APPLICATIONS FOR A GENERALIZATION OF TWO FUNDAMENTAL VARIATIONAL PRINCIPLES

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In this paper, the author proposes to involve a generalization of Ekeland principle in the solution of some mathematical physics equations issued from modeling of real phenomena. One makes remarks on some variants of Ekeland and Borwein-Preiss variational principles. In order to sustain links and other relations between namely variants of these fundamental statements, a series of theoretical results has been developed. An application of these statements at β-subderivative is also discussed in comparison with other results involving the last cited notion.

Keywords: Ekeland principle, Borwein-Preiss principle, β-derivative, β-subderivative, smooth variational principle, Dirichlet problem, β-viscosity solution.

1. Introduction

Ekeland principle is a perturbed variational principle discovered in 1972 [1] and nowadays, after more than 40 years, it was proved to be, the foundation of the modern Variational Calculus (see, for instance, the minimax theorems in Banach spaces or in the Finsler manifolds, in which the key step of demonstration is made by the application of the Ekeland principle).

As referring the applications, these are numerous and various: the geometry of Banach spaces, nonlinear analysis, differential equations and partial differential equations, global analysis, probabilistic analysis, differential geometry, fixed point theorems, nonlinear semigroups, dynamical systems, optimization, mathematical programming, optimal control.

The author worked in *variational principles* in [2], discussions on theoretical results in [3], [4], applications of variational principles in [5-12], even though applications in problems evolved from modelling of real phenomena and usage of β-differentiability in [2] and [13].

This paper aims to compare some variants of Ekeland principle [1], [14] and Borwein-Preiss principle [15], involving the way from [16]. We improve some results from [16], establish also links, connections and applications *via* some statement of β-differentiability highlited in [2] and also open the way to apply generalizations of these principles in solving some mathematical physics problems.

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In [5], [6] and [7] the author involved Ekeland variational principle in solving problems issued from models of real phenomena.

2. Preliminaries

We start by introducing the original variant of Ekeland principle and some other variants necessary for our goal. Give also some variants of Borwein-Preiss variational principle to have a support for generalization and unification of them.

2.1 Ekeland principle ([1], [14]). *Let* (X, d) *be a complete metric space and* φ: X $\to$ $(-\infty, +\infty]$ *bounded from below, lower semicontinuous and proper. For any* $\varepsilon > 0$ *and* u *of* X *with*

$$\varphi(u) \le \inf \varphi(X) + \varepsilon$$

and for any $\lambda > 0$, *there exists* v_ε *in* X *such that*

$$\varphi(v_\varepsilon) < \varphi(w) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, w)\ \forall w \in X \setminus \{v_\varepsilon\} \tag{1}$$

and

(2) $\varphi(v_\varepsilon) \le \varphi(u)$, (3) $d(v_\varepsilon, u) \le \lambda$.

An improved formulation of the Ekeland principle (Jean-Paul Penot, [17]) follows.

2.2 *Under the conditions of the theorem* 2.1, *there exists* v_ε *in* X *verifying* (1) *and* (3), *while* (2) *is replaced by*

$$\varphi(v_\varepsilon) \le \varphi(u) - \frac{\varepsilon}{\lambda} d(v_\varepsilon, u). \tag{2_0}$$

A generalization of Ekeland principle, when the perturbing function is smooth in the case of a Banach space (*in the following, all the Banach spaces are real*) is given in [15].

Begin the preliminaries in order to introduce Borwein-Preiss principle. Let X be a real normed space. For any p in $[1, +\infty)$ consider the set Σ_p of the functions Δ_p: $X \to \mathbf{R}$,

$$\Delta_p(x) = \sum_{n=0}^{\infty} \mu_n \|x - v_n\|^p,\ \mu_n \ge 0,\ \sum_{n=0}^{\infty} \mu_n = 1,\ \lim_{n\to\infty} v_n = v, \tag{$*$}$$

convergent series since we have, $\forall n \ge 0$, $|\mu_n \|x - v_n\|^p| \le M^p \mu_n$, $M > 0$. Moreover, *if* β *is a bornology*[2] *on* X, p > 1, *and* ν: $\nu(x) = \|x\|$ *is* β*-differentiable*[3] *on* $X \setminus \{0\}$, *then* Δ_p *is* β *- differentiable on* X.

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[2] Let X be a real normed space. A nonempty set β of bounded parts of X, with the properties:
1° $\bigcup_{A\in\beta} A = X$, 2° $A \in \beta \Rightarrow -A \in \beta$ and $\lambda A \in \beta$ ($\lambda > 0$), 3° for every A, B in β there exists C in β such that $A \subset C$ and $B \subset C$, is called *bornology on* X.

[3] Let β be a bornology on X and $f: X \to \overline{\mathbf{R}}$ locally finite in the point a (there is a neighborhood of a on which f is finite). By definition, f is β*-differentiable in* a, if there exists φ in the dual X^* such

2.3 Borwein - Preiss principle (smooth variational principle). *Let* X *be real Banach space*, φ: X → (– ∞, + ∞] *bounded from below, lower semicontinuous and proper*. *For any* $\varepsilon > 0$, $\lambda > 0$, $p \geq 1$ *and* x_0 *of* X *with*

(I) $$\varphi(x_0) < \inf \varphi(X) + \varepsilon,$$

there exists Δ_p *in* Σ_p *and* v_ε *in* X *so that*

(II) $$\varphi(v_\varepsilon) + \frac{\varepsilon}{\lambda^p}\Delta_p(v_\varepsilon) \leq \varphi(x) + \frac{\varepsilon}{\lambda^p}\Delta_p(x) \quad \forall x \in X,$$

(III) $\varphi(v_\varepsilon) < \inf \varphi(X) + \varepsilon$, (IV) $\|v_\varepsilon - x_0\| < \lambda$.

Moreover, if the norm of X *is* β - *differentiable on* X \ {0}*and* p > 1, *then*

(V) $$0 \in \partial_\beta \varphi(v_\varepsilon) + p(\frac{\varepsilon}{\lambda}S^*),\ S^* := \{\xi \in X^* : \|\xi\| \leq 1\}.$$

Δ_p is called *perturbing function*.

2.4 Borwein-Preiss principle (**complete metric space**). *Let* (X, d) *be complete metric space*, φ: X → (– ∞, + ∞] *bounded from below l. s. c. proper*. *For any* $\varepsilon > 0$, $\lambda > 0$, $p \geq 1$ *and* x_0 *of* X *with*

(I) $$\varphi(x_0) < \inf \varphi(X) + \varepsilon,$$

there exists Δ_p *in* Σ_p *and* v_ε *in* X *so that*

(II) $$\varphi(v_\varepsilon) + \frac{\varepsilon}{\lambda^p}\Delta_p(v_\varepsilon) \leq \varphi(x) + \frac{\varepsilon}{\lambda^p}\Delta_p(x) \ \forall x \in X,$$

(III) $\varphi(v_\varepsilon) < \inf \varphi(x) + \varepsilon$, (IV) $d(v_\varepsilon, x_0) < \lambda$.

Explanation.

$\Sigma_p := \{\Delta_p : X \to \mathbf{R} : \Delta_p(x) = \sum_{n=0}^{\infty} \mu_n d^p(x, v_n),\ \mu_n \geq 0,\ \sum_{n=0}^{\infty} \mu_n = 1,\ (v_n)_{n \geq 0}$ convergent sequence$\}$.

Ekeland principle cannot be recovered in all its force from the Borwein-Preiss principle. The following theorem, due to Li Yongxin and Shi Shuzhong [16]), generalizes and unifies these two principles.

3. Generalization and unification of the two principles

Definition. Let (*X*, d) be a metric space. $\rho: X \times X \to [0, +\infty]$ is called *gauge function* on *X* if

(1) for any *x* from *X*, $\rho(x, x) = 0$;

(2) for any sequence $(x_n, y_n)_{n\geq 1}$ from $X \times X$, $\rho(x_n, y_n) \to 0 \Rightarrow d(x_n, y_n) \to 0$;

(3) for any *y* from *X*, $x \to \rho(x, y)$ is lower semicontinuous.

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that for every *S* in β we have $\lim_{\substack{t\to 0\\ h\in S}} u \frac{f(a+th) - f(a)}{t} = \varphi(h)$ (uniform limit on *S* for $t \to 0$). φ is the β-*derivative of* f *in* a, and it is denoted $\nabla_\beta f(a)$.

For instance, $f \circ d$ with $f: \mathbf{R}_+ \to \mathbf{R}_+$ continuous strictly increasing and $f(0) = 0$ is a gauge function on X (attention to the inverse function of f).

3.1 Li Yongxin - Shi Shuzhong theorem. *Let* (X, d) *be a complete metric space and* φ: X $\to$ $(-\infty, +\infty]$ *bounded from below, lower semicontinuous and proper. For any* $\varepsilon > 0$ *and* x_0 *from X with*

$$\varphi(x_0) \le \inf \varphi(X) + \varepsilon, \tag{4}$$

for any gauge function $\rho: X \times X \to [0, +\infty]$ *and for any sequence* $(\delta_n)_{n\ge 0}$, $\delta_0 > 0$ *and* $\delta_n \ge 0$ $\forall n \ge 1$, *there exists a convergent sequence* $(x_n)_{n\ge 1}$ *from* X, $\lim_{n\to\infty} x_n = v_\varepsilon$, *such that*

$$\rho(v_\varepsilon, x_n) \le \frac{\varepsilon}{2^n \delta_0} \quad \forall n \ge 0; \tag{5}$$

If $\delta_n > 0$ *for an infinity of indices, then*

$$\varphi(v_\varepsilon) \le \varphi(x_0) - \sum_{n=0}^{\infty} \delta_n \rho(v_\varepsilon, x_n) \tag{6}$$

and

$$\varphi(v_\varepsilon) + \sum_{n=0}^{\infty} \delta_n \rho(v_\varepsilon, x_n) < \varphi(x) + \sum_{n=0}^{\infty} \delta_n \rho(x, x_n) \ \forall x \in X \setminus \{v_\varepsilon\}; \tag{7}$$

If $\delta_N > 0$, *where* $N \ge 0$, *and* $\delta_k = 0$ *for* $k > N$, *there exists* $m > N$ *so that* (6) *is preserved and* (7) *is replaced by*

$$\varphi(v_\varepsilon) + \sum_{k=0}^{N-1} \delta_k \rho(v_\varepsilon, v_k) + \delta_N \rho(v_\varepsilon, x_m) < \varphi(x) + \sum_{k=0}^{N-1} \rho_k \rho(x, x_k) + \delta_N \rho(x, x_m), \tag{8}$$

$\forall x \in X \setminus \{v_\varepsilon\}$.

Specification. If $N = 0$ the terms with $\sum_{k=0}^{N-1}$ disappear.

The proof of this result, which is simple, ingenious and elementary can be found in [2], page 446. Some embarrassments from [16], which are in the demonstration of the case I, have been eliminated by the author.

Ekeland principle under the form 2.2 *is a particular case of the Li-Shi theorem.*

Indeed, place us in the conditions and notations from 2.1 and apply 3.1 with $x_0 = u$, $\rho(x, y) = \frac{\varepsilon}{\lambda} d(x, y)$ and $(\delta_n)_{n\ge 0}$ with $\delta_0 = 1$, $\delta_n = 0$ for $n \ge 1$, hence $N = 0$. (5) from 3.1 gives, for $n = 0$, $\frac{\varepsilon}{\lambda} d(v_\varepsilon, u) \le \varepsilon$, that is (3) from 2.1. (6) from 3.1 gives (2_0) from 2.2 and (8) from 3.1 becomes

$$x \in X \setminus \{v_\varepsilon\} \Rightarrow \varphi(v_\varepsilon) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, x_m) < \varphi(x) + \frac{\varepsilon}{\lambda} d(x, x_m) \ ,$$

hence $\varphi(v_\varepsilon) < \varphi(x) + \frac{\varepsilon}{\lambda}[d(x, x_m) - d(v_\varepsilon, x_m)]$ and consequently (1) from 2.1.

Borwein-Preiss principle 2.4 *is a particular case of the Li-Shi theorem.*

Place us in the conditions and the notations from 2.4 and apply 3.1 with $\varphi(x_0) = \inf \varphi(X) + \varepsilon'$, $\rho(x, y) = \frac{\varepsilon}{\lambda^p} d^p(x, y)$ and $(\delta_n)_{n\geq 0} = (\mu_n)_{n\geq 0}$, where $\mu_n > 0$ $\forall n$, $\sum_{n=0}^{\infty} \mu_n = 1$. There is $(v_n)_{n\geq 0}$ a convergent sequence, $\lim_{n\to\infty} v_n = v_{\varepsilon'} (=: v_\varepsilon)$, so that $\frac{\varepsilon}{\lambda^p} d^p(v_\varepsilon, x_0) \overset{(5)}{\leq} \varepsilon' < \varepsilon$, hence $d(v_\varepsilon, x_0) < \lambda$, i.e. (IV) of 2.4,

$$\varphi(v_\varepsilon) + \frac{\varepsilon}{\lambda^p} \sum_{n=0}^{\infty} \mu_n d^p(v_\varepsilon, v_n) \overset{(6)}{\leq} \varphi(x_0) < \inf \varphi(X) + \varepsilon$$

and hence (III) from 2.4 (improved!).

$$x \in X \setminus \{v_\varepsilon\} \Rightarrow \varphi(v_\varepsilon) + \frac{\varepsilon}{\lambda^p} \sum_{n=0}^{\infty} \mu_n d^p(v_\varepsilon, v_n) < \varphi(x) + \frac{\varepsilon}{\lambda^p} \sum_{n=0}^{\infty} \mu_n d(x, v_n),$$

that is (II) from 2.4 (improved!). The last inequality is justified by

$$\varphi(x_n) + \sum_{i=0}^{n-1} \delta_i \rho(x_n, x_i) < \inf_{x\in T(x_{n-1})} \left[\varphi(x) + \sum_{i=0}^{n-1} \delta_i \rho(x, x_i) \right] + \delta_n \frac{\varepsilon}{2^n \delta_0}$$

for which one can regrde, for instance in [2], page 447, (17).

Remark. As it was observed above, Li-Shi theorem gives an improved statement of Borwein-Preiss principle.

Pass to a variant of the theorem 3.1 for Banach spaces.

3.2 *Let* X *be Banach space and* φ: X $\to (-\infty, +\infty]$ *lower bounded l.s.c. proper*. *For any* $\varepsilon > 0$ *and* x_0 *from* X *so that*

$$\varphi(x_0) \leq \inf \varphi(X) + \varepsilon, \tag{9}$$

for any sequence $(\delta_n)_{n\geq 0}$ *with* $\delta_0 > 0$, $\delta_n \geq 0$ $\forall n \geq 1$ *and for any* $r : X \to [0, +\infty]$ *l.s.c. with the properties* $r(0) = 0$, $r(x_n) \to 0 \Rightarrow \|x_n\| \to 0$, *there exists* $(x_n)_{n\geq 0}$ *convergent sequence in* X, $\lim_{n\to\infty} x_n = v_\varepsilon$, *so that*

$$r(v_\varepsilon - x_n) \leq \frac{\varepsilon}{2^n \delta_0} \quad \forall n \geq 0; \tag{10}$$

When $\delta_n > 0$ *for an infinity of indices, we have*

$$\varphi(v_\varepsilon) \leq \varphi(x_0) - \sum_{n=0}^{\infty} \delta_n r(v_\varepsilon - x_n) \tag{11}$$

and

$$\varphi(v_\varepsilon) + \sum_{n=0}^{\infty} \delta_n r(v_\varepsilon - x_n) < \varphi(x) + \sum_{n=0}^{\infty} \delta_n r(x - x_n) \quad \forall x \in X \setminus \{v_\varepsilon\}; \tag{12}$$

When $\delta_N > 0$, *where* $N \geq 0$, *and* $\delta_k = 0$ *for* $k > N$, *there exists* $m > N$ *so that* (11) *is preserved and* (12) *is replaced by*

$$\varphi(v_\varepsilon) + \sum_{k=0}^{N-1} \delta_k r(v_\varepsilon - x_k) + \delta_N r(v_\varepsilon - x_m) < \varphi(x) + \sum_{k=0}^{N-1} \delta_k r(x - x_k) + \delta_N r(x - x_m) \quad (13)$$

$\forall\, x \in X \setminus \{v_\varepsilon\}$ ([16]).

Specification. If $N = 0$, the terms $\sum_{k=0}^{N-1}$ disappear.

And now pass to a generalization, *via* 3.2, of the Borwein-Preiss principle for Banach spaces 2.3 in the following variant. But firstly, as in 2.3 (β is a bornology on X),

3.3 *Let* X *be real normed space*, r *a real Lipschitz function* β*-differentiable on* $S_\alpha : = \{x \in X: \|x\| < \alpha\}$, v_ε *from* X, $(x_n)_{n\geq 0}$ *a sequence of points from* X *with*

$$\| v_\varepsilon - x_n \| < \frac{\alpha}{2} \ \forall n \geq 0, \quad (14)$$

$(\delta_n)_{n\geq 0}$ *with* $\delta_n > 0$ $\forall n$ *and* $\sum_{n=0}^{\infty} \delta_n$ *convergent and*

$$\Phi : \Phi(x) = \sum_{n=0}^{\infty} \delta_n r(x - x_n). \quad (15)$$

Then Φ *is* β *- differentiable in* v_ε *and*

$$\nabla_\beta \Phi(v_\varepsilon) = \sum_{n=0}^{\infty} \delta_n \nabla_\beta r(v_\varepsilon - x_n). \quad (16)$$

■ Use the rule on β - differentiability of function series (see, for instance [2], page 110, [13]) with $U = v_\varepsilon + S_{\frac{\alpha}{2}}$. $x_n \overset{(14)}{\in} U$ $\forall n \geq 0$, hence $x \in U \Rightarrow \|x - x_n\| < \alpha$ $\forall n \geq 0$, which imposes, r being Lipschitz on S_α, that the functions $x \to r(x - x_n)$, $\forall n \geq 0$ are uniformly bounded on U and consequently the series from (15) is convergent on U.

Show that the series $\sum_{n=0}^{\infty} \delta_n \xi_n(x)$, $\xi_n(x) : = \nabla_\beta\, r(x - x_n)$ converges uniformly on U. Let S be any from β.

$$\xi_n(x)(h) = \lim_{\substack{t\to 0 \\ h\in S}} u \frac{r(x - x_n + th) - r(x - x_n)}{t} .$$

But for t sufficiently small and $x \in U$ we have $x - x_n + th \in S_\alpha$ $\forall n \geq 0$ (S is bounded), hence

$$\left| \frac{r(x - x_n + th) - r(x - x_n)}{t} \right| \leq L \| h \| \quad \forall\, n \geq 0 ,$$

L the Lipschitz constant, consequently

$$|\xi_n(x)(h)| \le L\|h\| \quad \forall x \in U, \forall h \in S, \forall n \ge 0 \ .$$

As $X = \bigcup_{S \in \beta} S$, it results

$$|\xi_n(x)(h)| \le L\|h\| \quad \forall h \in X, \forall x \in U, \forall n \ge 0 \ ,$$

that imposes

$$\|\nabla_\beta r(x - x_n)\| \le L \quad \forall x \in U, \forall n \ge 0 . \tag{17}$$

Applying the Weierstrass rule we find that the demand (2) from [2], I, 3.12[4] is also verified for the series of (15), it can β-differentiate on U term by term. ■

REMARK. In [16] the demonstration for the proposition 1, 3.3 here, must be repelled as totally false. This is the reason why we given here the above proof, since it is original.

And now

3.4 *Let* X *be Banach space,* $\varphi : X \to (-\infty, +\infty]$ *lower bounded l.s.c. proper,* $\alpha, \lambda, k > 0$ *and* $r : X \to [0, +\infty]$ *l. s. c. with the properties*

$$r(0) = 0, \tag{18}$$

$$r(x_n) \to 0 \Rightarrow \|x_n\| \to 0, \tag{19}$$

r *is Lipschitz with the constant* L *and* β *- differentiable on*

$$S_{r,\lambda} := \{x \in X : r(x) < \lambda\}, \tag{20}$$

$$\|x\| < \alpha \Rightarrow x \in S_{r,\lambda},\ r(x) \le \frac{\lambda}{k} \Rightarrow \|x\| < \frac{\alpha}{2} . \tag{21}$$

Then, for any $\varepsilon > 0$ *and* x_0 *from* X *with*

$$\varphi(x_0) \le \inf \varphi(X) + \varepsilon, \tag{22}$$

there exists v_ε *in* X *so that*

$$(23)\ \ \|v_\varepsilon - x_0\| \le \frac{\alpha}{2},\ \ (24)\ \ \varphi(v_\varepsilon) \le \varphi(x_0),\ \ (25)\ \ 0 \in \partial_\beta \varphi(v_\varepsilon) + \frac{2\varepsilon k L}{\lambda} S^*,$$

$S^* := \{\xi \in X^* : \|\xi\| \le 1\}$ ([16]).

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[4] β-*differentiability of function series. Let be* $\sum_{n=1}^{\infty} f_n$, $f_n : U \to \mathbf{R}$, U *an open convex bounded set in a real normed space. If* 1° f_n *is,* $\forall n \in \mathbf{N}$, β-*differentiable on* U; 2° $\sum_{n=1}^{\infty} \nabla_\beta f_n$ *is uniformly convergent on* U; 3° *there is* x_0 *in* U *such that* $\sum_{n=1}^{\infty} f_n(x_0)$ *is convergent, then the given series is uniformly convergent on* U, *its sum is* β-*differentiable on* U *and*

$$\nabla_\beta\Big(\sum_{n=1}^{\infty} f_n\Big)(x) = \sum_{n=1}^{\infty} \nabla_\beta f_n(x) \ \ \forall x \in U \text{ (we can see lso in [13]).}$$

We discuss now about the connexion between the conditions

$$(P) : = (18) + (19) + (20) + (\|x\| < \alpha \Rightarrow x \in S_{r,\lambda})$$

(see 3.4) and

(H) *There exists* $\Phi : X \to [0, 1]$ *Lipschitz* β-*differentiable having the properties*

$$(26) \quad \Phi(0) = 1, \qquad (27) \quad \|x\| > 1 \Rightarrow \Phi(x) = 0.$$

■ (H) $\Rightarrow$ (18) + (19) + (20). Take $r : X \to [0, +\infty)$,

$$r(x) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \left[1 - \Phi(2^n x) \right] .$$

The definition is correct, each term is majorized in absolute value by $\frac{1}{2^{2n}}$. r is Lipschitz on X with the constant L the same as for Φ:

$$| r(x) - r(y) | \le \sum_{n=1}^{\infty} \frac{1}{2^{2n}} | \Phi(2^n x) - \Phi(2^n y) | \le \sum_{n=1}^{\infty} \frac{1}{2^n} L \| x - y \| \text{etc.}$$

r is β-differentiable on X: apply I, 3.12 from [2] (see this one in the above note reference 4), the series of β-derivatives is $-\sum_{n=1}^{\infty} \frac{1}{2^n} \nabla_\beta \Phi(2^n x)$ (see in the following the calculus of the β-derivatives) and the calculus continues as for (16), Φ being Lipschitz and β - differentiable on X. So (20) is *con brio* ensured. Obviously $r(x) \ge 0$ and $r(0) \overset{(26)}{=} 0$. Moreover,

$$\| x \| > \frac{1}{2^n} \Rightarrow r(x) \ge \frac{1}{2^{2n}} [1 - \Phi(2^n x)] \overset{(27)}{=} \frac{1}{2^{2n}}$$

and this implication justifies (*ad absurdum*!) (19).

Complete the demonstration. For the function Ψ: $\Psi(x) = \Phi(2^n x)$ we have

$$\nabla_\beta \Psi(x) = 2^n \nabla_\beta \Phi(2^n x) \ .$$

Indeed, setting $\xi : = \nabla_\beta \Phi(2^n x)$ we have

$$\frac{\Psi(x + th) - \Psi(x)}{t} - 2^n \xi(h) = 2^n \left[\frac{\Phi(2^n x + 2^n th) - \Phi(2^n x)}{2^n t} - \xi(h) \right]$$

and the uniform limit on S of the bracket for $\tau : = 2^n t \to 0$ is equal to 0.

(P) $\Rightarrow$ (H). Let α' be from $(0, \alpha)$. $\exists\ \delta > 0$ so that

$$\|x\| > \alpha' \Rightarrow r(x) > \delta$$

(*ad absurdum,* use (19)). Take $f : \mathbf{R}_+ \to [0,1]$ of the C^1 class, $t > \delta \Rightarrow f(t) = 0$, $f(0) = 1$. The function Φ: $X \to [0,1]$, $\Phi(x) = f(r(\alpha' x))$ verifies (H). Indeed, the conditions (26) and (27) are obviously verified. Φ is Lipschitz on X : consider the cases $\|x\| \le 1$ and $\|y\| \le 1$, $\|x\| > 1$ and $\|y\| > 1$, $\|x\| \le 1$ and $\|y\| > 1$ which respectively give the cases $\|\alpha' x\| \le \alpha'$ and $\|\alpha' y\| \le \alpha'$, $\|\alpha' x\| > \alpha'$ and $\|\alpha' y\| > \alpha'$, $\|\alpha' x\| \le \alpha'$ and $\|\alpha' y\| > \alpha'$; apply the elementary finite increment formula, f' is bounded. Φ is β-differentiable on X: use the rule for the β-derivative of the composed function ([2], I, §3, 3.11[5]). ■

4. Applications

4.1 Applications to the β-subderivative, β-derivative and discussions

4.1 *Let* X *be Banach space,* β *bornology on* X *and* φ: X → (–∞, +∞] *lower bounded l.s.c. proper. If* X *has the property* (H), *then* φ *is* β-*subdifferentiable on a set* A *so that*

$$\{(x, f(x)) : x \in A\}$$

is dense in the graph of φ *in* X × **R** ([16]).

Compare this result with the following one. But firstly introduce the next:

Definition. Let X be a real normed space, φ: $X \to (-\infty, +\infty]$ proper, ε > 0 and v from X. φ is ε - *supported in* v if there exists ξ in X^* and δ > 0 so that

$$\|w - v\| \le \delta \Rightarrow \varphi(v) \le \varphi(w) + \varepsilon\|v - w\| + \xi(v - w). \tag{1}$$

We see that, for ξ = 0 and δ whatever positive, (1) becomes the fundamental inequality (1) of the Ekeland principle 2.1.

And now, other variant of 2.1 –

4.2 *Let* X *be a real Banach space and* φ: X → (–∞, +∞] *lower semicontinuous and proper. Assume there is* F: X → **R** *with Fréchet derivative,* F(0) > 0 *and* F(x) ≤ 0 *on* $\{x \in X : \|x\| \ge 1\}$. *Then, whatever* ε > 0, *the set of points where* φ *is* ε-*supported is dense in* dom φ (*I. Ekeland - G. Lebourg*).

■ Let be $u \in$ dom φ and $r > 0$. It follows to find in $S := S(u, r)$ (open ball) a point $v \in$ dom φ in which φ is ε - supported. φ being l.s.c. in u, r can be chosen small enough so that φ is bounded from below on S. Consider the function Φ: $X \to (0, +\infty]$,

$$\Phi(x) = \frac{1}{\max\left(0, F\left(\frac{x-u}{r}\right)\right)}. \tag{2}$$

Φ is l.s.c.: let x_0 be of X and $a < \dfrac{1}{\max\left(0, F\left(\frac{x_0-u}{r}\right)\right)}$; when $a \le 0$, the inequality is valid on X; when $a > 0$, then $\dfrac{1}{a} > \max\left(0, F\left(\frac{x_0-u}{r}\right)\right)$ and the inequality

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[5] β-*derivative of the composed function. Let be* f : X → **R** β-*differentiable in* x_0 *and* F : **R** → **R** *of* C^1 *class on* J *open interval with* $f(x_0) \in J$. *Then* g : = F o f *is* β-*differentiable in* x_0 *and*

$$\nabla_\beta\, g(x_0) = F'(f(x_0))\, \nabla_\beta f(x_0) \text{ (see also in [13]).}$$

remains valid on a neighborhood of x_0, as F is continuous. We have (3) dom $\Phi \subset S$, since $\Phi(x) = +\infty$ on $X \setminus S$. Obviously (4) $u \in$ dom Φ. Moreover, dom Φ is open, as $x \in$ dom $\Phi \Rightarrow F\left(\frac{x-u}{r}\right) > 0$ and F is continuous, Φ is Fréchet differentiable on dom Φ.

Therefore, the function $\Phi + \varphi$: $X \to (-\infty, +\infty]$ is bounded from below ($\Phi(x) + \varphi(x) = +\infty$ on $X \setminus S$, $\Phi + \varphi$ is lower bounded on S), l.s.c. (Φ is l.s.c.) and proper ($u \overset{(4)}{\in}$ dom $\Phi \cap$ dom φ).

According to 2.1, $\exists\, v \in X$ so that

$$\Phi(v) + \varphi(v) \le \Phi(w) + \varphi(w) + \frac{\varepsilon}{2}\|v - w\| \;\forall w \in X. \tag{5}$$

Taking $w = u$ in (5) we find $v \in$ dom $\Phi \cap$ dom φ (*ad absurdum*, use (4)). So, Φ being Fréchet differentiable in v, for $\xi := -\Phi'(v)$ and for $\delta > 0$ we have

$$\|w - v\| \le \delta \Rightarrow \Phi(w) - \xi(v - w) - \frac{\varepsilon}{2}\|v - w\| \le \Phi(v). \tag{6}$$

By adding (5) to (6) it results (1). ■

Remark 1. The demand "φ bounded from below" does not exist in the statement 4.2.

Remark 2. The Hilbert spaces have got the functions F like in 4.2 ($F(x) = 1 - x \cdot x$), also the spaces $\mathbf{L}^p$, $p \in (1, +\infty)$ ($F(x) = 1 - \|x\|^p$). $\mathbf{L}^1$ and $\mathbf{L}^\infty$ have not this property ([18]).

4.3 *Corollary. Let* X *be Banach space*, β *bornology on* X *and* φ: X $\to$ **R** *continuous convex upper bounded.*

If X *has the property* (H), *then* φ *is* β-*differentiable on a subset everywhere dense.*

■ As dom $(-\varphi) = X$, $-\varphi$ is β-subdifferentiable on a set A dense in X (4.1), i.e. φ is β-superdifferentiable on A. But φ is β-subdifferentiable in X (apply [2], I, 5.9) and by applying: "*Let* $f: X \to \overline{\mathbf{R}}$ *be locally finite at* x_0. f *is* β-*differentiable at* $x_0 \Leftrightarrow$ f *is* β-*superdifferentiable and* β-*subdifferentiable at* x_0 . *In this case* $\{\nabla_\beta f(x_0)\} = \partial^\beta f(x_0) = \partial_\beta f(x_0)$", we find φ β-differentiable on A. ■

To clarify, by definition, f is β-*superdifferentiable at* x_0 if $\partial^\beta f(x_0) \ne \emptyset$. f is β-*superdifferentiable* on A iff it is β-*superdifferentiable* at any point in A.

By usage of this generalization of Borwein-Preiss variational principle, we can give generalization for some results regarding viscosity β-solutions of the differential equation of *Hamilton - Jacobi type*: $u + F(u') = f$ as in [19] and [20]. We can weaken the conditions from the statements concerning the existence and characterization of the viscosity β-subsolution, viscosity β-supersolution and viscosity β-solution with their proofs based on Borwein-Preiss variational principle for: $H(x, u(x), \nabla_\beta u(x)) = 0 \;\forall x \in \Omega$.

To follow these proofs (for which here is proposed the generalization), we can search in [2], pages 422-426.

4.2 Applications in solution of mathematical physics problems evolved from modeling of real phenomena

We can generalize the results from [5] and [7] concerning the sequence of statements in obtaining weak solutions, weak subsolutions and weak supersolutions for some elliptic type problems. As applications in real phenomena, we can cite generalization of problems from [5]: applications in glaciology, nonlinear elastic membrane, using the *p*-Laplacian and pseudo torsion problem, nonlinear elastic membrane with *p*-pseudo-Laplacian. We can propose, using the discussed variational principles, generalizations for results for critical points for nondifferentiable functionals ([5], [7]) in order to give characterizations of the solutions for the Dirichlet problems from issued by the movement of the glacier, nonlinear elastic membrane, pseudo torsion problem or nonlinear elastic membrane with *p*-pseudo-Laplacian. As in [5] and [6], we can give a generalization of the series of propositions from there *via* 3.1 instead of Ekeland principle in order to arrive to a solution for the mixed problem involving the *p*-Laplacian which models the injection mould filling.

5. Conclusions

In this paper, some weakening of conditions for existence and characterization results for the viscosity β-solution, viscosity β-subsolution, viscosity β-supersolution and for some Dirichlet problems involving the *p*-Laplacian and the *p*-pseudo-Laplacian have been proposed.

For this reason, two variants each for the Ekeland and Borwein-Preiss variational principles have been presented together with a generalization of them given in [16]. The author recovered a proposition from [16] which is important to generalize and unify the two variational principles.

Via the generalizations of these two fundamental variational principles, we propose to replace Ekeland principle in corresponding three series of results due to the author from [5], [6] and [7] aiming to solve and / or chareaterize weak solutions for some mathematical physics problems involving the *p*-Laplacian and the *p*-pseudo-Laplacian issued from modelling of real phenomena. We highlight solution and characterizations of solutions for the movement of the glacier, nonlinear elastic membrane, pseudo torsion problem or nonlinear elastic membrane with *p*-pseudo-Laplacian or injection mould filling.

Acknowledgements

This research was partially supported by the project PN-III-P1-1.2-PCCDI-2017-0404/ 31PCCDI/2018, **HORESEC**.

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