points ξ_i are chosen such that the first point ξ_1 is near $x = 0$ and the last point ξ_n is located near the end of the beam ($x = L$), in order to avoid the null columns or rows values in the matrix $[G_w]$.

One can reduce the dimension of the eigenvalue problem by using collocation functions. The displacement w is written in this case as:

$$w(x) = \sum_{k=1}^{p} C_k \cdot f_k(x), \qquad (12)$$

where $f_k(x)$ are p known functions corresponding to the boundary conditions and C_k are constant coefficients. One obtains relations of the form:

$$\{w\} = [F]\{C\}; \quad \{w^{'}\} = [F_1]\{C\}; \quad \{w^{''}\} = [F_2]\{C\}. \qquad (13)$$

The matrices $[F]$, $[F_1]$, $[F_2]$ contain the values f_k, $f_k^{'}$, $f_k^{''}$ in the collocation points. Their dimensions are (n, p). An advantage of this formulation is that the differentiating matrices are no more necessary and (10) can be written as:

$$[F]\{C\} = -P[G_w][W][F_2]\{C\}. \qquad (14)$$

Multiplying left with transpose of matrix $[F]$ one obtains matrix relations as:

$$[A]\{C\} = -P[B]\{C\}, \qquad (15)$$

or:

$$[B]^{-1}[A]\{C\} = -P\{C\}, \qquad (16)$$

which can be written as an eigenvalue problem:

$$[[A_2] + P[I]]\{C\} = \{0\}, \qquad (17)$$

where $[A_2]$ is now a $p \times p$ matrix ($p < n$), and its eigenvalues represent the critical buckling loads ($\lambda = - P_c$).

Another way considered in this paper is using a similar approach but formulation is written in terms of bending slopes. Starting from (9):

$$[EI(x)w'']' = -Pw' \qquad (18)$$

and using the notation $w' = \varphi$ for the local bending slope, the above equation becomes:

$$[EI(x)\varphi']' + P\varphi = 0 \qquad (19)$$

This equation can be considered of the form (3) and its integral form is similar with (4) if $m(\xi) = P\varphi(\xi)$:

$$\varphi(x) = \int_0^L G_\varphi(x,\xi) m(\xi) d\xi \,. \tag{20}$$

In the above relation, the Green's function values $G_\varphi(x, \xi)$ represents the bending deflection slopes $\varphi(x, \xi)$ at distances x due to unit bending moments applied at distances ξ (Fig. 3) and $m(\xi)$ is the distributed bending moment.

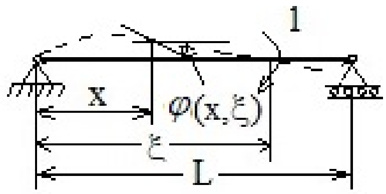

Fig. 3. Green's functions for bending slope in the case of pin-ended straight beam

In matrix form equation (20) becomes:

$$\{\varphi\} = P[G_\varphi][W]\{\varphi\}, \tag{21}$$

which is an eigenvalue problem:

$$[[A_3] - P[I]]\{\varphi\} = \{0\} \tag{22}$$

with $[A_3] = inv([G_\varphi][W])$. The eigenvalues of the matrix $[A_3]$ determine the buckling loads ($\lambda = P_c$).

As numerical application a beam having the constant bending rigidity $EI = 1$ and $L = 1$ is analyzed. According to [10], the analytical results concerning the first three critical buckling loads are the followings:

$$P_{c1} = \frac{\pi^2 EI}{L^2} = \pi^2; \quad P_{c2} = 4\pi^2; \quad P_{c3} = 9\pi^2. \tag{23}$$

The first critical buckling loads determined using the relation (11) based on the matrix $[G_w]$ and n collocation points are given in the next table.

Table 1

Results for the pin-ended straight beam - relation (11), collocation points

	n = 10	n = 20	n = 40	n = 60	n = 100	Exact
P_{c1}	11.246	10.422	10.119	10.03	9.963	π^2=9.869
P_{c2}	50.7	42.848	40.74	40.236	39.894	$4\pi^2$ = 39.478
P_{c3}	139.473	100.947	92.668	90.961	89.693	$9\pi^2$ = 88.826

The results are improved by increasing the collocation points number n. The numerical differentiation is a source of errors, so it is preferable to avoid the differentiating matrix $[D_2]$ using the collocation functions approach. This approach is especially efficient in the case of the calculation of critical buckling loads for non-uniform cross-section beams using the real buckling mode shapes for the uniform beam which are compatible with the boundary conditions. For the pin-ended straight beam these functions are, [10]:

$$w_k(x) = \sin\left(\frac{k\pi x}{L}\right) \qquad k = 1..p. \tag{24}$$

The next table shows the results for the uniform beam in the case of the use of relation (22) based on the matrix [G_φ] and n collocation points.

Table 2

Results for the pin-ended straight beam-relation (22), collocation points

	n = 10	n = 20	n = 40	n = 60	n = 100	Exact
P_{c1}	10.404	10.171	10.027	9.976	9.934	π^2=9.869
P_{c2}	39.959	40.320	40.024	39.868	39.724	$4\pi^2$ = 39.478
P_{c3}	83.214	89.339	89.732	89.563	89.329	$9\pi^2$ = 88.826

The convergence is slow as in this case the matrix [G_φ] can contain also very small positive and negative values.

The next example concerns the calculation of the first critical buckling loads for a non-uniform stepped cross-section beam having the total length L. (see fig. 4).

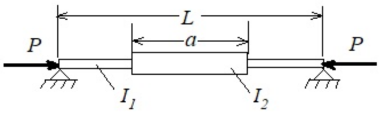

Fig. 4. Non-uniform pin-ended stepped cross-section beam

The idea is to use the approach with collocation points and the collocation functions representing in fact the buckling mode shapes for the uniform beam given by (24). The table 3 presents the results obtained for n = 100 collocation points and p = 5 collocation functions, in comparison with those of [10], in terms of the parameter λ calculated using the formula giving the first critical buckling load:

$$P_{c1} = \frac{\lambda E I_2}{L^2} \tag{25}$$

The results obtained in the present work show good agreement when compared with those of [10]. This reference obtains the buckling loads based on the resolution of transcendental equations.

Table 3

λ values for pin-ended stepped beam -relation (17), collocation functions

I_1/I_2	a/L=0.2	a/L=0.4	a/L=0.6	a/L=0.8	Source
0.01	0.15	0.27	0.60	2.26	[10]
	0.150	0.271	0.601	2.287	present
0.1	1.47	2.40	4.50	8.59	[10]
	1.468	2.406	4.508	8.670	present
0.2	2.80	4.22	6.69	9.33	[10]
	2.796	4.228	6.700	9.346	present

0.4	5.09	6.68	8.51	9.67	[10]
	5.089	6.680	8.510	9.675	present
0.6	6.98	8.19	9.24	9.78	[10]
	6.979	8.185	9.243	9.782	present
0.8	8.55	9.18	9.63	9.84	[10]
	8.550	9.175	9.630	9.836	present

4. Buckling analysis of clamped-free straight beam

A clamped-free straight beam compressed with the force P and the corresponding Green's functions values $G_w(x,\xi) = w(x,\xi)$ are shown in Fig. 5.

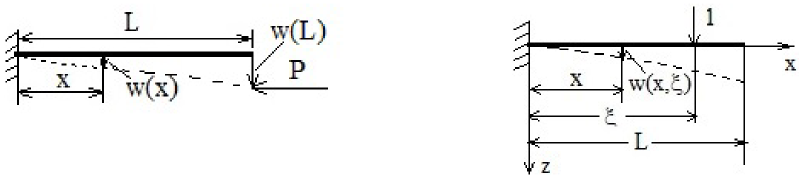

Fig. 5. Buckling of a clamped-free straight beam and the used Green function

In this case, the equation governing the bending behavior can be written as:

$$EI(x)w'' = -P[w - w(L)] \qquad (26)$$

Comparing equations (26) and (9), it has a supplementary term, the bending moment $Pw(L)$. After a first differentiation with respect to x this equation becomes:

$$[EI(x)w'']' = -P[w' - w'(L)] \qquad (27)$$

where the supplementary term $Pw'(L)$ can be considered as a transverse tip concentrated force. The second differentiation with respect to x leads to:

$$[EI(x)w'']'' = -P[w'' - w''(L)] \qquad (28)$$

Neglecting the term in $w''(L)$, the integral form of the equation (28) regarded as of the form (1) becomes:

$$w(x) = -P\int_0^L G_w(x,\xi)w''(\xi)d\xi + P\cdot G_w(x,L)\cdot w'(L) \qquad (29)$$

where the second term of (29) gives the influence of the concentrated tip transverse force $Pw'(L)$ from (27) before the second differentiation. The relation (29) takes the following matrix form:

$$\{w\} = -P[G_w][W][D_2]\{w\} + P[G_w^*][D_1]\{w\}, \qquad (30)$$

where

$[D_1]$ is a differentiating matrix used to obtain the first derivative of the bending deflection,

$\left[G_w^*\right]$ is a (n,n) correction matrix used only to obtain the terms containing the first derivative of the bending deflection $w'(L)$ at beam end, considered as given by the relation:

$$w'(L) = \frac{w(L) - w(L - \Delta x)}{\Delta x} = \frac{w_n - w_{n-1}}{x_n - x_{n-1}} \tag{31}$$

The collocation points ξ_i are chosen equally spaced with the step $\Delta\xi = L/n$, the first point ξ_1 being near $x = 0$ and the last point $\xi_n = L$ being exactly at the tip of the beam. The Green functions can be computed using the formula:

$$G_w(x,\xi) = \begin{cases} \int_0^\xi \frac{(x-\xi_1)(\xi-\xi_1)d\xi_1}{EI(\xi_1)} & for\ \xi \le x \\ \int_0^x \frac{(x-\xi_1)(\xi-\xi_1)d\xi_1}{EI(\xi_1)} & for\ x \le \xi \end{cases} \tag{32}$$

Following a similar procedure one can consider the equation written in terms of bending deflection slopes. Starting from (26) written with $w' = \varphi$ one can obtain:

$$[EI(x)\varphi']' + P\varphi - P\varphi'(L) = 0 \tag{33}$$

Neglecting the last term, the integral form of (33) is the same as (20):

$$\varphi(x) = \int_0^L G_\varphi(x,\xi)m(\xi)d\xi \tag{34}$$

but now the Green's function values $G_\varphi(x,\xi)$ are the bending deflection slopes $\varphi(x,\xi)$ according to Fig. 6 and $m(\xi)$ is the distributed bending moment.

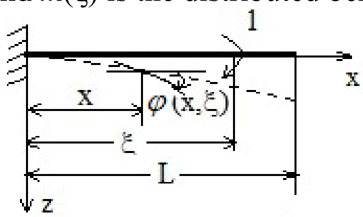

Fig. 6. Green's functions for bending deflection slope in the case of clamped-free straight beam

The matrix form is similar to equation (21) with a different matrix $[G_\varphi(x,\xi)]$. This represents an eigenvalue problem allowing the buckling loads determination. In this case, the Green functions can be computed using the formula:

$$G_\varphi(x,\xi) = \begin{cases} \int_0^\xi \frac{d\xi_1}{EI(\xi_1)} & for\ \xi \le x \\ \int_0^x \frac{d\xi_1}{EI(\xi_1)} & for\ x \le \xi \end{cases} \tag{35}$$

As numerical application a beam having the constant bending rigidity $EI = 1$ and $L = 1$ is considered. In this case, the first three critical buckling loads are according to [10]:

$$P_{c1} = \frac{\pi^2 EI}{4L^2} = \frac{\pi^2}{4}; \quad P_{c2} = \frac{9\pi^2}{4}; \quad P_{c3} = \frac{25\pi^2}{4}. \tag{36}$$

The first three critical buckling loads determined using the relation (30) based on the matrix [G_w] and n collocation points are given in table 4.

Table 4

Results for the clamped-free beam - relation (30), collocation points

	n = 10	n = 20	n = 40	n = 60	n = 100	Exact
P_{c1}	2.460	2.465	2.466	2.467	2.467	$\pi^2/4$=2.467
P_{c2}	21.652	22.069	22.172	22.191	22.201	$9\pi^2/4$ =22.206
P_{c3}	57.295	60.620	61.620	61.567	61.642	$25\pi^2/4$= 61.685

The precision increases with the number of the collocation points number n.

The next table shows the results in the case of the use of relation (22) based on the matrix [G_φ] and n collocation points.

Table 5

Results for the clamped-free beam - relation (22), collocation points

	n = 10	n = 20	n = 40	n = 60	n = 100	Exact
P_{c1}	2.460	2.465	2.466	2.467	2.467	$\pi^2/4$=2.467
P_{c2}	21.652	22.069	22.172	22.191	22.201	$9\pi^2/4$ =22.206
P_{c3}	57.295	60.620	61.620	61.567	61.642	$25\pi^2/4$= 61.685

The results are in good agreement as in this case, according to the relation (35), matrix [G_φ] contains only positive values.

Another example considering a non-uniform beam and using the collocation function approach is presented next. As appropriate collocation functions one can take the family of the real vibration mode shapes of a clamped-free uniform beam, [11]:

$$w_k(x) = T(\beta_k)\cdot U\left(\beta_k \frac{x}{L}\right) - S(\beta_k)\cdot V\left(\beta_k \frac{x}{L}\right), \tag{37}$$

with the functions:

$$S(x) = \frac{ch\, x + \cos x}{2}, T(x) = \frac{sh\, x + \sin x}{2},$$
$$U(x) = \frac{ch\, x - \cos x}{2}, V(x) = \frac{sh\, x - \sin x}{2} \tag{38}$$

and:

$$\beta_1 = 1.8751, \beta_2 = 4.6941, \beta_i = \frac{2i-1}{2}\pi, k = 3...n. \tag{39}$$

The example concerns the calculation of the first critical buckling loads for a non-uniform cross-section clamped-free beam (fig. 7).

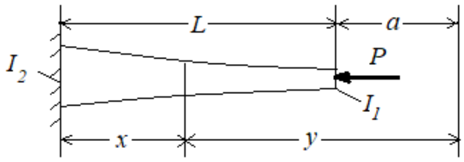

Fig. 7. Non-uniform clamped-free beam

The bending stiffness distribution for this example from [10] is in terms of distances *a* and *y*:

$$EI(y) = EI_1\left(\frac{y}{a}\right)^k \tag{40}$$

Here we will use the variable *x*, so:

$$EI(x) = EI_1\left(\frac{a+L-x}{a}\right)^k \tag{41}$$

with EI_2 given at $x = 0$:

$$EI_2 = EI_1\left(\frac{a+L}{a}\right)^k \tag{42}$$

For given values of I_1/I_2 and *k* one can obtain the distance *a*.

The first approach uses the relation (17) with collocation points and the collocation functions (37). The second one based on bending slopes uses only collocation points and relation (22).

The table below presents the results obtained for $n = 100$ points and $p = 5$ functions, in comparison with those presented in [10] in terms of the parameter λ from the formula:

$$P_{c1} = \frac{\lambda EI_2}{L^2} \tag{43}$$

Table 6

λ values for the clamped-free non-uniform cross section beam

I_1/I_2	$k = 2$			$k = 4$		
	[10]	(17)	(22)	[10]	(17)	(22)
0.1	1.350	1.341	1.3359	1.202	1.194	1.1897
0.2	1.593	1.5829	1.5796	1.505	1.4964	1.4935
0.3	1.763	1.7566	1.754	1.710	1.7026	1.7002
0.4	1.904	1.8974	1.895	1.870	1.8635	1.8614
0.5	2.023	2.0179	2.016	2.002	1.9973	1.9955
0.6	2.128	2.1246	2.123	2.116	2.1129	2.1112
0.7	2.223	2.2211	2.219	2.217	2.2151	2.2135

0.8	2.311	2.3096	2.308	2.308	2.3072	2.3057
0.9	2.392	2.3918	2.3903	2.391	2.3912	2.3898

The results obtained for both formulations agree well with the analytical ones obtained in [10] by solutions of Bessel type differential equations.

6. Conclusions

This work presents two simple integral formulations for stability analysis of straight Euler-Bernoulli beams. These approaches are based on the use of *flexibility influence functions* (Green's functions). These functions are numerically computed, taking into account the fact that they are displacements in some points of a beam due to unit forces applied in other points. The symmetric matrix containing the Green's functions values is computed in a number of *collocation points*. An *integration matrix* based of Simpson's method of integration is also employed. *Differentiating matrices* are used in order to obtain the first and the second derivative of the bending deflections. In order to avoid these differentiating matrices one can use *collocation functions* depending upon the boundary conditions. In fact the collocation functions are suitable especially in the case of non-uniform cross-section beams when the buckling mode shapes or natural mode shapes of vibrations of uniform beams can be employed.

The second approach based on the use of the buckling differential equations written in terms of bending slopes is a specific contribution of this work. This approach is suitable especially for the clamped-free beam critical buckling loads calculation, as no differentiation matrices or collocation functions are necessary. In this work the standard cases of the pin-ended beam and of the clamped-free beam have been analyzed. The Green's functions are computed numerically. In the case of the clamped-free beam they are given by simple relations listed in the text. The numerical examples, for several critical buckling loads calculations show good agreement with known analytical results obtained for uniform or non-uniform cross section beams, the accuracy depending on the number of the used collocation points. Both approaches are in fact matrix formulations, suitable for Matlab/Octave implementation, an appropriate calculation software for numerical and data manipulation point of view.

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