GEOMETRY OF A WEAK PARA-f-STRUCTURE

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Dedicated to the memory of Prof. Emeritus Dr. Constantin Udrişte

Introducing the weak para-f-structure on a smooth $(2n+s)$-dimensional manifold allows us to take a fresh look at the geometry of the para-f-structure by A. Bucki and A. Miernowski and find new applications. We demonstrate this by generalizing some known results on para-f-manifolds.

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Introduction

Totally geodesic foliations have the simplest extrinsic geometry of the leaves and appear in Riemannian geometry, e.g., in the theory of $\mathfrak{g}$-foliations, as kernels of degenerate tensors, see [1, 6]. We are motivated by the problem of finding structures on manifolds, which lead to totally geodesic foliations and Killing vector fields, see [5]. A well-known source of totally geodesic foliations is the para-f-structure on a smooth manifold M^{2n+p}, defined using (1,1)-tensor field f satisfying $f^3 = f$ and having constant rank $2n$, see [3, 7]. The paracontact geometry (a counterpart to the contact geometry) is a higher dimensional analog of almost product ($p = 0$) and almost paracontact ($p = 1$) structures [4]. A para-f-structure with $p = 2$ arises on hypersurfaces in almost contact manifolds, e.g., [2]. Interest in para-Sasakian manifolds is due to their connection with para-Kähler manifolds and their role in mathematical physics. If there exists a set of vector fields $\xi_1, \ldots, \xi_p$ with certain properties, then M^{2n+p} is said to have a para-f-structure with complemented frames. In this case, the tangent bundle TM splits into three complementary subbundles: ± 1-eigen-distributions for f composing a $2n$-dimensional distribution $f(TM)$ and a p-dimensional distribution $\ker f$ (the kernel of f).

In [9], we introduced the "weak" metric structures that generalize an f-structure and a para-f-structure, and allow us to take a fresh look at the classical theory. In [8], we studied geometry of the weak f-structure and its satellites that are analogs of $\mathcal{K}$- $\mathcal{S}$- and $\mathcal{C}$-manifolds. In this paper, using a similar approach, we study geometry of the weak para-f-structure and its important cases related to a pseudo-Riemannian manifold endowed with a totally geodesic foliation. A natural question arises: how rich are weak para-f-structures compared to the classical ones? We study this question for weak analogs of para-$\mathcal{K}$-, para-$\mathcal{S}$- and para-$\mathcal{C}$- structures. The proofs of main results use the properties of new tensors, as well as the constructions required in the classical case. The theory presented here can be used to deepen our knowledge of pseudo-Riemannian geometry of manifolds equipped with distributions.

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1. **Preliminaries**

Here, we describe "weak" metric structures generalizing certain classes of para-f-manifolds and discuss their properties. A *weak para-f-structure* on a smooth manifold M^{2n+p} is defined by a $(1,1)$-tensor field f of rank $2\,n$ and a nonsingular $(1,1)$-tensor field Q satisfying, see [9],

$$f^3 - fQ = 0, \qquad Q\,\xi = \xi \quad (\xi \in \ker f). \tag{1}$$

If $\ker f = \{X \in TM : f(X) = 0\}$ is parallelizable, then we fix vector fields ξ_i $(1 \le i \le p)$, which span $\ker f$, and their dual one-forms η^i. We get a *weak almost para-f-structure* (a weak almost paracontact structure for $p = 1$), see [9],

$$f^2 = Q - \sum\nolimits_i \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j\,. \tag{2}$$

By (2), $f(TM) = \bigcap_i \ker \eta^i$ and $f(TM)$ is f-invariant., i.e., $fX \in f(TM)$, $X \in f(TM)$. Thus, $f(TM)$ is invariant for Q. A weak almost f-structure is called *normal* if the following tensor (known for $Q = \mathrm{id}_{TM}$, e.g., [6]) is zero:

$$N^{(1)}(X,Y) = [f,f](X,Y) - 2\sum\nolimits_i d\eta^i(X,Y)\,\xi_i. \tag{3}$$

The Nijenhuis torsion of f and the exterior derivative of η^i are given by

$$[f,f](X,Y) = f^2[X,Y] + [fX,fY] - f[fX,Y] - f[X,fY], \ X,Y \in \mathfrak{X}_M, \tag{4}$$

$$d\eta^i(X,Y) = (1/2)\,\{X(\eta^i(Y)) - Y(\eta^i(X)) - \eta^i([X,Y])\}, \quad X,Y \in \mathfrak{X}_M. \tag{5}$$

If there exists a pseudo-Riemannian metric g such that

$$g(fX,fY) = -g(X,Q\,Y) + \sum\nolimits_i \eta^i(X)\,\eta^i(Y), \quad X,Y \in \mathfrak{X}_M, \tag{6}$$

then (f,Q,ξ_i,η^i,g) is called a *metric weak para-f-structure*, $M(f,Q,\xi_i,\eta^i,g)$ is called a *metric weak para-f-manifold*, and g is called a *compatible metric*. Putting $Y = \xi_i$ in (6) and using (1), we get $g(X,\xi_i) = \eta^i(X)$, thus, $f(TM) \perp \ker f$ and $\{\xi_i\}$ is an orthonormal frame of $\ker f$.

We can rewrite (4) in terms of the Levi-Civita connection ∇ (of g) as

$$[f,f](X,Y) = (f\nabla_Y f - \nabla_{fY} f)X - (f\nabla_X f - \nabla_{fX} f)Y. \tag{7}$$

Proposition 1.1. (a) *For a weak almost para-f-structure the following hold:*

$$f\,\xi_i = 0, \quad \eta^i \circ f = 0, \quad \eta^i \circ Q = \eta_i \quad (1 \le i \le p), \quad [Q,\,f] = 0.$$

(b) *For a metric weak almost para-f-structure, we get*

$$g(fX,Y) = -g(X,fY), \quad g(QX,Y) = g(X,QY). \tag{8}$$

Proof. (a) By (1) and (2), $f^2\xi_i = 0$. Applying (1) to $f\xi_i$, we get $f\xi_i = 0$. To show $\eta^i \circ f = 0$, note that $\eta^i(f\,\xi_i) = \eta^i(0) = 0$, and we get $\eta^i(fX) = 0$ for $X \in f(TM)$. Next, using (2) and $f(Q\,\xi_i) = f\,\xi_i = 0$, we get

$$f^3X = f(f^2X) = f\,QX - \sum\nolimits_i \eta^i(X)\,f\xi_i = f\,QX,$$

$$f^3X = f^2(fX) = Q\,fX - \sum\nolimits_i \eta^i(fX)\,\xi_i = Q\,fX$$

for any $X \in f(TM)$. This and $[Q,\,f]\,\xi_i = 0$ provide $[Q,\,f] = Q\,f - fQ = 0$.

(b) By (6), the restriction $Q_{|\,f(TM)}$ is self-adjoint. This and (1) provide (8b) – the symmetry of Q. For any $Y \in f(TM)$ there is $\tilde Y \in f(TM)$ such that $fY = \tilde Y$. From (2) and (6) with $X \in f(TM)$ and $\tilde Y$ we get

$$g(fX,\tilde Y) = g(fX,fY) \overset{(6)}{=} -g(X,QY) \overset{(2)}{=} -g(X,f^2Y) = -g(X,f\tilde Y),$$

and the skew-symmetry of f, see (8a), follows. $\square$

Remark 1.1. If we assume that the symmetric tensor Q is positive definite, then $f(TM)$ decomposes into the sum of two n-dimensional subbundles: $f(TM)=\mathcal{D}_+\oplus\mathcal{D}_-$, corresponding to positive and negative eigenvalues of f, and in this case we get $TM=\mathcal{D}_+\oplus\mathcal{D}_-\oplus\ker f$.

Definition 1.1. A metric weak para-f-structure is called a *weak para-$\mathcal{K}$-structure* if it is normal and the fundamental 2-form $\Phi(X,Y)=g(X,fY)$ is closed, i.e., $d\Phi=0$. We define two subclasses of weak para-$\mathcal{K}$-manifolds as follows: *weak para-$\mathcal{C}$-manifolds* if $d\eta^i=0$ for any i, and *weak para-$\mathcal{S}$-manifolds* if

$$d\eta^i=\Phi,\quad 1\le i\le p. \tag{9}$$

Omitting the normality condition, we get the following: a metric weak para-f-structure is called (i) a *weak almost para-$\mathcal{S}$-structure* if (9) is valid; (ii) a *weak almost para-$\mathcal{C}$-structure* if Φ and η^i are closed forms.

For $p=1$, weak para-$\mathcal{C}$- and weak para-$\mathcal{S}$- manifolds reduce to weak para-cosymplectic manifolds and weak para-Sasakian manifolds, respectively. Recall the formulas with the Lie derivative $\pounds_Z$ in the Z-direction and $X,Y\in\mathfrak{X}_M$:

$$(\pounds_Z f)X = [Z,fX]-f[Z,X], \tag{10}$$
$$(\pounds_Z\,\eta^j)X = Z(\eta^j(X))-\eta^j([Z,X]), \tag{11}$$
$$\begin{aligned}(\pounds_Z\,g)(X,Y) &= Z(g(X,Y))-g([Z,X],Y)-g(X,[Z,Y])\\ &= g(\nabla_X Z,Y)+g(\nabla_Y Z,X).\end{aligned} \tag{12}$$

The following tensors are known in the theory of para-f-manifolds, e.g., [6]:

$$N^{(2)}_i(X,Y)=(\pounds_{fX}\,\eta^i)Y-(\pounds_{fY}\,\eta^i)X\overset{(5)}{=}2\,d\eta^i(fX,Y)-2\,d\eta^i(fY,X), \tag{13}$$

$$N^{(3)}_i(X)=(\pounds_{\xi_i}f)X\overset{(10)}{=}[\xi_i,fX]-f[\xi_i,X], \tag{14}$$

$$N^{(4)}_{ij}(X)=(\pounds_{\xi_i}\,\eta^j)X\overset{(11)}{=}\xi_i(\eta^j(X))-\eta^j([\xi_i,X])=2\,d\eta^j(\xi_i,X). \tag{15}$$

Define the difference tensor $\tilde Q=Q-\mathrm{id}_{TM}$ (vanishing on a para-f-structure). By the above, $\widetilde Q\,\xi_i=0$ and $[\tilde Q,f]=0$.

Remark 1.2. Let $M^{2n+p}(\varphi,Q,\xi_i,\eta^i)$ be a framed weak para-f-manifold. Consider the product manifold $\bar M=M^{2n+p}\times\mathbb{R}^p$, where $\mathbb{R}^p$ is a Euclidean space with a basis $\partial_1,\dots,\partial_p$, and define tensor fields $\bar f$ and $\bar Q$ on $\bar M$ putting

$$\bar f(X,\sum a^i\partial_i)=(fX-\sum a^i\xi_i,\sum\eta^j(X)\partial_j),\ \ \bar Q(X,\sum a^i\partial_i)=(QX,\sum a^i\partial_i).$$

Then $\bar f^{\,2}=-\bar Q$. The tensors $N^{\,(i)}$ appear when we use the integrability $[\bar f,\bar f]=0$ of $\bar f$ to express the normality of a weak almost para-f-structure.

2. The geometry of a metric weak para-f-structure

Here, we study the geometry of the characteristic distribution $\ker f$, supplement the sequence of tensors (3) and (13)–(15) with a new tensor $N^{\,(5)}$ and calculate the covariant derivative of f on a metric weak para-f-structure.

A distribution $\mathcal{D}\subset TM$ is *totally geodesic* if and only if $\nabla_XY+\nabla_YX\in\mathcal{D}$ for any vector fields $X,Y\in\mathcal{D}$ – this is the case when any geodesic of M that is tangent to $\mathcal{D}$ at one point is tangent to $\mathcal{D}$ at all its points. Any integrable and totally geodesic distribution determines a totally geodesic foliation. A foliation, whose orthogonal distribution is totally geodesic, is called a Riemannian foliation. For example, a foliation is Riemannian if it is invariant under isometries generated by Killing vector fields. Note that $X=X^\top+X^\perp$, where $X^\top$ is the projection of the vector $X\in TM$ onto $f(TM)$, and $X^\perp=\sum_i\eta^i(X)\,\xi_i$.

The next statement generalizes [6, Proposition 3], i.e., $Q=\mathrm{id}_{TM}$.

Proposition 2.1. *Let a metric weak para-f-structure be normal. Then the distribution* $\ker f$ *is totally geodesic, the tensors* $N_i^{(3)}$ *and* $N_{ij}^{(4)}$ *vanish and*

$$N_i^{(2)}(X,Y)=\eta^i([\widetilde{Q}X,\,fY]). \tag{16}$$

Proof. Assume $N^{\,(1)}=0$. Taking ξ_i instead of Y and using (4), we get

$$0=[f,f](X,\xi_i)-2\sum\nolimits_j d\eta^j(X,\xi_i)\xi_j=f^2[X,\xi_i]-f[fX,\xi_i]-2\sum\nolimits_j d\eta^j(X,\xi_i)\xi_j. \tag{17}$$

For the scalar product of (17) with ξ_j, using $f\,\xi_i=0$, we get

$$d\eta^j(\xi_i,\,\cdot)=0; \tag{18}$$

hence, $N_{ij}^{(4)}=0$, see (15). Next, combining (17) and (18), we get $0=[f,f](X,\xi_i)=f^2[X,\xi_i]-f[fX,\xi_i]=f\,(\pounds_{\xi_i}f)X$. Applying f and using (2) and $\eta^i\circ f=0$, we achieve

$$0=f^2(\pounds_{\xi_i}f)X=Q(\pounds_{\xi_i}f)X-\sum\nolimits_j\eta^j([\xi_i,fX])\,\xi_j. \tag{19}$$

Further, (18) and (5) yield

$$0=2\,d\eta^j(fX,\xi_i)=(fX)(\eta^j(\xi_i))-\xi_i(\eta^j(fX))-\eta^j([fX,\xi_i])=\eta^j([\xi_i,fX]). \tag{20}$$

Since Q is non-singular, from (19)–(20) we get $\pounds_{\xi_i}f=0$, i.e, $N_i^{(3)}=0$, see (14). Replacing X by fX in our assumption $N^{\,(1)}=0$ and using (4) and (5), we get

$$\begin{aligned}0&=g([f,f](fX,Y)-2\sum\nolimits_j d\eta^j(fX,Y)\,\xi_j,\ \xi_i)\\&=g([f^2X,fY],\xi_i)-(fX)(\eta^i(Y))+\eta^i([fX,Y]),\quad 1\le i\le p.\end{aligned} \tag{21}$$

Using (2) and $[fY,\eta^j(X)\xi_i]=(fY)(\eta^j(X))\xi_i+\eta^j(X)[fY,\xi_i]$, we rewrite (21) as

$$0=\eta^i([QX,fY])-\sum\eta^j(X)\,\eta^i([\xi_j,fY])+fY(\eta^i(X))-fX(\eta^i(Y))+\eta^i([fX,Y]).$$

Since (20) gives $\eta^i([fY,\xi_j])=0$, the above equation becomes

$$\eta^i([QX,fY])+(fY)(\eta^i(X))-(fX)(\eta^i(Y))+\eta^i([fX,Y])=0. \tag{22}$$

Finally, combining (22) with (13), we get (16). Using the identity

$$\pounds_{\xi_i}=\iota_{\xi_i}\,d+d\,\iota_{\xi_i}, \tag{23}$$

from (18) and $\eta^i(\xi_j)=\delta^i_j$ we obtain $\pounds_{\xi_i}\eta^j=d(\eta^j(\xi_i))+\iota_{\xi_i}d\eta^j=0$. On the other hand, by (11) we have $(\pounds_{\xi_i}\eta^j)X=g(X,\nabla_{\xi_i}\xi_j)+g(\nabla_X\,\xi_i,\,\xi_j),\ X\in\mathfrak{X}_M$. Symmetrizing this and using $\pounds_{\xi_i}\eta^j=0$ and $g(\xi_i,\,\xi_j)=\delta_{ij}$ yield

$$\nabla_{\xi_i}\xi_j+\nabla_{\xi_j}\xi_i=0, \tag{24}$$

thus, the distribution $\ker f$ is totally geodesic. □

Recall the co-boundary formula for exterior derivative d on a 2-form Φ,

$$\begin{aligned}d\Phi(X,Y,Z)\ &=\ (1/3)\big\{X\,\Phi(Y,Z)+Y\,\Phi(Z,X)+Z\,\Phi(X,Y)\\&\quad-\Phi([X,Y],Z)-\Phi([Z,X],Y)-\Phi([Y,Z],X)\big\}.\end{aligned} \tag{25}$$

The following result generalizes [6, Proposition 4].

Theorem 2.1. *On a weak para-$\mathcal{K}$-manifold the vector fields $\xi_1,\ldots,\xi_p$ are Killing and the following is valid:*

$$\nabla_{\xi_i}\xi_j=0,\qquad 1\le i,j\le p; \tag{26}$$

thus, the distribution $\ker f$ *is integrable and defines a totally geodesic Riemannian foliation with flat leaves.*

Proof. By Proposition 2.1, the distribution $\ker f$ is totally geodesic, see (24), and $N^{(3)}_i = \pounds_{\xi_i} f = 0$. Using $\iota_{\xi_i}\Phi = 0$ and condition $d\Phi = 0$ in the identity (23), we get $\pounds_{\xi_i}\Phi = 0$. By direct calculation we get the following:

$$(\pounds_{\xi_i}\Phi)(X,Y) = (\pounds_{\xi_i} g)(X, fY) + g(X, (\pounds_{\xi_i} f)Y). \tag{27}$$

From (27) we obtain $(\pounds_{\xi_i} g)(X, fY) = 0$. To show $\pounds_{\xi_i} g = 0$, we will examine $(\pounds_{\xi_i} g)(fX, \xi_j)$ and $(\pounds_{\xi_i} g)(\xi_k, \xi_j)$. Using $\pounds_{\xi_i}\eta^j = 0$, we get

$$(\pounds_{\xi_i} g)(fX, \xi_j) = (\pounds_{\xi_i}\eta^j) fX - g(fX, [\xi_i, \xi_j]) = -g(fX, [\xi_i, \xi_j]) = 0.$$

Next, using (24), we get $(\pounds_{\xi_i} g)(\xi_k, \xi_j) = -g(\xi_i, \nabla_{\xi_k}\xi_j + \nabla_{\xi_j}\xi_k) = 0$. Thus, ξ_i is a Killing vector field, i.e., $\pounds_{\xi_i} g = 0$. By $d\Phi(X, \xi_i, \xi_j) = 0$ and (25) we obtain $g([\xi_i, \xi_j], fX) = 0$, i.e., $\ker f$ is integrable. From this and (24) we get $\nabla_{\xi_k}\xi_j = 0$; thus, the sectional curvature is $K(\xi_i, \xi_j) = 0$. □

Theorem 2.2. *For a weak almost para-$\mathcal{S}$-structure, we get $N^{(2)}_i = N^{(4)}_{ij} = 0$ and*

$$(N^{(1)}(X,Y))^{\perp} = 2\, g(X, f\widetilde{Q}Y)\,\bar\xi\,; \tag{28}$$

moreover, $N^{(3)}_i$ vanishes if and only if ξ_i is a Killing vector field.

Proof. Applying (9) in (13) and using skew-symmetry of f we get $N^{(2)}_i = 0$. Equation (9) with $Y = \xi_i$ yields $d\eta^j(X, \xi_i) = g(X, f\xi_i) = 0$ for any $X \in \mathfrak{X}_M$; thus, we get (18), i.e., $N^{(4)}_{ij} = 0$. Using (9) and

$$g([f,f](X,Y), \xi_i) = g([fX, fY], \xi_i) = -2\, d\eta^i(fX, fY) = -2\,\Phi(fX, fY)$$

for all i, we also calculate

$$\begin{aligned}(1/2)\, g(N^{(1)}(X,Y), \xi_i) &= -d\eta^i(fX, fY) - g(\textstyle\sum_j d\eta^j(X,Y)\,\xi_j, \xi_i)\\ &= -\Phi(fX, fY) - \Phi(X,Y) = g(X, (f^3 - f)Y) = g(X, \widetilde{Q} fY),\end{aligned}$$

that proves (28). Using (9) in the equality $(\pounds_{\xi_i} d\eta^j)(X,Y) = \xi_i(d\eta^j(X,Y)) - d\eta^j([\xi_i, X], Y) - d\eta^j(X, [\xi_i, Y])$, and using (12), we obtain for all i, j

$$(\pounds_{\xi_i} d\eta^j)(X,Y) = (\pounds_{\xi_i} g)(X, fY) + g(X, (\pounds_{\xi_i} f)Y). \tag{29}$$

Since $\pounds_V = \iota_V \circ d + d \circ \iota_V$, the exterior derivative d commutes with the Lie-derivative, i.e., $d \circ \pounds_V = \pounds_V \circ d$, and as in the proof of Theorem 2.1, we get that $d\eta^i$ is invariant under the action of ξ_i, i.e., $\pounds_{\xi_i} d\eta^j = 0$. Therefore, (29) implies that ξ_i is a Killing vector field if and only if $N^{(3)}_i = 0$. □

Theorem 2.3. *For a weak almost para-$\mathcal{C}$-structure, we get $N^{(2)}_i = N^{(4)}_{ij} = 0$, $N^{(1)} = [f,f]$, and (26) is valid; thus, the distribution $\ker f$ is tangent to a totally geodesic foliation with the sectional curvature $K(\xi_i, \xi_j) = 0$. Moreover, $N^{(3)}_i = 0$ $(1 \le i \le p)$ if and only if each ξ_i is a Killing vector field.*

Proof. By (13) and (15) and $d\eta^i = 0$, the tensors $N^{(2)}_i$ and $N^{(4)}_{ij}$ vanish. Moreover, by (3) and (29), respectively, the tensor $N^{(1)}$ coincides with $[f,f]$, and $N^{(3)}_i = \pounds_{\xi_i} f$ $(1 \le i \le p)$ vanish if and only if each ξ_i is a Killing vector. From the equalities $3\, d\Phi(X, \xi_i, \xi_j) = g([\xi_i, \xi_j], fX)$ and $2\, d\eta^k(\xi_j, \xi_i) = g([\xi_i, \xi_j], \xi_k)$ and conditions $d\Phi = 0$ and $d\eta^i = 0$ we obtain $[\xi_i, \xi_j] = 0$, $1 \le i, j \le p$. Next, from $d\eta^i = 0$ and the equality $2\, d\eta^i(\xi_j, X) + 2\, d\eta^j(\xi_i, X) = g(\nabla_{\xi_i}\xi_j + \nabla_{\xi_j}\xi_i, X)$ we obtain (24): $\nabla_{\xi_i}\xi_j + \nabla_{\xi_j}\xi_i = 0$. From the above we get (26). □

The following assertion generalizes [6, Proposition 1].

Proposition 2.2. *For a metric weak para-f-structure we get*

$$2\,g((\nabla_X f)Y,Z)=-3\,d\Phi(X,fY,fZ)-3\,d\Phi(X,Y,Z) \\ -\,g(N^{(1)}(Y,Z),fX)+\sum\nolimits_i(N_i^{(2)}(Y,Z)\eta^i(X) \\ +\,2\,d\eta^i(fY,X)\eta^i(Z)-2\,d\eta^i(fZ,X)\eta^i(Y))+N^{(5)}(X,Y,Z),$$

where a skew-symmetric w.r.t. Y and Z tensor $N^{(5)}(X,Y,Z)$ is defined by

$$\begin{aligned} N^{(5)}(X,Y,Z) &= (fZ)\,(g(X,\widetilde QY))-(fY)\,(g(X,\widetilde QZ))+g([X,fZ],\widetilde QY) \\ &\quad -g([X,fY],\widetilde QZ)+g([Y,fZ]-[Z,fY]-f[Y,Z],\ \widetilde QX). \end{aligned}$$

Proof. Using the skew-symmetry of f, one can compute

$$\begin{aligned} 2\,g((\nabla_X f)Y,Z) &= X\,g(fY,Z)+(fY)\,g(X,Z)-Z\,g(X,fY)+g([X,fY],Z) \\ &\quad +g([Z,X],fY)-g([fY,Z],X)+X\,g(Y,fZ)+Y\,g(X,fZ) \\ &\quad -(fZ)\,g(X,Y)+g([X,Y],fZ)+g([fZ,X],Y)-g([Y,fZ],X). \end{aligned} \tag{30}$$

Using (6), we obtain

$$\begin{aligned} g(X,Z) &= -\Phi(fX,Z)-g(X,\widetilde QZ)+\sum\nolimits_i\big(\eta^i(X)\,\eta^i(Z)+\eta^i(X)\,\eta^i(\widetilde QZ)\big) \\ &= -\Phi(fX,Z)+\sum\nolimits_i\eta^i(X)\,\eta^i(Z)-g(X,\widetilde QZ). \end{aligned} \tag{31}$$

By the skew-symmetry of f and using (31) six times, (30) can be written as

$$\begin{aligned} 2\,g((\nabla_X f)Y,Z) &= X\,\Phi(Y,Z)+(fY)\,\big(-\Phi(fX,Z)+\sum\nolimits_i\eta^i(X)\,\eta^i(Z)\big) \\ &-(fY)\,g(X,\widetilde QZ)-Z\,\Phi(X,Y) \\ &+\Phi([X,fY],fZ)+\sum\nolimits_i\eta^i([X,fY])\eta^i(Z)-g([X,fY],\widetilde QZ)+\Phi([Z,X],Y) \\ &-\Phi([fY,Z],fX)-\sum\nolimits_i\eta^i([fY,Z])\,\eta^i(X)+g([fY,Z],\widetilde QX)+X\,\Phi(Y,Z) \\ &+Y\,\Phi(X,Z)-(fZ)\,\big(-\Phi(fX,Y)+\sum\nolimits_i\eta^i(X)\,\eta^i(Y)\big)+(fZ)g(X,\widetilde QY) \\ &+\Phi([X,Y],Z)+g(f[-fZ,X],fY)+\sum\nolimits_i\eta^i([fZ,X])\eta^i(Y)-g([fZ,X],\widetilde QY) \\ &+g(f[Y,fZ],fX)-\sum\nolimits_i\eta^i([Y,fZ])\,\eta^i(X)+g([Y,fZ],\widetilde QX). \end{aligned}$$

We also have

$$\begin{aligned} g(N^{(1)}(Y,Z),fX) &= g(f^2[Y,Z]+[fY,fZ]-f[fY,Z]-f[Y,fZ],fX) \\ &= -g(f[Y,Z],\widetilde QX)+g([fY,fZ]-f[fY,Z]-f[Y,fZ]-[Y,Z],fX). \end{aligned}$$

From this and (25) we get the required result. □

Remark 2.1. For particular values of the tensor $N^{(5)}$ we get

$$\begin{aligned} N^{(5)}(X,\xi_i,Z) &= -N^{(5)}(X,Z,\xi_i)=g(N_i^{(3)}(Z),\,\widetilde QX), \\ N^{(5)}(\xi_i,Y,Z) &= g([\xi_i,fZ],\widetilde QY)-g([\xi_i,fY],\widetilde QZ), \\ N^{(5)}(\xi_i,Y,\xi_j) &= N^{(5)}(\xi_i,\xi_j,Y)=0. \end{aligned} \tag{32}$$

The following corollary of Proposition 2.2 and Theorem 2.2 generalizes well-known results with $Q=\mathrm{id}_{TM}$.

Corollary 2.1. *For a weak almost para-$\mathcal S$-structure we get*

$$\begin{aligned} 2\,g((\nabla_X f)Y,Z) &= -g(N^{(1)}(Y,Z),fX)+2\,g(fX,fY)\,\bar\eta(Z) \\ &\quad -2\,g(fX,fZ)\,\bar\eta(Y)+N^{(5)}(X,Y,Z), \end{aligned} \tag{33}$$

where $\bar\eta=\sum_i\eta^i$*. In particular, taking* $x=\xi_i$ *and then* $Y=\xi_j$ *in* (33)*, we get*

$$2\,g((\nabla_{\xi_i}f)Y,Z)=N^{(5)}(\xi_i,Y,Z),\quad 1\le i\le p, \tag{34}$$

and (26)*; thus, the characteristic distribution is tangent to a totally geodesic foliation with the sectional curvature* $K(\xi_i,\xi_j)=0$.

Proof. By Theorem 2.2, we have $d\eta^i=\Phi$ and $N_i^{(2)}=N_{ij}^{(4)}=0$. Invoking (9) and using Theorem 2.2 and Proposition 2.2, we get (33). From (34) with $Y=\xi_j$ we get $g(f\nabla_{\xi_i}\xi_j,Z)=0$, thus $\nabla_{\xi_i}\xi_j\in\ker f$. Also, $\eta^k([\xi_i,\xi_j])=-2\,d\eta^k(\xi_i,\xi_j)=-2\,g(\xi_i,f\xi_j)=0$; hence, $[\xi_i,\xi_j]=0$, i.e., $\nabla_{\xi_i}\xi_j=\nabla_{\xi_j}\xi_i$. Finally, from $g(\xi_j,\xi_k)=\delta_{jk}$, using the ξ_i-derivation and the above equality, we get $\nabla_{\xi_i}\xi_j\in f(TM)$. This and $\nabla_{\xi_i}\xi_j\in\ker f$ prove (26). □

3. **The tensor field h**

Here, we apply for a weak almost para-$\mathcal{S}$-manifold the tensor field $h=(h_1,\ldots,h_p)$, where $h_i=\frac12N_i^{(3)}=\frac12\pounds_{\xi_i}f$. By Theorem 2.2, $h_i=0$ if and only if ξ_i is a Killing field. First, we calculate

$$(\pounds_{\xi_i}f)X\overset{(10)}{=}\nabla_{\xi_i}(fX)-\nabla_{fX}\xi_i-f(\nabla_{\xi_i}X-\nabla_X\xi_i)=(\nabla_{\xi_i}f)X-\nabla_{fX}\xi_i+f\nabla_X\xi_i. \tag{35}$$

For $X=\xi_i$ in (35), using $g((\nabla_{\xi_i}f)\,\xi_j,Z)=\frac12N^{(5)}(\xi_i,\xi_j,Z)=0$, see (34) and (26), we get $h_i\,\xi_j=0$. The following result generalizes the fact that for an almost para-$\mathcal{S}$-structure, each tensor h_i is self-adjoint and commutes with f.

Proposition 3.1. *For a weak almost para-*$\mathcal{S}$*-structure, the tensor* h_i *and its conjugate* h_i^* *satisfy*

$$g((h_i-h_i^*)X,Y)=(1/2)\,N^{(5)}(\xi_i,X,Y), \tag{36}$$

$$\nabla\xi_i=Q^{-1}f\,h_i^*-f, \tag{37}$$

$$h_if+f\,h_i=-(1/2)\,\pounds_{\xi_i}\widetilde{Q}. \tag{38}$$

Proof. (i) The scalar product of (35) with Y, using (34), gives

$$g((\pounds_{\xi_i}f)X,Y)=N^{(5)}(\xi_i,X,Y)+g(f\nabla_X\xi_i-\nabla_{fX}\xi_i,\ Y). \tag{39}$$

Similarly,

$$g((\pounds_{\xi_i}f)Y,X)=N^{(5)}(\xi_i,Y,X)+g(f\nabla_Y\xi_i-\nabla_{fY}\xi_i,\ X). \tag{40}$$

Using (13) and $(fX)(\eta^i(Y))-(fY)(\eta^i(X))\equiv0$ (this vanishes if X or Y equals ξ_j and also for X and Y in $f(TM)$), we get $N_i^{(2)}(X,Y)=\eta^i([fY,X]-[fX,Y])$. Thus, the difference of (39) and (40) gives $2\,g((h_i-h_i^*)X,Y)=N^{(5)}(\xi_i,X,Y)-N_i^{(2)}(X,Y)$. From this and $N_i^{(2)}=0$ (see Theorem 2.2) we get (36).

(ii) From Corollary 2.1 with $Y=\xi_i$, we find

$$g((\nabla_Xf)\xi_i,Z)=-\,(1/2)g(N^{(1)}(\xi_i,Z),fX)-g(fX,fZ)+(1/2)\,N^{(5)}(X,\xi_i,Z). \tag{41}$$

Note that $\frac12N^{(5)}(X,\xi_i,Z)=g(h_iZ,\widetilde{Q}X)$, see (32). By (4) with $Y=\xi_i$, we get

$$[f,f](X,\xi_i)=f^2[X,\xi_i]-f[fX,\xi_i]=fN_i^{(3)}(X). \tag{42}$$

Using (6), (10) and (42), we calculate

$$\begin{aligned}g([f,f](\xi_i,Z),fX)&=g(f^2\,[\xi_i,Z]-f[\xi_i,fZ],fX)=-g(f(\pounds_{\xi_i}f)Z,fX)\\&=g((\pounds_{\xi_i}f)Z,QX)-\sum\nolimits_j\eta^j(X)\,\eta^j((\pounds_{\xi_i}f)Z).\end{aligned} \tag{43}$$

From (9) we have $g([X,\xi_i],\xi_k) = 2\,d\eta^k(\xi_i,X) = 2\,\Phi(\xi_i,X) = 0$. By (26), we get $g(\nabla_X\,\xi_i,\xi_k) = g(\nabla_{\xi_i}X,\xi_k) = -g(\nabla_{\xi_i}\xi_k,X) = 0$ for $X\in f(TM)$, thus

$$g(\nabla_X\,\xi_i,\ \xi_k) = 0,\quad X\in TM,\ 1\le i,k\le p. \tag{44}$$

Using (35), we get

$$2\,g((\nabla_{\xi_i}f)Y,\xi_j) \overset{(34)}{=} N^{(5)}(\xi_i,Y,\xi_j) \overset{(32)}{=} 0. \tag{45}$$

From (35), (44) and (45) we obtain

$$g((\pounds_{\xi_i}f)X,\xi_j) = -g(\nabla_{fX}\,\xi_i,\xi_j) = 0. \tag{46}$$

Since $f\,\xi_i = 0$, we find

$$(\nabla_X f)\,\xi_i = -f\,\nabla_X\,\xi_i. \tag{47}$$

Thus, combining (41), (43) and (46), gives

$$\begin{aligned}-g(f\,\nabla_X\xi_i,Z) &= g(X,QZ) - g(h_iZ,QX) - \textstyle\sum_j\eta^j(X)\eta^j(Z) + g(h_iZ,\widetilde{Q}X)\\ &= g(h_iZ,X) + g(X,QZ) - \textstyle\sum_j\eta^j(X)\,\eta^j(Z) + g(h_iZ,\widetilde{Q}X).\end{aligned} \tag{48}$$

Replacing Z by fZ in (48) and using (2), (44) and $f\,\xi_i=0$, we achieve (37): $g(Q\,\nabla_X\,\xi_i,Z) = g((fQ-h_if)Z,X) = g(f(h_i^*-Q)X,Z)$.

(iii) Using (2), we get $f\nabla_{\xi_i}f + (\nabla_{\xi_i}f)f = \nabla_{\xi_i}(f^2) = \nabla_{\xi_i}\widetilde{Q} - \nabla_{\xi_i}(\sum_j\eta^j\otimes\xi_j)$, where $\nabla_{\xi_i}(\sum_j\eta^j\otimes\xi_j) = 0$ by (26). From the above and (35), we get (38):

$$\begin{aligned}2(h_if + fh_i)X &= f(\pounds_{\xi_i}f)X + (\pounds_{\xi_i}f)fX = f(\nabla_{\xi_i}f)X + (\nabla_{\xi_i}f)fX\\ &+ f^2\nabla_X\,\xi_i - \nabla_{f^2X}\,\xi_i = [\widetilde{Q}X,\xi_i] - \widetilde{Q}\,[X,\xi_i] = -(\pounds_{\xi_i}\widetilde{Q})X.\end{aligned}$$

We used (26) and (44) to show $\sum_j\big(g(\nabla_X\,\xi_i,\xi_j)\,\xi_j - g(X,\xi_j)\nabla_{\xi_j}\,\xi_i\big) = 0$. □

Remark 3.1. For a weak almost para-$\mathcal{S}$-structure, we get $2\,g(h_iX,\xi_j) = -g(\nabla_{fX}\,\xi_i,\xi_j) = 0$ by (45); thus, $f(TM)$ is h_i-invariant; moreover, $h_i^*\,\xi_j = 0$.

The next statement follows from Propositions 2.1 and 2.2.

Corollary 3.1. *For a weak para-$\mathcal{K}$-structure, we have*

$$\begin{aligned}2\,g((\nabla_X f)Y,Z) &= \textstyle\sum_i\big(2\,d\eta^i(fY,X)\,\eta^i(Z) - 2\,d\eta^i(fZ,X)\,\eta^i(Y)\\ &+ \eta^i([\widetilde{Q}Y,\ fZ])\,\eta^i(X)\big) + N^{(5)}(X,Y,Z).\end{aligned}$$

In particular, using (36), *gives* $2\,g((\nabla_{\xi_i}f)Y,Z) = \eta^i([\widetilde{Q}Y,\ fZ])$ *for* $1\le i\le p$.

4. The rigidity of a para-$\mathcal{S}$-structure

Here, we prove the rigidity theorem for para-$\mathcal{S}$-manifolds.

Proposition 4.1. *For a weak para-$\mathcal{S}$-structure we get*

$$\begin{aligned}g((\nabla_X f)Y,Z) &= g(QX,Z)\,\bar\eta(Y) - g(QX,Y)\,\bar\eta(Z) + \tfrac12 N^{(5)}(X,Y,Z)\\ &- \textstyle\sum_j\eta^j(X)\big(\bar\eta(Y)\,\eta^j(Z) - \eta^j(Y)\,\bar\eta(Z)\big).\end{aligned} \tag{49}$$

Proof. Since $N^{(1)} = 0$, by Corollary 2.1, we get (49). □

Remark 4.1. Using $Y=\xi_i$ in (49), we get $f\nabla_X\xi_i = -f^2X - \frac12(N^{(5)}(X,\xi_i,\cdot))^\flat$, which generalizes the equality $\nabla_X\,\xi_i = -fX$ for a para-$\mathcal{S}$-structure, e.g., [6].

It was shown in [9] that a weak almost para-$\mathcal{S}$-structure with positive partial Ricci curvature can be deformed to an almost para-$\mathcal{S}$-structure.

Theorem 4.1. *A metric weak para-f-structure is a weak para-$\mathcal{S}$-structure if and only if it is a para-$\mathcal{S}$-structure.*

Proof. Let (f, Q, ξ_i, η^i, g) be a weak para-$\mathcal{S}$-structure. Since $N^{(1)} = 0$, by Proposition 2.1, we get $N^{(3)}_i = 0$. By (32), we then obtain $N^{(5)}(\cdot\,, \xi_i, \,\cdot\,) = 0$. Recall that $\tilde{Q}X = QX - X$ and $\eta^j(\widetilde{Q}X) = 0$. Using the above and $Y = \xi_i$ in (49), we get

$$\begin{aligned} g((\nabla_X f)\,\xi_i, Z) &= g(QX, Z) - \eta^i(QX)\,\bar\eta(Z) + \sum\nolimits_j \eta^j(X)\big(\eta^j(Z) - \delta^j_i\,\bar\eta(Z)\big) \\ &= g(QX^\top, Z) + \sum\nolimits_j \eta^j(Z)\big(\eta^j(QX) - \eta^i(QX)\big) - \sum\nolimits_j \eta^j(Z)\big(\eta^j(X) - \eta^i(X)\big) \\ &= g(QX^\top, Z) + \sum\nolimits_j \eta^j(Z)\big(\eta^j(\widetilde{Q}X) - \eta^i(\widetilde{Q}X)\big) = g(QX^\top, Z). \end{aligned} \tag{50}$$

Using (47), we rewrite (50) as $g(\nabla_X\, \xi_i, fZ) = g(QX^\top, Z)$. By the above and (2), we find for all i,

$$g(\nabla_X\, \xi_i + fX,\, f\,Z) = 0. \tag{51}$$

Since f is skew-symmetric, applying (49) with $Z = \xi_i$ in (7), we obtain

$$\begin{aligned} g([f, f](X, Y), \xi_i) &= g([fX, fY], \xi_i) = g((\nabla_{fX} f)Y, \xi_i) - g((\nabla_{fY} f)X, \xi_i) \\ &= g(Q\,fY, X) - g(Q\,fY, \xi_i)\,\bar\eta(X) - g(Q\,fX, Y) + g(Q\,fX, \xi_i)\,\bar\eta(Y). \end{aligned} \tag{52}$$

Recall that $[Q,\, f] = 0$ and $f\,\xi_i = 0$. Thus, (52) yields $g([f, f](X, Y), \xi_i) = 2\,g(QX, fY)$. From this, using the definition of $N^{(1)}$, we get

$$g(N^{(1)}(X, Y), \xi_i) = 2\,g(\widetilde{Q}X, fY). \tag{53}$$

From $N^{(1)} = 0$ and (53) we get $g(\widetilde{Q}X, fY) = 0$ $(X, Y \in \mathfrak{X}_M)$; thus, $\widetilde{Q} = 0$. □

For a weak almost para-$\mathcal{S}$-structure all ξ_i are Killing if and only if $h = 0$, see Theorem 2.2. The equality $h = 0$ holds for a weak para-$\mathcal{S}$-structure since it is true for a para-$\mathcal{S}$-structure, see Theorem 4.1.

Corollary 4.1. *For a weak para-$\mathcal{S}$-structure, $\xi_1, \ldots, \xi_p$ are Killing vector fields; moreover,* $\ker f$ *defines a Riemannian totally geodesic foliation.*

5. The characteristic of a weak para-$\mathcal{C}$-structure

Here, we show that a weak para-f-structure with parallel tensor f reduces to a weak para-$\mathcal{C}$-structure, Recall that $\nabla_X\, \xi_i = 0$ holds on para-$\mathcal{C}$-manifolds.

Proposition 5.1. *Let (f, Q, ξ_i, η^i, g) be a weak para-$\mathcal{C}$-structure. Then*

$$2\,g((\nabla_X f)Y, Z) = N^{(5)}(X, Y, Z). \tag{54}$$

Using (54) with $Y = \xi_i$ and (2), we get $g(\nabla_X\, \xi_i,\, QZ) = -\frac{1}{2}N^{(5)}(X, \xi_i, fZ)$.

Proof. Using Theorem 2.3, from Proposition 2.2 we get

$$2\,g((\nabla_X f)Y, Z) = -g([f, f](Y, Z), fX) + N^{(5)}(X, Y, Z). \tag{55}$$

From (55), using condition $[f, f] = 0$ we get (54). □

Theorem 5.1. *A metric weak para-f-structure with $\nabla f = 0$ and condition $[\xi_i, \xi_j]^\perp = 0$ is a weak para-$\mathcal{C}$-structure with $N^{(5)} = 0$.*

Proof. Using condition $\nabla f = 0$, from (7) we obtain $[f, f] = 0$. Hence, from (3) we get $N^{(1)}(X,Y) = -2\sum_i d\eta^i(X,Y)\,\xi_i$, and from (7) with $Y = \xi_i$ we obtain

$$\nabla_{fX}\,\xi_i - f\,\nabla_X\,\xi_i = 0, \quad X \in \mathfrak{X}_M. \tag{56}$$

By (25), $3\,d\Phi(X,Y,Z) = g((\nabla_X f)Z, Y) + g((\nabla_Y f)X, Z) + g((\nabla_Z f)Y, X)$; hence, using condition $\nabla f = 0$ again, we get $d\Phi = 0$. Next,

$$N_i^{(2)}(Y,\xi_j) = -\eta^i([fY,\xi_j]) = g(\xi_j, f\nabla_{\xi_i} Y) = 0.$$

Setting $Z = \xi_j$ in Proposition 2.2, and using $\nabla f = 0$ and the properties

$$d\Phi = 0, \quad N_i^{(2)}(Y,\xi_j) = 0, \quad N^{(1)}(X,Y) = -2\sum\nolimits_i d\eta^i(X,Y)\,\xi_i,$$

we find $0 = 2\,d\eta^j(fY,X) - N^{(5)}(X,\xi_j,Y)$. By (32) and (56),

$$N^{(5)}(X,\xi_j,Y) = g([\xi_j, fY] - f[\xi_j,Y],\, \widetilde{Q}X) = g(\nabla_{fY}\,\xi_j - f\,\nabla_Y\,\xi_j,\, \widetilde{Q}X) = 0;$$

hence, $d\eta^j(fY,X) = 0$. From this and $g([\xi_i,\xi_j],\xi_k) = 2\,d\eta^k(\xi_j,\xi_i) = 0$ we get $d\eta^j = 0$. By the above, $N^{(1)} = 0$. Thus, (f,Q,ξ_i,η^i,g) is a weak para-$\mathcal{C}$-structure. Finally, from (54) and condition $\nabla f = 0$ we get $N^{(5)} = 0$. □

Corollary 5.1. *A normal metric weak para-f-structure with $\nabla f = 0$ is a weak para-$\mathcal{C}$-structure with $N^{(5)} = 0$.*

Proof. By $N^{(1)}{=}0$, we get $d\eta^i{=}0$, $\forall i$. As in Theorem 5.1, we get $d\Phi = 0$. □

Example 5.1. Let M be a $2n$-dimensional smooth manifold and $\tilde f : TM \to TM$ an endomorphism of rank $2n$ such that $\nabla\tilde f = 0$. To construct a weak para-$\mathcal{C}$-structure on $M\times\mathbb{R}^p$, take any point $(x, t_1,\ldots,t_p)$ and set $\xi_i = (0, d/dt_i)$, $\eta^i = (0, dt_i)$, $f(X,Y) = (\tilde fX,\, 0)$ and $Q(X,Y) = (\tilde f^{\,2}X,\, Y)$, where $X \in T_xM$ and $Y = \sum_i Y^i\xi_i \in \mathbb{R}^p_t$. Then (2) holds and Theorem 5.1 can be used.

6. Conclusions

It was shown that the weak para-f-structure is a useful tool for studying totally geodesic foliations and Killing vector fields. We proved that a weak para-$\mathcal{S}$-structure is a para-$\mathcal{S}$-structure (the rigidity theorem) and that a weak para-f-structure with parallel tensor f reduces to a weak para-$\mathcal{C}$-structure.

REFERENCES

[1] *D. Alekseevsky and P. Michor*, Differential geometry of $\mathfrak{g}$-manifolds, Differential Geom. Appl. **5** (1995), 371-403.

[2] *D. E. Blair and G. D. Ludden*, Hypersurfaces in almost contact manifolds. Tohoku Math. J. **21** (1969), 354-362.

[3] *A. Bucki and A. Miernowski*, Almost r-paracontact structures. Ann. Univ. Mariae Curie-Sklodowska **39** (1985), No. 2, 13-26.

[4] *V. Cruceanu, P. Fortuny and P. M. Gadea*, A survey on paracontact geometry. Rocky Mt. J. Math. **26** (1996), No. 1, 83-115.

[5] *M. Falcitelli, S. Ianus and A. M. Pastore*, Riemannian Submersions and Related Topics, World Scientific, 2004.

[6] *L. M. Fernández and A. Prieto-Martín*, On η-Einstein para-S-manifolds. Bull. Malays. Math. Sci. Soc. **40** (2017), 1623-1637.

[7] *K. Matsumoto*, On a structure defined by a tensor field f of type (1,1) satisfying $f^3 - f = 0$, Bull. Yamagata Univ. **1** (1976), 33-47.

[8] *V. Rovenski*, On the geometry of a weakened f-structure, arXiv:2205.02158, 2022, 14 pp. Accepted in Journal of Geometry (JGEO).

[9] *V. Rovenski and R. Wolak*, New metric structures on $\mathfrak{g}$-foliations, Indagationes Mathematicae, **33** (2022), 518-532.