OPERATOR - VALUED MAZUR – ORLICZ AND MOMENT PROBLEMS IN SPACES OF ANALYTIC FUNCTIONS

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The aim of the present work is to prove new applications of some earlier general abstract results on the subject to spaces of analytic functions. The Cauchy inequalities are used systematically, as well as constrained extension of linear operators. One gives necessary and sufficient conditions and only sufficient conditions for the existence of solutions of an operator valued moment problem and of Mazur-Orlicz problems. The upper constraint appears naturally from the corresponding computations, while the lower constraint is sometimes the positivity of the solution. Considering Markov moment problem, one solves a concrete interpolation problem with two constraints. In the case of Mazur – Orlicz problems, the interpolation conditions are replaced by the corresponding inequalities mentioned in section 2. Operator valued solutions are obtained.

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Keywords: constrained extension of linear operators, Markov moment problem, Mazur – Orlicz theorem, spaces of analytic functions

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1. Introduction

In solving moment problems, three aspects are usually studied: the existence of the solution, the uniqueness and the construction of the solution. Concerning the first of these problems, Hahn – Banach type results and order structures have been used. In solving classical moment problems, extension results are strongly related to the explicit form of polynomials on closed subsets of $\mathbb{R}$ and $\mathbb{R}^n$. The reason is that the first idea is to extend a positive linear form defined on polynomials and satisfying the interpolation (moment) conditions to a larger space. This space should contain both the polynomials and the compactly supported functions. Thus one obtains a representing suitable positive measure. Conversely, the form of strictly positive polynomials on a compact subset $K \subset \mathbb{R}^n$ has been deduced by solving firstly the moment problem on that compact. We recall the statement of the classical real moment problem. Let $\left(y_j\right)_{j\in\mathbb{N}^n}$ be a

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(eventually multi - indexed) sequence of real numbers, and $\varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n}$, where $t = (t_1, \dots, t_n) \in A$, and A is a closed subset in $\mathbb{R}^n, j = (j_1, \dots, j_n) \in \mathbb{N}^n$. Find necessary and sufficient conditions on the sequence $(y_j)_{j \in \mathbb{N}^n}$ for the existence of a positive Borel measure μ on A such that

$$\int_A \varphi_j d\mu = y_j, \forall j \in \mathbb{N}^n.$$

If such a measure exists, the sequence $(y_j)_{j \in \mathbb{N}^n}$ is called a moment sequence (with respect to the representing measure μ). So, the lower constraint on the measure μ is its positivity. The Markov moment problem requires an additional upper constraint. In the multidimensional case, the $K-$ moment problem (when the subset $A = K$ is a compact with nonempty interior) has been intensively studied. An important particular case is that of semi algebraic compact subsets. Some generalizations of Hahn – Banach theorem have been applied in solving Markov moment problems. Here the polynomial functions $(\varphi_j)_{j \in \mathbb{N}^n}$ defined above are replaced by arbitrary elements of an order vector space, and the index set is an arbitrary set J, which might be uncountable. Exact statements are recalled in Section 2 from below. Several results on these subjects can be found in [1] – [19]. The uniqueness of the solution has been also intensively studied (see [5], [7] – [9]). Constructions of the solutions are mentioned in [13], [17]. Connections to other fields, including fixed point theory (see [6]) appear in [1] – [18]. The first aim of the present work is to find sufficient conditions for the existence of a solution of a Markov moment problem, involving a subspace which is distanced with respect to a convex set. Secondly, we solve Mazur – Orlicz problems in concrete spaces of functions. To this aim, we apply the general abstract results from [14] - [16] to concrete spaces. In some statements, the target space Y is an order complete vector lattice of self - adjoint operators, which is also a commutative algebra. Hence one solves vector valued and operator valued moment and Mazur – Orlicz problems (sections 3 and 4). The solutions will be operators satisfying interpolation conditions (moment conditions), with two constraints. One continues partially the study from [19]. The background of this paper is contained in some chapters of [20] – [23], as well as in the papers [14] – [16]. The rest of this work is organized as follows. In Section 2 we recall the general abstract results applied in the sequel. Section 3 is devoted to a concrete moment problem. In Section 4, Mazur – Orlicz problems are discussed. The Conclusions are mentioned in the end, in Section 5.

2. General results

We start be recalling some general type results on the abstract moment problem and Mazur – Orlicz theorem. If X is an order vector space, one denotes by X_+ its positive cone. If X, Y are real vector spaces, one denotes by $L(X, Y)$ the real vector space of all linear operators (with respect to the real field) from X into Y.

Theorem 2.1. (Theorem II. 5 [15]). *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and let S be a vector subspace of X. Assume that $A \subset X$ is a convex subset, such that the following two conditions are fulfilled:*

(a) *there exists a neighborhood V of the origin such that $(S + V) \cap A = \emptyset$ (A and S are distanced);*

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_i\}_{i \in I} \subset L(S, Y)$ and any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_i\}_{i \in I} \subset L(X, Y)$ such that $F_i(s) = f_i(s), \forall s \in S, \forall i \in I$ and $F_i(a) \geq \tilde{y}, \forall a \in A$. Moreover, if V is a convex, circled neighborhood of the origin with the properties $f_i(V \cap S) \subset [-u_0, u_0], \forall i \in I, (S + V) \cap A = \emptyset$, let p_V be the gauge (Minkowski functional) attached to V, $\alpha > 0$ such that $p_V(a) \leq \alpha, \forall a \in A$, and $\alpha_1 > 0$ such that $\tilde{y} \leq \alpha_1 u_0$. Then the following relations hold

$$F_i(x) \leq (1 + \alpha + \alpha_1) p_V(x) u_0, \forall x \in X, \forall i \in I.$$

Recall that the gauge p_V attached to V is defined by

$$p_V(x) = inf\{\lambda > 0, x \in \lambda V\}, \forall x \in X.$$

Finally, we recall the following variant of Mazur – Orlicz theorem.

Theorem 2.2. (Theorem 5 [16]). *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ arbitrary families in X, respectively in Y and $P: X \to Y$ a sublinear operator. The following statements are equivalent*

(a) $\exists F \in L(X, Y)$ *such that* $F(x_j) \geq y_j, \forall j \in J, F(x) \geq 0 \; \forall x \in X_+,$

$$F(x) \leq P(x), \forall x \in X;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}, \lambda_j \geq 0 \; \forall j \in J_0$, we have*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

3. Solving a Markov moment problem

Let $n \neq 0$ be a natural number and X be the space of absolutely convergent power series in the unit closed poly - disc $\overline{D}_1 = \{z = (z_1, \dots, z_n): |z_p| \leq 1, p \in \langle 1, \dots, n \rangle\}$, with real coefficients. The norm on X is defined by

$$\|\varphi\|_\infty = sup\{|\varphi(z)|: z \in \overline{D}_1\}.$$

Denote

$$h_k(z) = z_1^{k_1} \cdots z_n^{k_n}, k = (k_1, \dots, k_n) \in \mathbb{N}^n, z \in \overline{D}_1.$$

One denotes $|k| := k_1 + \cdots + k_n$. On the other hand, let H be a complex Hilbert space, $\mathcal{A}$ the real vector space of all self adjoint operators acting on $H, U_0 \in \mathcal{A}$. Define

$$Y_1 := \{A \in \mathcal{A}: AU_0 = U_0 A\}, Y := \{U \in Y_1: UV = VU, \forall\, V \in Y_1\},$$

$$Y_+ := \{U \in Y: < Uh, h > \geq 0, \forall h \in H\}.$$

Endowed with the order relation defined by this cone, and with the operatorial norm, Y is an order complete Banach lattice [20], [11] (which is also a commutative algebra of self - adjoint operators). Let $(B_k)_{k \in \mathbb{N}^n}$ be a multi - indexed sequence of operators in Y. Let $\tilde{B} \in Y_+ \backslash \{0\}$.

Theorem 3.1. *Assume that $A_1, \dots, A_n$ are elements of Y such that there exists a real number $M > 0$, so that*

$$|B_k| \leq M \frac{A_1^{2k_1}}{k_1!} \cdots \frac{A_n^{2k_n}}{k_n!}, \forall k \in \mathbb{N}^n, \sum_{p=1}^{n} A_p^2 \leq I,$$

where I is the identity operator. Let $\{\varphi_k\}_{k \in \mathbb{N}^n} \subset X$ be such that $1 = \|\varphi_k\| = \varphi_k(0), \forall k \in \mathbb{N}^n$. Then there exists a linear bounded operator $F \in B(X, Y)$ such that

$$F(h_k) = B_k, |k| \geq 1, F(\varphi_k) \geq \widetilde{B}, \forall k \in \mathbb{N}^n,$$

$$F(h) \leq \left(2 + \|\tilde{B}\| M^{-1} e^{-1}\right) \|h\|_\infty u_0, \forall h \in X, u_0 := MeI.$$

In particular, the following evaluation holds: $\|F\| \leq 2Me + \|\tilde{B}\|$.

Proof. One applies Theorem 2.1. The subspace generated by $\{h_k: |k| \geq 1\}$ stands for S of Theorem 2.1, and the convex hull of the set of the functions $\varphi_k, k \in \mathbb{N}^n$, stands for the set A. The following remark is essential:

$$\|s - \varphi\|_\infty \geq |s(0) - \varphi(0)| = |0 - 1| = 1, \forall s \in S, \forall \varphi \in A.$$

This proves that $(S + B(0,1)) \cap A = \emptyset$, so that $B(0,1)$ stands for V and $\|\cdot\|_\infty$ stands for p_V from Theorem 2.1. The operator $\tilde{B}$ will stand for $\tilde{y}$. Now let

$$\varphi = \sum_{j \in J_0} \beta_j\, h_j \in S \cap B(0,1),$$

where J_0 is a finite subset of $\mathbb{N}^n$. The following relations hold

$$\left|\sum_{j\in J_0}\beta_j B_j\right| \le \sum_{j\in J_0}|\beta_j||B_j| \le \|\varphi\|_\infty \sum_{j\in J_0}\frac{1}{r_1^{j_1}\cdots r_n^{j_n}}|B_j|,$$

for any $0 < r_p < 1, p \in \{1,\dots,n\}$, thanks to Cauchy inequalities. Passing to the limit with $r_p \uparrow 1, p \in \{1,\dots,n\}$ and using the fact that $\varphi \in B(0,1)$, as well as the hypothesis in the statement, the preceding relation further yields

$$\left|\sum_{j\in J_0}\beta_j B_j\right| \le \sum_{j\in J_0}|B_j| \le M\sum_{j\in J_0}\frac{A_1^{2j_1}}{j_1!}\cdots\frac{A_n^{2j_n}}{j_n!} \le M\left(\sum_{k_1\in\mathbb{N}}\frac{A_1^{2k_1}}{k_1!}\right)\cdots\left(\sum_{k_n\in\mathbb{N}}\frac{A_n^{2k_n}}{k_n!}\right) =$$
$$= Mexp\left(\sum_{p=1}^{n}A_p^2\right) \le Mexp(I) = MeI = u_0.$$

The conclusion is that denoting by $f: S \to Y$ the linear operator which satisfies the moment conditions $f(h_k) = B_k, k \in \mathbb{N}^n, |k| > 1$, we have

$$-MeI \le f(s) \le MeI = u_0, \forall s \in S \cap B(0,1).$$

On the other hand, the following relations hold

$$\tilde{B} \le \|\tilde{B}\| I = \|\tilde{B}\| M^{-1}e^{-1}u_0 = \alpha_1 u_0,$$

where $\alpha_1 := \|\tilde{B}\| M^{-1}e^{-1}$. The conditions on the norms of the functions $\varphi_k, k \in \mathbb{N}^n$ lead to

$$\|\varphi\| \le 1, \forall \varphi \in A.$$

So, the constant 1 stands for α from Theorem 2.1. Now all the conditions from the statement of theorem 2.1 are accomplished. Application of the latter theorem, leads to the existence of a linear mapping $F: X \to Y$, such that

$$F(h_k) = f(h_k) = B_k, k \in \mathbb{N}^n, |k| > 1, F(\varphi_k) \ge \tilde{B}, \forall k \in \mathbb{N}^n,$$
$$F(h) \le \left(2 + \|\tilde{B}\| M^{-1}e^{-1}\right)\|h\|_\infty MeI, \forall h \in X.$$

From the last inequality, we derive

$$|F(h)| \le \left(2Me + \|\tilde{B}\|\right)\|h\|_\infty I, \qquad \forall h \in X.$$

Since the norm on Y is solid, we infer that

$$\|F(h)\| \le \left(2Me + \|\tilde{B}\|\right)\|h\|_\infty, \forall h \in X \Rightarrow \|F\| \le 2Me + \|\tilde{B}\|.$$

This concludes the proof. □

For some other applications of Theorem 2.1 see [10].

4. Mazur – Orlicz problems

In the next result, the space $\boldsymbol{X}$ is formed by all absolutely convergent power series in the closed polydisc

$$\overline{\boldsymbol{D}}_{\boldsymbol{r}} = \left\{\boldsymbol{z} = (\boldsymbol{z}_{\boldsymbol{1}}, \dots, \boldsymbol{z}_{\boldsymbol{n}}) : |\boldsymbol{z}_{\boldsymbol{p}}| \leq \boldsymbol{r}_{\boldsymbol{p}}, \boldsymbol{p} \in \{\boldsymbol{1}, \dots, \boldsymbol{n}\}\right\}, \boldsymbol{r} := (\boldsymbol{r}_{\boldsymbol{1}}, \dots, \boldsymbol{r}_{\boldsymbol{n}}),$$

The order relation on the space $\boldsymbol{X}$ is defined by the convex cone of those power series with all their coefficients nonnegative numbers. The space $\boldsymbol{Y}$ is that used in Theorem 3.1. Assume that $\boldsymbol{A}_{\boldsymbol{p}} \in \boldsymbol{Y}, \boldsymbol{0} \leq \boldsymbol{A}_{\boldsymbol{p}} \leq \boldsymbol{r}_{\boldsymbol{p}} \boldsymbol{I}, \boldsymbol{p} \in \{\boldsymbol{1}, \dots, \boldsymbol{n}\}$. Let $(\boldsymbol{B}_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}}$ be a sequence of positive operators in $\boldsymbol{Y}$.

Theorem 4.1. *Let* $\boldsymbol{\alpha} > \boldsymbol{0}$. *The following statements are equivalent*

(a) *there exists a linear positive operator* $\boldsymbol{F}$ *applying* $\boldsymbol{X}$ *into* $\boldsymbol{Y}$, *such that*

$$\boldsymbol{F}(\boldsymbol{h}_{\boldsymbol{k}}) \geq \boldsymbol{B}_{\boldsymbol{k}}, \forall \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}, |\boldsymbol{F}(\boldsymbol{\varphi})| \leq \boldsymbol{\alpha} |\boldsymbol{\varphi}|(\boldsymbol{A}_{\boldsymbol{1}}, \dots, \boldsymbol{A}_{\boldsymbol{n}}), \|\boldsymbol{F}(\boldsymbol{\varphi})\| \leq \boldsymbol{\alpha} \||\boldsymbol{\varphi}|\|_{\infty},$$

$$\forall \boldsymbol{\varphi} \in \boldsymbol{X};$$

(b) $\boldsymbol{B}_{\boldsymbol{k}} \leq \boldsymbol{\alpha} \boldsymbol{A}_{\boldsymbol{1}}^{\boldsymbol{k}_1} \cdots \boldsymbol{A}_{\boldsymbol{n}}^{\boldsymbol{k}_n}, \forall \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}$.

Proof. The implication (a)⇒(b) is obvious. Namely, since $\boldsymbol{h}_{\boldsymbol{k}} \in \boldsymbol{X}_+$, we have $|\boldsymbol{h}_{\boldsymbol{k}}| = \boldsymbol{h}_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}$; the positivity of the operators $\boldsymbol{B}_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}$ and the conditions of (a) yield

$$\boldsymbol{0} \leq \boldsymbol{B}_{\boldsymbol{k}} \leq \boldsymbol{F}(\boldsymbol{h}_{\boldsymbol{k}}) = |\boldsymbol{F}(\boldsymbol{h}_{\boldsymbol{k}})| \leq \boldsymbol{\alpha} \boldsymbol{h}_{\boldsymbol{k}}(\boldsymbol{A}_{\boldsymbol{1}}, \dots, \boldsymbol{A}_{\boldsymbol{n}}) = \boldsymbol{\alpha} \boldsymbol{A}_{\boldsymbol{1}}^{\boldsymbol{k}_1} \cdots \boldsymbol{A}_{\boldsymbol{n}}^{\boldsymbol{k}_n}, \forall \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}.$$

For the converse, we apply Theorem 2.2, where $\mathbb{N}^{\boldsymbol{n}}$ stands for $\boldsymbol{J}$, $\boldsymbol{h}_{\boldsymbol{k}}$ stands for $\boldsymbol{x}_{\boldsymbol{j}}$. The following implication holds for a finite subset $\boldsymbol{J}_{\boldsymbol{0}} \subset \mathbb{N}^{\boldsymbol{n}}$ and a corresponding finite set $(\boldsymbol{\lambda}_{\boldsymbol{j}})_{\boldsymbol{j} \in \boldsymbol{J}_{\boldsymbol{0}}}$ of nonnegative real scalars:

$$\sum_{\boldsymbol{j} \in \boldsymbol{J}_{\boldsymbol{0}}} \boldsymbol{\lambda}_{\boldsymbol{j}} \boldsymbol{h}_{\boldsymbol{j}} \leq \boldsymbol{\varphi} = \sum_{\boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}} \boldsymbol{\gamma}_{\boldsymbol{k}} \boldsymbol{h}_{\boldsymbol{k}} \Rightarrow \boldsymbol{\lambda}_{\boldsymbol{j}} \leq \boldsymbol{\gamma}_{\boldsymbol{j}}, \boldsymbol{j} \in \boldsymbol{J}_{\boldsymbol{0}}, \boldsymbol{\gamma}_{\boldsymbol{k}} \geq \boldsymbol{0}, \forall \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}} \Rightarrow$$

$$\sum_{\boldsymbol{j} \in \boldsymbol{J}_{\boldsymbol{0}}} \boldsymbol{\lambda}_{\boldsymbol{j}} \boldsymbol{B}_{\boldsymbol{j}} \leq \sum_{\boldsymbol{j} \in \boldsymbol{J}_{\boldsymbol{0}}} \boldsymbol{\gamma}_{\boldsymbol{j}} \boldsymbol{B}_{\boldsymbol{j}} \leq \boldsymbol{\alpha} \sum_{\boldsymbol{j} \in \boldsymbol{J}_{\boldsymbol{0}}} \boldsymbol{\gamma}_{\boldsymbol{j}} \boldsymbol{A}_{\boldsymbol{1}}^{\boldsymbol{j}_1} \cdots \boldsymbol{A}_{\boldsymbol{n}}^{\boldsymbol{j}_n} \leq \boldsymbol{\alpha} \sum_{\boldsymbol{k} \in \mathbb{N}^{\boldsymbol{n}}} \boldsymbol{\gamma}_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{1}}^{\boldsymbol{k}_1} \cdots \boldsymbol{A}_{\boldsymbol{n}}^{\boldsymbol{k}_n}$$

$$= \boldsymbol{\alpha} \boldsymbol{\varphi}(\boldsymbol{A}_{\boldsymbol{1}}, \dots, \boldsymbol{A}_{\boldsymbol{n}}) = \boldsymbol{\alpha} |\boldsymbol{\varphi}|(\boldsymbol{A}_{\boldsymbol{1}}, \dots, \boldsymbol{A}_{\boldsymbol{n}}) =: \boldsymbol{P}(\boldsymbol{\varphi}) = \boldsymbol{P}(-\boldsymbol{\varphi}).$$

Notice that the series $\sum_{k\in\mathbb{N}^n}\gamma_k A_1^{k_1}\cdots A_n^{k_n}$ is absolutely convergent since

$$\sum_{k\in\mathbb{N}^n}\left\|\gamma_k A_1^{k_1}\cdots A_n^{k_n}\right\| \le \sum_{k\in\mathbb{N}^n}\gamma_k r_1^{k_1}\cdots r_n^{k_n} = \varphi(r_1,\dots,r_n) < \infty$$

(see the assumptions on $A_p, p\in\{1,\dots,n\}$ specified just before the statement). Application of Theorem 2.2 leads to the existence of a linear positive operator F from X into Y such that

$$F(h_k)\ge B_k, \forall k\in\mathbb{N}^n, |F(\varphi)| \le \alpha|\varphi|(A_1,\dots,A_n), \forall\varphi\in X.$$

Since the norm on Y is solid, we derive

$$\|F(\varphi)\| \le \alpha|\varphi|(r_1,\dots,r_n) = \alpha\||\varphi|\|_\infty.$$

This concludes the proof. □

For the next result, let X be the space of all absolutely convergent power series in the closed polydisc

$$\bar{D}_R = \left\{z=(z_1,\dots,z_n): |z_p|\le R_p, p\in\{1,\dots,n\}\right\}, R:=(R_1,\dots,R_n),$$

with real coefficients. The positive cone of X consists in all power series in X, having all the coefficients nonnegative numbers. The space Y is the same as in Theorems 3.1 and 4.1. Denote

$$\|h\|_\infty := \sup_{z\in\bar{D}_R}|h(z)|, h\in X.$$

Theorem 4.2. *Let* $0<r_p<R_p, p=1,\dots,n, h_k(z)=z_1^{k_1}\cdots z_n^{k_n}, k\in\mathbb{N}^n, z\in\bar{D}_R, \alpha>0$. *Let* $(B_k)_{k\in\mathbb{N}^n}$ *be a multi indexed sequence of positive operators in* Y. *Consider the following statements*

(a) *there exists a linear positive bounded operator* F *applying* X *into* Y *such that*

$$F(h_k)\ge B_k, \forall k\in\mathbb{N}^n, |F(\varphi)| \le \alpha\prod_{p=1}^{n}\frac{R_p}{R_p-r_p}\|\varphi\|_\infty I,$$

$$\|F\| \leq \alpha \prod_{p=1}^{n} \frac{R_p}{R_p - r_p};$$

(b) $B_k \leq \alpha r_1^{k_1} \cdots r_n^{k_n} I, \forall k \in \mathbb{N}^n$, *where* I *is the identity operator.*

Then (b) *implies* (a).

Proof. Let $J_0 \subset \mathbb{N}^n$ be a finite subset and $(\lambda_j)_{j\in J_0}$ be a set of nonnegative real scalars, such that $\sum_{j\in J_0} \lambda_j h_j \leq \varphi = \sum_{k\in\mathbb{N}^n} \gamma_k h_k \Rightarrow \lambda_j \leq \gamma_j, j \in J_0, \gamma_k \geq 0, \forall k \in \mathbb{N}^n$. Let ε be an arbitrary number such that $0 < \varepsilon < \min_{1\leq p\leq n}\{R_p - r_p\}$. The Cauchy's inequalities for the analytic function φ lead to

$$\gamma_k = |\gamma_k| \leq \frac{\|\varphi\|_\infty}{(R_1-\varepsilon)^{k_1}\cdots(R_n-\varepsilon)^{k_n}}, k \in \mathbb{N}^n.$$

Using these relations and the preceding ones, as well as the hypothesis on $B_k, k \in \mathbb{N}^n$, we infer that

$$\sum_{j\in J_0} \lambda_j B_j \leq \sum_{j\in J_0} \gamma_j B_j \leq \alpha\|\varphi\|_\infty \sum_{k\in\mathbb{N}^n} \left(\frac{r_1}{R_1-\varepsilon}\right)^{k_1} \cdots \left(\frac{r_n}{R_n-\varepsilon}\right)^{k_n} I =$$

$$= \alpha\|\varphi\|_\infty \prod_{p=1}^{n}\left(\sum_{k_p\in\mathbb{N}}\left(\frac{r_p}{R_p-\varepsilon}\right)^{k_p}\right) I = \alpha\|\varphi\|_\infty \prod_{p=1}^{n} \frac{R_p-\varepsilon}{R_p-\varepsilon-r_p} I,$$

$$\forall \varepsilon \in (0, R_p - r_p), p = 1, \dots, n.$$

Passing through the limit with $\varepsilon \downarrow 0$, the following basic relation follows

$$\sum_{j\in J_0} \lambda_j B_j \leq \alpha\|\varphi\|_\infty \prod_{p=1}^{n} \frac{R_p}{R_p-r_p} I =: P(\varphi) = P(-\varphi).$$

Application of Theorem 2.2 leads to the existence of a linear positive operator F,

$$F(h_k) \geq B_k, \forall k \in \mathbb{N}^n, |F(\varphi)| \leq \alpha \prod_{p=1}^{n} \frac{R_p}{R_p-r_p} \|\varphi\|_\infty I, \forall \varphi \in X.$$

Since the norm on $\boldsymbol{Y}$ is solid, we derive

$$\|\boldsymbol{F}(\boldsymbol{\varphi})\| \leq \boldsymbol{\alpha} \prod_{p=1}^{n} \frac{\boldsymbol{R_p}}{\boldsymbol{R_p} - \boldsymbol{r_p}} \|\boldsymbol{\varphi}\|_{\infty}, \forall \boldsymbol{\varphi} \in \boldsymbol{X}.$$

This concludes the proof. □

5. Conclusions

The present work focuses on Markov moment and Mazur – Orlicz problems in spaces of analytic functions. Classical operator valued problems are discussed in this framework. The results are proved in some special spaces and are new. Interesting such results can be proved in spaces related to measure theory [19]. Further improvements are very probable. For example, Theorem 4.2 can be applied to analytic functions in the whole space $\mathbb{C}^n$, with real coefficients. Then keeping r_p bounded and making $R_p \to \infty, p = 1, \dots, n$ one obtains $\|F\| \to \alpha$. New results in L^p spaces, $1 \leq p < \infty$, are in course.

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