MODULE AMENABILITY AND WEAK MODULE AMENABILITY OF BANACH ALGEBRAS

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In this paper we compare the notions of (weak) amenability and (weak) module amenability of Banach algebras which are Banach modules over another Banach algebras with compatible actions. We find conditions that (weak) module amenability implies (weak) amenability or vice versa. In particular, we show that if S is an inverse semigroup with finitely idempotents, then S is amenable and $l^1(S)$ is unital, when $l^1(S)$ is module amenable as a Banach $l^1(E_S)$-module with arbitrary commutative compatible actions.

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1. Introduction and preliminaries

The notion of module amenability for a Banach algebra was introduced by the third author in [1]. In [2] the authors have introduced the concept of weak module amenability and proved that $l^1(S)$ is weakly module amenable, when S is a commutative inverse semigroup with the set of idempotents E_S and $l^1(E_S)$ acts on $l^1(S)$ by the commutative compatible actions $\delta_e.\delta_s = \delta_s.\delta_e = \delta_{se}$ $(s \in S, e \in E_S)$. In general, the (weak) module amenability is defined for a Banach algebra $\mathcal{A}$ which is also a Banach $\mathfrak{A}$-module, on another Banach algebra $\mathfrak{A}$ [1], [2].

The notion of (weak) module amenability is very young compared to (weak) amenability, and the structure of (weakly) module amenable Banach algebras is rather unknown. The aim of the present paper is to investigate the relation between (weakly) module amenable and (weakly) amenable Banach algebra and finding conditions that these notions coincide. We mainly focus on the case where $\mathfrak{A}$ is commutative and (weakly) amenable. These are rather natural conditions which are automatic in the classical case where $\mathfrak{A} = \mathbb{C}$.

This paper is organized as follows. This section is devoted to the notations and definitions which are needed throughout the paper. In section 2 assume that $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module and show that if $\mathcal{A}$ has an identity, then weak module amenability of $\mathcal{A}$ implies its weak amenability, when $\mathfrak{A}$ is commutative and

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weakly amenable. Also, we give an example of an inverse semigroup S such that there is no any commutative compatible action which the Banach algebra $l^1(S)$ is weakly module amenable as an $l^1(E_S)$-module. In the main theorem of section 3, we suppose that $\mathfrak{A}$ is commutative and amenable and prove that amenability of $\mathcal{A}$ follows from its module amenability, when $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module.

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha.(ab) = (\alpha.a)b \ , \ (ab).\alpha = a(b.\alpha) \quad (\alpha \in \mathfrak{A} \ , \ a, b \in \mathcal{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions

$$\alpha.(a.x) = (\alpha.a)x \ , \ a.(\alpha.x) = (a.\alpha).x \ , \ (\alpha.x).a = \alpha.(x.a) \quad (\alpha \in \mathfrak{A} \ , \ a \in \mathcal{A} \ , x \in X),$$

and similarly for the right or two-side actions. Then X is called a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, if

$$\alpha.x = x.\alpha \quad (\alpha \in \mathfrak{A} \ , \ x \in X),$$

then X is called a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it does not need to be a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition $(a.\alpha).b = a.(\alpha.b)$ for $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$. But when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then clearly it is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathfrak{A}$, $\mathcal{A}$ and X be as above. A bounded map $D : \mathcal{A} \longrightarrow X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b) \ , \ D(ab) = D(a).b + a.D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\alpha.a) = \alpha.D(a) \ , \ D(a.\alpha) = D(a).\alpha \quad (a \in \mathcal{A} \ , \ \alpha \in \mathfrak{A}).$$

Note that $D : \mathcal{A} \to X$ is bounded if there exists $M > 0$ such that $||D(a)|| \leq M||a||$, $(a \in \mathcal{A})$, and although D is not necessarily linear, but still its boundedness implies its norm continuity. For every $x \in X$ we define the inner module derivation ad_x, by

$$ad_x(a) = a.x - x.a \quad (a \in \mathcal{A}).$$

The Banach algebra $\mathcal{A}$ is called module amenable (more precisely, $\mathfrak{A}$-module amenable) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner.

If $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module, then $\mathcal{A}$ is called weakly module amenable (or, $\mathfrak{A}$-weakly module amenable) if every module derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is inner.

Recall that a semigroup S is called an *inverse semigroup*, if for each $s \in S$ there is a unique element $s^* \in S$ such that $s^*ss^* = s^*$ and $ss^*s = s$. The space of $l^1(S)$, consisting of functions $\sum_{s\in S} \alpha_s\delta_s$ on S such that $\sum_{s\in S} |\alpha_s| < \infty$, is a Banach algebra by the convolution production. If we denote the set of idempotents of S by E_S (for instance if G is a group with unit of e_G, then it is an inverse semigroup such that $E_G = \{e_G\}$), then $l^1(E_S)$ could be regarded as a commutative subalgebra of $l^1(S)$ [10]. Consequently, $l^1(S)$ is a Banach algebra and a Banach $l^1(E_S)$-module with compatible actions.

2. **Weak module amenability and weak amenability**

In this section for any Banach $\mathfrak{A}$-module $\mathcal{A}$ with commutative compatible actions, we investigate conditions under which weak module amenability of $\mathcal{A}$ implies its weak amenability and vice versa.

We start with the following observation:

Proposition 2.1. *Let $\mathfrak{A}$ be a weakly amenable, closed subalgebra of $\mathcal{A}$ which is contained in the center of $\mathcal{A}$. Then weak amenability of $\mathcal{A}$ follows from its weak module amenability, when $\mathfrak{A}$ acts on $\mathcal{A}$ with the algebra multiplication.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation. Since $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module so is $\mathcal{A}^*$. Then the restriction of D on $\mathfrak{A}$ is zero, by Theorem 2.8.63 in [7]. Hence, $D(\alpha.a) = \alpha.D(a)$ for all $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, i.e. D is a module derivation. Therefore, D is inner. □

As an immediate consequence of Proposition 2.1, we have the next result which is obtained in [4].

Corollary 2.1. *The Banach algebra $l^1(S)$ is weakly amenable, for every commutative inverse semigroup S.*

Proof. Since $l^1(E_S)$ is a commutative Banach algebra that is spanned by its idempotents, hence it is weakly amenable Banach algebra, by Proposition 2.8.72 in [7]. Also by Theorem 3.1 of [2], $l^1(S)$ is weakly module amenable as a Banach $l^1(E_S)$-module with the action defined by the algebra multiplication. Now, the result follows if we apply Proposition 2.1, to $\mathcal{A} = l^1(S)$ and $\mathfrak{A} = l^1(E_S)$. □

Definition 2.1. *A Banach $\mathcal{A}$-bimodule X is called pseudo-unital if*

$$X = \{a.x.b : a, b \in \mathcal{A}, x \in X\}.$$

Similarly, one defines pseudo-unital left and right Banach modules.
Obviously, if $\mathcal{A}$ is unital then $\mathcal{A}^*$ is a pseudo-unital Banach $\mathcal{A}$-bimodule. In the next lemma we suppose that $\mathcal{A}^*$ is a pseudo-unital left Banach $\mathcal{A}$-bimodule and we present an analogue of Proposition 2.1, for arbitrary commutative compatible actions.

Lemma 2.1. *Let $\mathfrak{A}$ be a commutative, weakly amenable Banach algebra and $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module which is weakly module amenable. If $\mathcal{A}$ has a bounded approximate identity consisting of central idempotents then $\mathcal{A}$ is weakly amenable.*

Proof. Let $(e_j)_{j\in J}$ be a bounded approximate identity such that each e_j is an idempotent in the center of $\mathcal{A}$. Let $\mathcal{L}_j$ be the closed linear span of $\{\alpha.e_j \ : \ \alpha \in \mathfrak{A}\}$.

For each $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathfrak{A}$, we have that $(\alpha.a)(\beta.b) = (\alpha\beta).(ab)$. Therefore, $\mathcal{L}_j$ is a closed subalgebra of $\mathcal{A}$ with the following multiplication:

$$(\alpha.e_j).(\beta.e_j) := (\alpha\beta).e_j \quad (\alpha, \beta \in \mathfrak{A}).$$

For each $j \in J$, define

$$\theta_j : \mathfrak{A} \longrightarrow \mathcal{L}_j \ , \ \theta_j(\alpha) = \alpha.e_j.$$

It is clear that, θ_j is a continuous algebra homomorphism with dense range such that $||\theta_j|| \leq ||e_j||$. Thus, weak amenability of $\mathcal{L}_j$ $(j \in J)$ follows from commutativity and weak amenability of $\mathfrak{A}$, by Proposition 2.8.64 of [7]. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation. Then the restriction of D vanishes on $\mathcal{L}_j$, because $(e_j)_{j\in J}$ is contained in the center of $\mathcal{A}$. Hence the derivation $D|_{\mathcal{L}_j} : \mathcal{L}_j \longrightarrow \mathcal{A}^*$ is zero. Therefore,

$$\begin{aligned} D(\alpha.a) &= D\big(\lim_j(\alpha.e_j).a\big), \\ &= \lim_j \big[D(\alpha.e_j).a + (\alpha.e_j).D(a)\big], \\ &= \alpha.D(a) \quad (\alpha \in \mathfrak{A}, a \in \mathcal{A}). \end{aligned}$$

This means that $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a module derivation. Thus, weak module amenability of $\mathcal{A}$ implies that D is inner. □

Corollary 2.2. *Let $\mathfrak{A}$ be a commutative, weakly amenable Banach algebra. If $\mathcal{A}$ is a unital Banach algebra and a commutative Banach $\mathfrak{A}$-module, then weak module amenability of $\mathcal{A}$ implies its weak amenability.*

In the next corollary we give necessary condition for weak module amenability of $l^1(S)$ as a Banach $l^1(E_S)$-module, by any commutative compatible actions.

Corollary 2.3. *Let $l^1(S)$ be a commutative Banach $l^1(E_S)$-module, where S is a unital inverse semigroup. If $l^1(S)$ is weakly module amenable, then it is weakly amenable.*

Consider the bicyclic semigroup $S_1 = \langle e, p, q | pq = e\rangle$. This is the inverse semigroup generated by an identity element e and two elements p, q subject to the condition $pq = e$. It is known that $l^1(S_1)$ is not weakly amenable [5]. Consequently, there is none commutative compatible action that $l^1(S_1)$ is $l^1(E_{S_1})$-weakly module amenable with it, by Corollary 2.3.

The next proposition may be considered as a converse of Corollary 2.2, in which $\mathfrak{A}$ is not commutative or weakly amenable in general.

Proposition 2.2. *If $\mathcal{A}$ is unital, then module amenability of $\mathcal{A}$ follows from its amenability. In the case that $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module, then weak amenability of $\mathcal{A}$ implies its weak module amenability.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and D be a module derivation from $\mathcal{A}$ into X^*. Then without loss of generality, as in the classical case, we may suppose that X is a pseudo-unital Banach $\mathcal{A}$-bimodule. Let e be an identity for $\mathcal{A}$, then $D(e) = 0$, hence additivity and continuity of D implies that $D(\lambda e) = 0$, for each $\lambda \in \mathbb{R}$. Also

$$0 = D(-e) = D(ie.ie) = 2iD(ie).$$

Thus, $D(\lambda e) = 0$ for each $\lambda \in \mathbb{C}$. Therefore

$$\begin{aligned} D(\lambda a) &= D(\lambda e).a + \lambda e.D(a) \\ &= \lambda D(a) \quad (\lambda \in \mathbb{C}, a \in \mathcal{A}). \end{aligned}$$

Consequently, D is $\mathbb{C}$-linear and the result follows. $\square$

3. Module amenability and amenability

In this section we investigate the inverse of Proposition 2.2. Also, we show that the converse of Proposition 2.2 is not valid, in general. Lets recall Proposition 2.2 in [1].

Lemma 3.1. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If $\mathcal{A}$ is module amenable, then it has a bounded approximate identity.*

Suppose that X is a Banach $\mathcal{A}$-bimodule and E is a w^*-closed submodule of X^*. Then there is a Banach $\mathcal{A}$-bimodule F equipped with the following actions,

$$\langle x.a, \varphi\rangle = \langle x, a.\varphi\rangle \ , \ \langle a.x, \varphi\rangle = \langle x, \varphi.a\rangle \quad (a \in \mathcal{A}, x \in F, \varphi \in F^*) \quad (3.1),$$

such that $F^* = E$, see Exercise 2.1.2 of [12]. If there is a module action of $\mathfrak{A}$ on F such that F is a Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is F^*, with the the following extra actions

$$\langle x, \alpha.\varphi\rangle = \langle x.\alpha, \varphi\rangle \ , \ \langle x, \varphi.\alpha\rangle = \langle \alpha.x, \varphi\rangle \quad (\alpha \in \mathfrak{A}, x \in F, \ \varphi \in F^*).$$

Moreover it is clear that, if F^* is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is F. In the following lemma we assume that $(e_j)_{j\in J}$ is a bounded approximate identity for $\mathcal{A}$, which exists by Lemma 3.1.

Lemma 3.2. *Let $\mathcal{A}$ be a module amenable, commutative Banach $\mathfrak{A}$-module. If there is an amenable, closed subalgebra $\mathfrak{L}$ of $\mathcal{A}$ such that $\alpha.e_j \in \mathfrak{L}$ $(\alpha \in \mathfrak{A}, j \in J)$, then $\mathcal{A}$ is amenable.*

Proof. Let X be a Banach $\mathcal{A}$-bimodule; by Proposition 2.1.5 of [12] there is no loss of generality if we suppose that X is pseudo-unital. Let $D : \mathcal{A} \longrightarrow X^*$ be a derivation, then so is $D|_{\mathfrak{L}} : \mathfrak{L} \longrightarrow X^*$. By the amenability of $\mathfrak{L}$, there is $\varphi \in X^*$ such that

$$D(l) = l.\varphi - \varphi.l \quad (l \in \mathfrak{L}),$$

i.e. $D|_{\mathfrak{L}} = ad_\varphi$. Let $\tilde{D} = D - ad_\varphi$ and E be the w^*-closed linear span of the following set,

$$Y = \left\{a.\tilde{D}(b).c \ : \ a, b, c \in \mathcal{A}\right\}.$$

It is clear that, $\tilde{D} : \mathcal{A} \longrightarrow X^*$ is a derivation with $\tilde{D}|_{\mathfrak{L}} = 0$. Since X is pseudo-unital, we conclude that E is a Banach $\mathcal{A}$-bimodule such that $\tilde{D}(\mathcal{A}) \subseteq E \subseteq X^*$.

Now we show that $\tilde{D}$ is an inner derivation. Let F be a Banach $\mathcal{A}$-bimodule, such that $F^* = E$. For $x \in X$, let $a, b \in \mathcal{A}$ and $z \in X$ be such that $x = a.z.b$. For $\alpha \in \mathfrak{A}$, define

$$\alpha \circ x := (\alpha.a).(z.b)\ ,\ x \circ \alpha := (a.z).(b.\alpha) \quad (3.2).$$

We claim that $\alpha \circ x$ is well defined, i.e. it is independent of the choices of a, b and z. Let $z' \in X$ and a', b' be in $\mathcal{A}$ such that $x = a'.z'.b'$. Then for each $\alpha \in \mathfrak{A}$, we have

$$(\alpha.a).(z.b) = \lim_j (\alpha.e_j).(a.z.b) = \lim_j (\alpha.e_j).(a'.z'.b') = (\alpha.a').(z'.b').$$

Similarly, $x \circ \alpha$ is well defined. Clearly, by the above actions of $\mathfrak{A}$ and the given actions of $\mathcal{A}$, X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. For $\alpha \in \mathfrak{A}$ and $x \in F$, we have

$$\alpha \circ x = \lim_j \alpha \circ (e_j.x) = \lim_j (\alpha.e_j).x \in F,$$

similarly, $x \circ \alpha \in F$. Thus F is a Banach $\mathfrak{A}$-submodule of X. Clearly, the actions (3.1) and (3.2), turn F into a Banach $\mathcal{A}$-$\mathfrak{A}$-module. So F^* is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. For all $b, c \in \mathcal{A}$, $\tilde{D}(b).c$ is an element of F^*, so for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, we have

$$\alpha \circ (a.\tilde{D}(b).c) = (\alpha.a).\tilde{D}(b).c;$$

similarly, $(a.\tilde{D}(b).c) \circ \alpha = a.\tilde{D}(b).(c.\alpha)$. Also, since $\tilde{D}|_{\mathcal{L}} = 0$ and $\mathcal{A}$ is a commutative $\mathfrak{A}$-module,

$$\begin{aligned}(\alpha.a).(\tilde{D}(b).c) &= \lim_j a.\left[(\alpha.e_j).\tilde{D}(b)\right].c,\\ &= \lim_j a.\left[\tilde{D}((\alpha.e_j).b)\right].c,\\ &= a.\left[\tilde{D}(\alpha.b)\right].c,\\ &= a.\left[\tilde{D}(b.\alpha)\right].c,\\ &= \lim_j a.\left[\tilde{D}(b).(\alpha.e_j)\right].c,\\ &= (a.\tilde{D}(b)).(c.\alpha).\end{aligned}$$

It follows that elements of Y commute with elements of $\mathfrak{A}$. Thus, by the w^*-continuity of compatible actions, we conclude that $E = F^*$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and so is F. Since X is pseudo-unital, as in the proof of Lemma 2.1, we obtain that

$$\begin{aligned}\tilde{D}(\alpha.a) &= \lim_j (\alpha.e_j).\tilde{D}(a),\\ &= \alpha \circ \tilde{D}(a) \quad (\alpha \in \mathfrak{A}, a \in \mathcal{A}).\end{aligned}$$

That is, $\tilde{D} : \mathcal{A} \longrightarrow F^*$ is a module derivation. Consequently, module amenability of $\mathcal{A}$ implies that $\tilde{D} = ad_\psi$, for some $\psi \in E$. Therefore, $D = ad_{\varphi+\psi}$, where $\varphi + \psi \in X^*$. $\square$

Corollary 3.1. *Let $\mathfrak{A}$ be an amenable Banach algebra and $\mathcal{A}$ be a unital Banach algebra and a commutative Banach $\mathfrak{A}$-module. Then the amenability of $\mathcal{A}$ follows from its module amenability.*

Proof. Let e be an identity for $\mathcal{A}$ and $\mathcal{L}$ be the closed linear span of $\{\alpha.e \ : \ \alpha \in \mathfrak{A}\}$. Then $\mathcal{L}$ is a closed subalgebra of $\mathcal{A}$ with the following multiplication:

$$(\alpha.e).(\beta.e) = (\alpha\beta).e \quad (\alpha, \beta \in \mathfrak{A}).$$

Consider the continuous algebra homomorphism $\theta : \mathfrak{A} \longrightarrow \mathcal{L}$ defined by $\theta(\alpha) = \alpha.e$, for $\alpha \in \mathfrak{A}$, then $\theta(\mathfrak{A})$ is dense in $\mathcal{L}$ and so $\mathcal{L}$ is amenable by Proposition 2.3.1 of [12]. Therefore the amenability of $\mathcal{A}$ follows from Lemma 3.2. □

It should be noted that the above corollary is different from [3, Prop. 3.2]. Because [3, Prop. 3.2] is valid only for a very specific compatible action.

Corollary 3.2. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If $\mathfrak{A}$ is amenable, then $\mathcal{A}^{**}$ is module amenable, if and only if it is amenable.*

Proof. Since $\mathcal{A}$ is w^*-dense in $\mathcal{A}^{**}$ and compatible actions are w^*-continuous, so $\mathcal{A}^{**}$ is a commutative Banach $\mathfrak{A}$-module. Thus, the module amenability or the amenability of $\mathcal{A}^{**}$ implies that it has a bounded approximate identity. Hence $\mathcal{A}^{**}$ is unital by Lemma 1.1 in [9]. Now, the result follows by Proposition 2.2 and Corollary 3.1. □

The proof of the following lemma is routine but we present it, briefly.

Lemma 3.3. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module and $\mathcal{L}$ be a closed ideal and $\mathfrak{A}$-submodule of $\mathcal{A}$. If $\mathcal{L}$ and $\mathcal{A}/\mathcal{L}$ are module amenable, then so is $\mathcal{A}$.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. Suppose that E is the space of all elements $\psi \in X^*$ such that $\psi.\mathcal{L} = \mathcal{L}.\psi = 0$, and F is the subspace of X generated by $\mathcal{L}.X + X.\mathcal{L}$. Since $\mathcal{L}.\frac{X}{F} = \frac{X}{F}.\mathcal{L} = 0$, the following module actions are well-defined

$$\alpha.(x + F) := \alpha.x + F \ , \ (a + \mathcal{L}).(x + F) := a.x + F \quad (\alpha \in \mathfrak{A}, a \in \mathcal{A}, x \in X),$$

and similar for the right actions. Therefore X/F is a commutative Banach $\mathcal{A}/\mathcal{L}$-$\mathfrak{A}$-module, and so is $E \cong (X/F)^*$.

Now, let $D : \mathcal{A} \to X^*$ be a module derivation. Consider $\varphi \in X^*$ such that $D|_{\mathcal{L}} = ad_\varphi$ and let $\tilde{D} := D - ad_\varphi$. $\tilde{D}$ vanishes on $\mathcal{L}$, therefore it induces a module derivation from $\mathcal{A}/\mathcal{L}$ to X^*, that we denote likewise by $\tilde{D}$. Also, For all $a \in \mathcal{A}$ and $l \in \mathcal{L}$ we obtain that $l.\tilde{D}(a) = \tilde{D}(a).l = 0$. Hence $\tilde{D}(\mathcal{A}/\mathcal{L}) \subseteq E$. Consequently, the module amenability of $\mathcal{A}/\mathcal{L}$ implies that $\tilde{D}$ is inner, and so is D. □

Now, we are ready to present the main theorem of this paper.

Theorem 3.1. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If $\mathfrak{A}$ is commutative and amenable, then the module amenability of $\mathcal{A}$ implies its amenability.*

Proof. First we suppose that $\mathfrak{A}$ has an identity e for itself and $\mathcal{A}$. Consider $\mathcal{B} = \mathcal{A} \oplus \mathfrak{A}$ with the following multiplication

$$(a, \alpha).(b, \beta) = (ab + \alpha.b + \beta.a, \alpha\beta) \quad (\alpha, \beta \in \mathfrak{A}, a, b \in \mathcal{A}).$$

Clearly, $\mathcal{B}$ is a unital Banach algebra and $\mathcal{A}$ is a closed ideal of $\mathcal{B}$. Also, $\mathcal{B}$ is a commutative Banach $\mathfrak{A}$-module with the following actions

$$\gamma.(a, \alpha) = (a, \alpha).\gamma := (\gamma.a, \gamma\alpha) \quad (\alpha, \gamma \in \mathfrak{A}, a \in \mathcal{A}).$$

Since $\mathcal{B}/\mathcal{A} \cong \mathfrak{A}$ is a unital, amenable Banach algebra, $\mathcal{B}/\mathcal{A}$ is module amenable, by Proposition 2.2. Therefore $\mathcal{B}$ is module amenable, by Lemma 3.3. Hence Corollary 3.1 implies that $\mathcal{B}$ is amenable and so is $\mathcal{A}$, by Proposition 2.3.3 in [12].

In the case $\mathfrak{A}$ is not unital we consider $\mathfrak{A}^\sharp$, as the unitization of $\mathfrak{A}$ and extend the module actions of $\mathcal{A}$ on $\mathfrak{A}$ by letting

$$(\alpha + \lambda e).a = a.(\alpha + \lambda e) := \alpha.a + \lambda a \quad (\alpha \in \mathfrak{A}, \lambda \in \mathbb{C}, a \in \mathcal{A}).$$

Then $\mathcal{A}$ is a commutative Banach $\mathfrak{A}^\sharp$-module and e is an identity for the action on $\mathcal{A}$. Since $\mathfrak{A} \subseteq \mathfrak{A}^\sharp$, any $\mathfrak{A}^\sharp$-module derivation on $\mathcal{A}$ is an $\mathfrak{A}$-module derivation. Therefore $\mathfrak{A}$-module amenability of $\mathcal{A}$ implies its $\mathfrak{A}^\sharp$-module amenability. Therefore, $\mathcal{A}$ is amenable. $\square$

Let $S = (\mathbb{N}, \wedge)$ be the inverse semigroup of positive integers with the minimum operation, $\mathcal{A} = l^1(S)$ and $\mathfrak{A} = l^1(E_S)$. Then $\mathcal{A}$ is not unital, but has an approximate bounded identity $(\delta_n)_{n\in\mathbb{N}}$. It is known that $\mathcal{A}$ is weak amenable, but $\mathcal{A}$ is not amenable, because each element of S is an idempotent. Consider $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module under the actions defined by the algebra multiplication. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and D be a module derivation from $\mathcal{A}$ into X^*. We may suppose that X is a pseudo-unital Banach $\mathcal{A}$-bimodule. Therefore, for all $f \in \mathcal{A}$ we have

$$\begin{aligned} D(f) &= \lim_{n\to\infty} D(f * \delta_n), \\ &= D(f) + \lim_{n\to\infty} f.D(\delta_n), \\ &= 2D(f), \end{aligned}$$

it follows that D is zero. Therefore $\mathcal{A}$ is module amenable. This example shows that the hypothesis of amenability of $\mathfrak{A}$ is a necessary condition in Theorem 3.1, and therefore cannot be omitted or exchange with the weaker condition of weak amenability.

In [11], we compared the notions of super amenability and super module amenability and found some results about the structure of $l^1(S)$, when it is super module amenable as a commutative Banach $l^1(E_S)$-module. Now, as an application of Theorem 3.1 we obtain some properties about the structure of $l^1(S)$, when it is module amenable.

A discrete semigroup S is left amenable if the space $l^\infty(S)$ admits a functional m such that $m(\mathbf{1}) = 1 = ||m||$ and $m(l_s\varphi) = m(\varphi)$, for any $s \in S$ and $\varphi \in l^\infty(S)$, where $l_s\varphi$ is the left translation of φ. Similarly for right amenable. If S is both left and right amenable, it is amenable [8].

We summarize some known structural implications of amenability of $l^1(S)$ [8, Prop. 10.5, Cor. 10.6].

Theorem 3.2. *Let S be a semigroup such that $l^1(S)$ is amenable. Then*
(i) *S is amenable.*
(ii) *$l^1(S)$ has an identity.*
(iii) *S has only finitely many idempotents.*

Now, suppose that $l^1(S)$ is module amenable as a Banach $l^1(E_S)$-module with arbitrary commutative compatible actions. The preceding example shows that (ii) and (iii) may fail. On the other hand,

Theorem 3.3. *Let S be an inverse semigroup such that E_S is finite. If $l^1(S)$ is module amenable as a commutative Banach $l^1(E_S)$-module, then*
(i) *S is amenable.*
(ii) *$l^1(S)$ has an identity.*

Proof. Since E_S is finite, $l^1(E_S)$ is amenable by [6]. Also $l^1(E_S)$ is a commutative Banach algebra. Consequently, the result follows from Theorem 3.1. □

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