EXISTENCE RESULTS OF INFINITELY MANY WEAK SOLUTIONS FOR $p(x)$-LAPLACIAN-LIKE OPERATORS

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The paper, in view of the variational approach, studies the existence of infinitely many weak solutions for $p(x)$-Laplacian-like operators, originated from a capillary phenomenon.

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1. Introduction

This paper is devoted to the study of the existence of infinitely many weak solutions for the nonlinear eigenvalue problems for $p(x)$-Laplacian-like operators originated from a capillary phenomenon of the following form:

$$\begin{cases} -\text{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2}\nabla u \right) = \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with boundary of class C^1, λ is a positive parameter, f is a Carathéodory function and $p \in C^0(\overline{\Omega})$ with $N < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < +\infty$.

In recent years, the study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electro-rheological fluids, etc.

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems.

In this paper, motivated by the above facts and the recent paper [14], we establish some new sufficient conditions under which the problem (1) possesses infinitely many weak solutions. To this aim, we require that the primitive F of f satisfies a suitable oscillatory

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behavior at infinity. Our approach is a fully variational method and the main tool is a general critical point theorem (see Lemma 2.1 below) contained in [2]; see also [13]. This lemma and its variations have been used in several works in order to obtain existence results for different kinds of problems (see, for instance, [1, 3, 4, 5, 6, 7, 8, 11] and references therein).

The remainder of the paper is organized as follows. In Section 2, we will recall the definitions and some properties of variable exponent Sobolev spaces. In Section 3 we will state and prove the main results of the paper.

2. Preliminaries

Our analysis is mainly based on a recent critical point theorem of Bonanno and Molica Bisci [2] (see Lemma (2.1) below) which is a more precise version of Ricceri's variational principle [13, Theorem 2.5].

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad and \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda \Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(b$_1$) *I_λ possesses a global minimum, or*

(b$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(c$_1$) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c$_2$) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

For the reader's convenience, we introduce some basic preliminary knowledge of the variable exponent spaces. For more details, we refer the reader to [10, 12, 15, 16].

Set

$$C_+(\Omega) := \left\{ h \in C(\overline{\Omega}) : h(x) > 1, \ \forall x \in \overline{\Omega} \right\}.$$

For $p(\cdot) \in C_+(\Omega)$, define

$$L^{p(\cdot)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\}.$$

We can introduce a norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf \Big\{ \beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)} dx \leq 1 \Big\}.$$

The space $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ is a Banach space called a variable exponent Lebesgue space. The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined as

$$W^{1,p(\cdot)}(\Omega) = \big\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \big\},$$

with the norm

$$\|u\|_{1,p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

and it also becomes a separable, reflexive Banach space (see [9]). We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Now we denote $X := W_0^{1,p(\cdot)}(\Omega)$. For any $u \in X$, define $\|u\| = |\nabla u|_{p(\cdot)}$. It is easy to see that X endowed with the above norm is a separable, reflexive Banach space. We denote by X^* its dual. In $W_0^{1,p(\cdot)}(\Omega)$ the Poincare inequality holds, so $|\nabla u|_{p(\cdot)}$ is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$.

Proposition 2.1 (see [10, 15]). *The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{q(\cdot)}(\Omega)$ where $q(\cdot)$ is the conjugate function of $p(\cdot)$; i.e.,*

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) |u|_{p(\cdot)} |v|_{q(\cdot)} \leq 2|u|_{p(\cdot)} |v|_{q(\cdot)},$$

where $p^- := \inf_{x \in \overline{\Omega}} p(x)$ and $q^- := \inf_{x \in \overline{\Omega}} q(x)$.

Proposition 2.2 (see [10, 15]). *Set $\rho(u) = \int_\Omega |u|^{p(x)} dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < (=; >)1 \Leftrightarrow \rho(u) < (=; >)1$,
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^+}$,
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^-}$,
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.2, for $u \in L^{p(\cdot)}(\Omega)$ the following inequalities hold:

$$\|u\|^{p^-} \leq \int_\Omega |\nabla u(x)|^{p(x)} dx \leq \|u\|^{p^+} \quad \text{if } \|u\| > 1; \tag{2}$$

$$\|u\|^{p^+} \leq \int_\Omega |\nabla u(x)|^{p(x)} dx \leq \|u\|^{p^-} \quad \text{if } \|u\| < 1. \tag{3}$$

Proposition 2.3 (see [12]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N < p^-$.*

From Proposition 2.3, there exists a positive constant c depending on $p(\cdot), N$ and Ω such that

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)| \leq c\|u\|, \quad \forall u \in X. \tag{4}$$

Corresponding to f we introduce the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Consider the following functional

$$\Phi(u) := \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\sqrt{1 + |\nabla u|^{2p(x)}}}{p(x)} \right) dx, \quad \forall u \in X.$$

Then from [14], we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \nabla v\,dx$$

for any $v \in X$.

Proposition 2.4 (see [14])**.** *The functional* $\Phi : X \to \mathbb{R}$ *is convex and the mapping* $\Phi' : X \to X^*$ *is a strictly monotone and bounded homeomorphism.*

We say that a function $u \in X$ is a *weak solution* of problem (1) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \nabla v\,dx - \lambda \int_\Omega f(x,u(x))v(x)\,dx = 0$$

holds for all $v \in X$.

3. Main results

Before introducing our result we observe that, setting

$$\varrho(x) = \sup\{\varsigma > 0 : B(x,\varsigma) \subseteq \Omega\}$$

for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $B(x_0,\tau) \subseteq \Omega$, where

$$\tau = \sup_{x \in \Omega} \varrho(x). \tag{5}$$

Fixed $r > 0$, we also denote by

$$\omega_r := r^N \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

the measure of the N-dimensional ball of radius r where Γ is the Gamma function.

$$\beta_+ := \frac{\sigma(p^+,N)}{\tau^{p^+}},$$

where

$$\sigma(p^+,N) = \frac{1 - \bar{\mu}^N}{\bar{\mu}^N(1 - \bar{\mu})^{p^+}} = \inf_{\mu \in]0,1[} \frac{1 - \mu^N}{\mu^N(1 - \mu)^{p^+}}.$$

Finally, we put

$$A := \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \sup_{|t| \leq \xi} F(x,t)\, dx}{\xi^{p^-}},$$

$$B := \limsup_{\xi \to +\infty} \frac{\displaystyle\int_{B(x_0, \bar{\mu}\tau)} F(x,\xi)\, dx}{\xi^{p^+}}.$$

Now, we present our main result.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that*

(i) $F(x,\xi) \geq 0$ *for every* $(x,\xi) \in \Omega \times [0, +\infty[$.

(ii) $A < \frac{p^-}{3\beta_+ p^+ c^{p^-} \omega_\tau \bar{\mu}^N} B$.

Then, for each $\lambda \in \Lambda :=]\frac{3\beta_+ \omega_\tau \bar{\mu}^N}{Bp^-}, \frac{1}{p^+ c^{p^-} A}[$ the problem (1) has an unbounded sequence of weak solutions in X.

Proof. Our aim is to apply Lemma 2.1. Fix $\lambda \in \Lambda$. For each $u \in X := W_0^{1,p(x)}(\Omega)$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\sqrt{1 + |\nabla u|^{2p(x)}}}{p(x)} \right) dx,$$

$$\Psi(u) := \int_\Omega F(x, u(x))\, dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \qquad \forall u \in X.$$

Note that the weak solutions of (1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Lemma 2.1. Indeed, we have already pointed out that Φ is C^1 on X and sequentially weakly lower semi-continuous. Furthermore, the differential $\Phi' : X \to X^*$ admits a continuous inverse (see Proposition 2.4) and the coercivity of Φ follows from (2). On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_\Omega f(x, u(x)) v(x)\, dx$$

for any $u, v \in W_0^{1,p(x)}(\Omega)$.

Now, we verify condition (b) of Theorem 2.1. Let $\{\xi_n\}$ be a real sequence of positive numbers such that $\lim_{n \to +\infty} \xi_n = +\infty$ and

$$\lim_{n \to +\infty} \frac{\displaystyle\int_\Omega \sup_{|t| \leq \xi_n} F(x,t)\, dx}{\xi_n^{p^-}} = A.$$

Put

$$r_n := \frac{1}{p^+} \left(\frac{\xi_n}{c} \right)^{p^-}$$

for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2) and (3) into account, one has

$$\|v\| \le \max\left\{(p^+ r_n)^{\frac{1}{p^+}}, (p^+ r_n)^{\frac{1}{p^-}}\right\}.$$

So, thanks to the embedding $X \hookrightarrow C^0(\overline{\Omega})$ (see (4)), one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\varphi(r_n) &= \inf_{u \in \Phi^{-1}(-\infty, r_n)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty, r_n)} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v \in \Phi^{-1}(-\infty, r_n)} \Psi(v)}{r_n} \\
&\le \frac{\int_\Omega \sup_{|t| \le \xi_n} F(x,t)\, dx}{\frac{1}{p^+}\left(\frac{\xi_n}{c}\right)^{p^-}} \\
&= p^+ c^{p^-} \frac{\int_\Omega \sup_{|t| \le \xi_n} F(x,t)\, dx}{\xi_n^{p^-}}.
\end{aligned}
$$

Since $A < +\infty$, this leads to

$$\gamma \le \liminf_{n \to +\infty} \varphi(r_n) \le p^+ c^{p^-} A < +\infty.$$

We claim that the functional I_λ is unbounded from below. Let $\{\tau_n\}$ be a sequence of positive numbers such that $\lim_{n \to +\infty} \tau_n = +\infty$ and

$$\frac{\int_{B(x_0, \bar\mu\tau)} F(x, \tau_n)\, dx}{\tau_n^{p^+}} = B.$$

Fix $n \in \mathbb{N}$, we consider the following function:

$$w_n(x) := \begin{cases} 0 & \text{if} \quad x \in \Omega \setminus B(x_0, \tau), \\ \tau_n & \text{if} \quad x \in B(x_0, \bar\mu\tau), \\ \frac{\tau_n}{\tau(1-\bar\mu)}(\tau - |x - x_0|) & \text{if} \quad x \in B(x_0, \tau) \setminus B(x_0, \bar\mu\tau). \end{cases} \tag{6}$$

We have, definitively,

$$
\begin{aligned}
\Phi(w_n) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla w_n|^{p(x)} + \sqrt{1 + |\nabla w_n|^{2p(x)}}\right) dx \\
&= \int_{B(x_0, \tau) \setminus B(x_0, \bar\mu\tau)} \frac{1}{p(x)}\left(|\nabla w_n|^{p(x)} + \sqrt{1 + |\nabla w_n|^{2p(x)}}\right) dx \\
&\le \int_{B(x_0, \tau) \setminus B(x_0, \bar\mu\tau)} \frac{1}{p^-}(2|\nabla w_n|^{p(x)} + 1) dx \\
&\le \frac{3}{p^-} \tau_n^{p^+} \omega_\tau \bar\mu^N \beta_+.
\end{aligned}
$$

Now, for each $n \in \mathbb{N}$, one has

$$\Psi(w_n) = \int_\Omega F(x, w_n(x))\, dx \ge \int_{B(x_0, \bar\mu\tau)} F(x, \tau_n)\, dx.$$

Now, fix $\lambda \in \left]\frac{3\beta_+ \omega_\tau \bar\mu^N}{B p^-}, \frac{1}{p^+ c^{p^-} A}\right[$ and consider the following two cases.

If $B < +\infty$, then, fixed $\epsilon \in]\frac{3\beta_+\omega_\tau\bar{\mu}^N}{\lambda B p^-}, 1[$, definitively one has

$$\int_{B(x_0,\bar{\mu}\tau)} F(x, \tau_n)\, dx > \epsilon B \tau_n^{p^+},$$

and so

$$
\begin{aligned}
I_\lambda(w_n) &\leq \frac{3}{p^-}\tau_n^{p^+}\beta_+ w_\tau \bar{\mu}^N - \lambda\epsilon B \tau_n^{p^+} \\
&= \tau_n^{p^+}\left\{\frac{3}{p^-}\beta_+ w_\tau\bar{\mu}^N - \lambda\epsilon B\right\}.
\end{aligned}
$$

Since $\frac{3}{p^-}\beta_+ w_\tau\bar{\mu}^N - \lambda\epsilon B < 0$, one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

If $B = +\infty$, then, fixed $M > \frac{3\beta_+\omega_\tau\bar{\mu}^N}{\lambda p^-}$, definitively, one has

$$\int_{B(x_0,\bar{\mu}\tau)} F(x, \tau_n)\, dx > M\tau_n^{p^+}$$

and so

$$I_\lambda(w_n) \leq \tau_n^{p^+}\left\{\frac{3}{p^-}\beta_+ w_\tau\bar{\mu}^N - \lambda M\right\},$$

and this leads to

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

Taking into account that

$$\left]\frac{3\beta_+\omega_\tau\bar{\mu}^N}{Bp^-}, \frac{1}{p^+c^{p^-}A}\right[\subseteq \left]1, \frac{1}{\gamma}\right[,$$

Then, the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of I_λ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty,$$

and the conclusion is achieved. $\qquad\square$

Remark 3.1. *Under the conditions $A = 0$ and $B = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ the problem (1) admits a sequence of weak solutions which is unbounded in X.*

Here we point out the following consequence of Theorem 3.1.

Corollary 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function satisfying condition* (i). *If*

$$A < \frac{1}{p^+c^{p^-}}, \qquad B > \frac{3\beta_+\omega_\tau\bar{\mu}^N}{p^-},$$

then, the problem

$$
\begin{cases}
-\mathrm{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = f(x, u), & x \in \Omega, \\
u = 0, & x \in \partial\Omega
\end{cases}
\tag{7}
$$

has an unbounded sequence of weak solutions in X.

Corollary 3.2. *Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $G_1(\xi) :=$*
$\int_0^\xi g_1(t)\,dt$ *for all $\xi \in \mathbb{R}$ and assume that*

(j) $\liminf\limits_{\xi \to +\infty} \dfrac{G_1(\xi)}{\xi^{p^-}} < +\infty$;

(jj) $\limsup\limits_{\xi \to +\infty} \dfrac{G_1(\xi)}{\xi^{p^+}} = +\infty$.

Then, for every $\alpha_i \in L^1(\Omega)$ for $1 \le i \le n$, with $\min_{x \in \Omega}\{\alpha_i(x) : 1 \le i \le n\} \ge 0$ and with $\alpha_1 \ne 0$, and for every non-negative continuous $g_i : \mathbb{R} \to \mathbb{R}$ for $2 \le i \le n$, satisfying

$$\max\left\{ \sup_{\xi \in \mathbb{R}} G_i(\xi) : 2 \le i \le n \right\} \le 0$$

and

$$\min\left\{ \liminf_{\xi \to +\infty} \frac{G_i(\xi)}{\xi^{p^-}} : 2 \le i \le n \right\} > -\infty,$$

where $G_i(\xi) := \int_0^\xi g_i(t)\,dt$ for all $\xi \in \mathbb{R}$ for $2 \le i \le n$, for each

$$\lambda \in \left] 0, \frac{1}{p^+ c^{p^-} \liminf_{\xi \to +\infty} \frac{G_1(\xi)}{\xi^{p^-}} \int_\Omega \alpha_1(x)\,dx} \right[,$$

the problem

$$\begin{cases} -\mathrm{div}\left(\left(1 + \dfrac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) = \lambda \sum_{i=1}^n \alpha_i(x) g_i(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

has an unbounded sequence of weak solutions in X.

Proof. Set $f(x,t) = \sum_{i=1}^n \alpha_i(x) g_i(t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. From the assumption (jj) and the condition

$$\min\left\{ \liminf_{\xi \to +\infty} \frac{G_i(\xi)}{\xi^{p^-}} : 2 \le i \le n \right\} > -\infty$$

we have

$$\limsup_{\xi \to +\infty} \frac{\int_\Omega F(x,\xi)\,dx}{\xi^{p^+}} = \limsup_{\xi \to +\infty} \frac{\sum_{i=1}^n \left(G_i(\xi) \int_\Omega \alpha_i(x)\,dx \right)}{\xi^{p^+}} = +\infty.$$

Moreover, from the assumption (j) and the condition

$$\max\left\{ \sup_{\xi \in \mathbb{R}} G_i(\xi) : 2 \le i \le n \right\} \le 0$$

we have

$$\liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{|t| \le \xi} F(x,t)\,dx}{\xi^{p^-}} \le \left(\int_\Omega \alpha_1(x)\,dx \right) \liminf_{\xi \to +\infty} \frac{G_1(\xi)}{\xi^2} < +\infty.$$

This completes the proof in view of the Theorem 3.1. $\qquad\qquad\qquad\qquad\square$

Corollary 3.3. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous and non-negative function such that*

$$\liminf_{\xi \to +\infty} \frac{H(\xi)}{\xi^{p^-}} < \frac{p^-}{3p^+ \beta_+ c^{p^-} |\Omega|} \limsup_{\xi \to +\infty} \frac{H(\xi)}{\xi^{p^+}},$$

where $H(\xi) := \int_0^\xi h(t)\, dt$. Then, for every

$$\lambda \in \left(\frac{3\beta_+}{p^- \limsup\limits_{\xi \to +\infty} \frac{H(\xi)}{\xi^{p^+}}}, \frac{1}{p^+ c^{p^-} |\Omega| \liminf\limits_{\xi \to +\infty} \frac{H(\xi)}{\xi^{p^-}}} \right),$$

the problem

$$\begin{cases} -\mathrm{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) = \lambda h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

admits an unbounded sequence of weak solutions.

We conclude this paper with the following example to illustrate our results.

Example 3.1. *Let $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 3\}$. Consider the problem*

$$\begin{cases} -\mathrm{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) = \lambda f(x, y, u) & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{8}$$

where $p(x, y) = x^2 + y^2 + 3$ for all $(x, y) \in \Omega$,

$$f(x, y, t) = \begin{cases} f^*(x, y)t^6 \Big(7 + \sin(\ln(|t|)) - 7\cos(\ln(|t|)) \Big) & \text{if} \quad (x, y, t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0 & \text{if} \quad (x, y, t) \in \Omega \times \{0\}, \end{cases}$$

where $f^* : \Omega \to \mathbb{R}$ is a non-negative continuous function. It is obvious that $p^- = 3$ and $p^+ = 6$. A direct calculation shows

$$F(x, y, t) = \begin{cases} f^*(x, y)t^7 \Big(1 - \cos(\ln(|t|)) \Big) & \text{if} \quad (x, y, t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0 & \text{if} \quad (x, y, t) \in \Omega \times \{0\}. \end{cases}$$

So,

$$\liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \le \xi} F(x, y, t)\, d\sigma}{\xi^3} = 0,$$

and

$$\limsup_{\xi \to +\infty} \frac{\displaystyle\int_{B((0,0),1)} F(x, y, \xi)\, dx}{\xi^6} = +\infty.$$

Hence, using Theorem 3.1, the problem (8) for every $\lambda \in]0, +\infty[$ admits infinitely many weak solutions in X.

Conclusion

Based on a recent critical point theorem of Bonanno and Molica Bisci [2], we established the existence of an open interval $]\lambda', \lambda''[$, such that for each $\lambda \in]\lambda', \lambda''[$, a class of nonlinear Dirichlet problems admits infinitely many weak solutions.

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<div align="center">R E F E R E N C E S</div>

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