FRACTIONAL ORDER OPTIMAL CONTROL PROBLEMS VIA THE OPERATIONAL MATRICES OF BERNSTEIN POLYNOMIALS

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In this paper a numerical method for solving a class of fractional optimal control problems is presented which is based on Bernstein polynomials approximation. Operational matrices of integration, differentiation, dual and product are introduced and are utilized to reduce the problem of solving a system of algebraic equations. The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

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1. Introduction

Fractional order dynamics appear in several problems in science and engineering such as viscoelasticity [2], dynamics of interfaces between nanoparticles and substrates [5], etc. It has also been shown that the materials with memory and hereditary effects and dynamical processes including gas diffusion and heat conduction in fractal porous media can be modeled by fractional order models better than integer models [21].
Bernstein polynomials (B-polynomials) have many useful properties. They play a prominent role in various areas of mathematics and have frequently been used in the solution of integral equations, differential equations and approximation theory; see e.g., [6, 13, 20]. In recent years various operational matrices for the polynomials have been developed to cover the numerical solution of differential, integral and integro-differential equations [10, 11, 14, 15]. The main advantage of the new method is that with the use of only few number of Bernstein basis we achieve satisfactory results.
In this paper, we focus on optimal control problems with the quadratic performance index and the dynamic system with the Caputo fractional derivative. We solve the problem directly without using Hamiltonian formulas. Our tools for this aim are the

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Bernstein basis and the operational matrix of fractional integration. The problem formulation is as follows:

$$J = \frac{1}{2}\int_0^1 [q(t)x^2(t) + r(t)u^2(t)]dt, \tag{1}$$

$${}_0^cD_t^\alpha x(t) = a(t)x(t) + b(t)u(t), \tag{2}$$

$$x(t_0) = x_0,$$

where $q(t) \geq 0, r(t) > 0, b(t) \neq 0$, and the fractional derivative is defined in the Caputo sense. We intend to extend the application of polynomials to solve fractional differential equations. Our main aim is to generalize Bernstein operational matrix to fractional calculus. Also illustrative examples are included to demonstrate the applicability of the new approach. In [1], the problem is solved by a discrete iterative method.

We refer the interested reader to [3, 8, 12, 18] for more studies on this subject. The paper is structured as follows: In Section 2, we present some preliminaries on fractional calculus. Section 3 describes the basic formulation of B-polynomials required for our subsequent development and Section 4 is devoted to the function approximation by using B-polynomials basis. In Sections 5–6, we explain the general procedure of forming of operational matrices of integration and product, respectively. Section 7 is devoted to the formulation of the fractional optimal control problems. In Section 8, we report our numerical finding and demonstrate the validity, accuracy and applicability of the operational matrices by considering numerical examples.

2. **Some preliminaries on fractional calculus**

In this section, we give some basic definitions and properties of the fractional calculus [16, 17] which are used further in this paper.

Definition 2.1. *A real function $f(t)$, $t > 0$ is said to be in the space $C_\alpha, \alpha \in \Re$ if there exists a real number $p\,(> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.*

Definition 2.2. *A function $f(x)$, $x > 0$ is said to be in the space $C_\alpha^m, m \in N \bigcup \{0\}$, if $f^{(m)} \in C_\alpha$.*

Definition 2.3. *The (left sided) Riemann - Liouville fractional integral of order $\alpha > 0$, of a function $f \in C_\alpha, \alpha \geq -1$ is defined as:*

$${}_0I_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, & \alpha > 0,\ t > 0, \\ \\ f(t), & \alpha = 0. \end{cases} \tag{3}$$

As a property for the left Riemann - Liouville fractional integration, we have

$$ {}_0I_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \qquad \alpha \in I\!N \cup \{0\}, \ t > 0. \tag{4} $$

Definition 2.4. *The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in I\!N \bigcup \{0\}$, is defined as:*

$$ {}_0^C D_t^{\alpha} f(t) = \begin{cases} \left[I^{m-\alpha} f^{(m)}(t)\right], & m-1 < \alpha < m, \quad m \in I\!N, \\ \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{5} $$

Note that

$$ (i) \qquad {}_0I^{\alpha}\, {}_0^C D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{t^k}{k!}, \qquad m-1 < \alpha \le m, \quad m \in I\!N. $$

$$ (ii) \qquad {}_0^C D^{\beta}\, {}_0I^{\alpha} f(t) = \begin{cases} I^{\alpha-\beta} f(t), & \text{if } \alpha > \beta, \\ f(t), & \text{if } \alpha = \beta, \\ D^{\beta-\alpha} f(t), & \text{if } \alpha < \beta. \end{cases} $$

3. Bernstein polynomials and their properties

The Bernstein polynomials of the mth degree on the interval $[0,1]$ are defined as [4]:

$$ B_{i,m} = \binom{m}{i} x^i (1-x)^{m-i}, \ 0 \le i \le m. \tag{6} $$

Bernstein polynomials defined above form a complete basis over the interval $[0,1]$. There are $m+1$ mth-degree polynomials. For convenience, we set $B_{i,m}(x) = 0$, if $i < 0$ or $i > m$. A recursive definition can also be used to generate the Bernstein polynomials over $[0,1]$ so that the ith mth-degree Bernstein polynomials can be written

$$ B_{i,m}(x) = -xB_{i,m-1}(x) + xB_{i-1,m-1}(x). \tag{7} $$

It can easily be shown that each of the Bernstein polynomials is positive and also the sum of all the Bernstein polynomials is unity for all real $x \in [0,1]$, i.e., $\sum_{i=0}^{m} B_{i,m}(x) = 1$. It is easy to show that any given polynomial of degree m can be expanded in terms of linear combination of the basis functions.
By using the binomial expansion of $(1-x)^{n-i}$, one can show that

$$ \begin{aligned} B_{i,m} &= \binom{m}{i} x^i (1-x)^{m-i} = \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} x^k \right) \\ &= \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^{i+k}, \qquad i = 0, 1, \dots, m. \end{aligned} \tag{8} $$

Now, if we define vector A_{i+1}, as

$$A_{i+1} = \left[0, 0, \dots, 0, (-1)^0 \binom{m}{i}, (-1)^1 \binom{m}{i}\binom{m-i}{1}, \dots, (-1)^{m-i}\binom{m}{i}\binom{m-i}{m-i}\right],$$

then $B_{i,m}(x) = A_{i+1}T_m(x)$, for $i = 0, 1, \dots, m$, where

$$T_m(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^m \end{bmatrix}.$$

Now if we define $(m+1) \times (m+1)$ matrix A such that

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m+1} \end{bmatrix},$$

then

$$\phi(x) = AT_m(x), \qquad (9)$$

where

$$\phi(x) = \begin{bmatrix} B_{0,m}(x) \\ B_{1,m}(x) \\ \vdots \\ B_{m,m}(x) \end{bmatrix},$$

and matrix A is an upper triangular matrix given by :

$$A = \begin{bmatrix} (-1)^0\binom{m}{0} & (-1)^1\binom{m}{0}\binom{m-0}{1-0} & \dots & (-1)^{m-0}\binom{m}{0}\binom{m-0}{m-0} \\ \ddots & & & \vdots \\ 0 & (-1)^0\binom{m}{i} & \dots & (-1)^{m-i}\binom{m}{i}\binom{m-i}{m-i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & (-1)^m\binom{m}{m} \end{bmatrix}$$

and $|Aj| = \Pi_{i=0}^{m}\binom{m}{i}$, so A is an invertible matrix.

4. **Approximation of function**

We recall here some theorems and lemma that are stated and proved in [9].

Theorem 4.1. *Let X be an inner product space and $M \neq \phi$ a convex subset which is complete (in the metric induced by the inner product). Then for every given $x \in X$ there exists a unique $y \in M$ such that*

$$\delta = inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|. \qquad (10)$$

Lemma 4.1. *In Theorem 4.1, let M be a complete subspace Y and $x \in X$ fixed. Then $z = x - y$ is orthogonal to Y.*

Theorem 4.1 and Lemma 4.1 together imply the following theorem.

Theorem 4.2. *For every given x in a Hilbert space H and every given closed subspace Y of H there is a unique best approximation to x out of Y.*
The Gram determinant of $y_1, y_2, \ldots, y_n$ is defined by

$$G(y_1, y_2, \ldots, y_n) = \begin{vmatrix} \langle y_1, y_1\rangle & \langle y_1, y_2\rangle & \ldots & \langle y_1, y_n\rangle \\ \langle y_2, y_1\rangle & \langle y_2, y_2\rangle & \ldots & \langle y_2, y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, y_1\rangle & \langle y_n, y_2\rangle & \ldots & \langle y_n, y_n\rangle \end{vmatrix},$$

where $\langle , \rangle$ denotes the inner product. We also note a useful criterion involving G.

Theorem 4.3. *Elements $y_1, y_2, \ldots, y_n$ of a Hilbert space H constitute a linearly independent set in H if and only if*

$$G(y_1, y_2, \ldots, y_n) \neq 0$$

It is interesting that the distance $\|x - y_0\|$ between x and the best approximation y_0 to x (error of approximation) can also be expressed by Gram determinants.

Theorem 4.4. *Suppose that H be a Hilbert space and Y be a closed subspace of H such that $dimY < \infty$ and $y_1, y_2, \ldots, y_n$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then*

$$\|x - y_0\|^2 = \frac{G(x, y_1, y_2, \ldots, y_n)}{G(y_1, y_2, \ldots, y_n)}, \tag{11}$$

where

$$G(x, y_1, y_2, \ldots, y_n) = \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \ldots & \langle x, y_n\rangle \\ \langle y_1, x\rangle & \langle y_1, y_2\rangle & \ldots & \langle y_1, y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, x\rangle & \langle y_n, y_1\rangle & \ldots & \langle y_n, y_n\rangle \end{vmatrix}.$$

Now, Suppose that $H = L^2[0,1]$ and $\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\} \subset H$ be the set of Bernstein polynomials of mth-degree and

$$Y = Span\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\}, \tag{12}$$

and f be an arbitrary element in H. Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $y_0 \in Y$, that is

$$\exists y_0 \in Y; \quad \forall y \in Y \; \|f - y_0\|_2 \leq \|f - y\|_2, \tag{13}$$

where $\|f\|_2 = \sqrt{\langle f, f\rangle}$.
Since $y_0 \in Y$, there exist unique coefficients $c_0, c_1, \ldots, c_m$ such that

$$f(x) \simeq y_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T \phi, \tag{14}$$

where $\phi^T = [B_{0,m}, B_{1,m}, \ldots, B_{m,m}]$ and $c^T = [c_0, c_1, \ldots, c_m]$. Then c^T can be obtained by

$$c^T \langle \phi, \phi \rangle = \langle f, \phi \rangle, \tag{15}$$

where

$$\langle f, \phi \rangle = \int_0^1 f(x)\phi(x)^T dx = [\langle f, B_{0,m}\rangle, \langle f, B_{1,m}\rangle, \ldots, \langle f, B_{m,m}\rangle], \tag{16}$$

and $\langle \phi, \phi \rangle$ is a $(m+1) \times (m+1)$ matrix which is called the dual matrix of ϕ, denoted by Q, and will be introduced in the following. Therefore

$$Q = \langle \phi, \phi \rangle = \int_0^1 \phi(x)\phi(x)^T dx, \tag{17}$$

and then

$$c^T = \left(\int_0^1 f(x)\phi(x)^T dx \right) Q^{-1}. \tag{18}$$

The elements of the dual matrix, Q, are given explicitly by

$$\begin{aligned} Q_{i+1,j+1} &= \int_0^1 B_{i,m}(x) B_{j,m}(x) dx \\ &= \binom{n}{i}\binom{n}{j} \int_0^1 (1-x)^{2n-(i+j)} x^{i+j} dx \\ &= \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}} \qquad i,j = 0,1,\ldots,m. \end{aligned}$$

By (9), we have

$$\begin{aligned} Q &= \int_0^1 \phi(x)\phi(x)^T dx = \int_0^1 (AT_m(x))(AT_m(x))^T dx, \\ &= A \left[\int_0^1 T_m(x) T_m(x)^T dx \right] A^T = AHA^T, \end{aligned}$$

where H is a well-known Hilbert matrix,

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{m+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{m+1} & \frac{1}{m+2} & \frac{1}{m+3} & \cdots & \frac{1}{2m+1} \end{bmatrix}.$$

5. **The operational matrix of integration**

In this section we describe breifley about the Bernstein polynomials operational matrices of fractional integration of the vector ϕ. The operational matrices can be approximated as [19]

$${}_0I_x^{\alpha}\phi(x) \simeq I^{\alpha}\phi(x), \tag{19}$$

where I^α is the $(m+1)\times(m+1)$ Riemann-Liouville fractional operational matrix of integration. We construct I^α as follows:

$$ {}_0I_x^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1}\phi_m(\tau)d\tau = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * \phi(x), \tag{20} $$

where $*$ denotes the convolution product and from (9) and by using (4) we get

$$ \begin{aligned} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * \phi(x) &= \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * (AT_m(x)) = \frac{1}{\Gamma(\alpha)} A(x^{\alpha-1} * T_m(x)) \\ &= \frac{A}{\Gamma(\alpha)} \left[x^{\alpha-1} * 1, x^{\alpha-1} * x, \ldots, x^{\alpha-1} * x\right]^T = A\left[I^\alpha 1, I^\alpha x, \ldots, I^\alpha x\right]^T \\ &= A\left[\frac{0!}{\Gamma(\alpha+1)} x^\alpha, \frac{1!}{\Gamma(\alpha+2)} x^{\alpha+1}, \ldots, \frac{m!}{\Gamma(\alpha+m+1)} x^{\alpha+m}\right]^T \\ &= AD\overline{T}_m, \end{aligned} \tag{21} $$

where D is an $(m+1)\times(m+1)$ matrix given by

$$ D = \begin{bmatrix} \frac{0!}{\Gamma(\alpha+1)} & 0 & \ldots & 0 \\ 0 & \frac{1!}{\Gamma(\alpha+2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \frac{m!}{\Gamma(\alpha+m+1)} \end{bmatrix} \text{ or } D_{i,j} = \begin{cases} \frac{i!}{\Gamma(\alpha+i+1)}, & i=j, \\ 0, & i \neq j, \end{cases} $$

where $i,j = 0,1,\ldots,m$ and $\overline{T}_m$ is given by:

$$ \overline{T}_m = \begin{bmatrix} x^\alpha \\ x^{\alpha+1} \\ \vdots \\ x^{\alpha+m} \end{bmatrix}. $$

Now we approximate $x^{k+\alpha}$ by $m+1$ terms of the Bernstein basis

$$ x^{\alpha+i} \simeq E_i^T \phi_m(x) \tag{22} $$

Therefore we have

$$ \begin{aligned} E_i &= Q^{-1}\left(\int_0^1 x^{\alpha+i}\phi(x)dx\right) \\ &= Q^{-1}\left[\int_0^1 x^{\alpha+i}B_{0,m}(x)dx, \int_0^1 x^{\alpha+i}B_{1,m}(x)dx, \ldots, \int_0^1 x^{\alpha+i}B_{m,m}(x)dx\right]^T = Q^{-1}\overline{E}_i, \end{aligned} \tag{23} $$

where $\overline{E}_i = [\overline{E}_{i,0}, \overline{E}_{i,1}, \ldots, \overline{E}_{i,m}]$ and

$$ \overline{E}_{i,j} = \int_1^0 x^{\alpha+i}B_{i,j}(x)dx = \frac{m!\Gamma(i+j+\alpha+1)}{\Gamma(j!+m+\alpha+2)}, \; i,j = 0,1,\ldots,m, \tag{24} $$

where E is an $(m+1)\times(m+1)$ matrix that has vector $Q^{-1}\overline{E}_i$ for ith columns. Therefore, we can write

$$\begin{aligned} I^{\alpha}\phi(x) &= AD[E_0^T\phi(x), E_1^T\phi(x), \dots, E_m^T\phi(x)]^T \\ &= ADE^T\phi(x). \end{aligned} \tag{25}$$

Finally, we obtain

$${}_0I_x^{\alpha}\phi(x) \simeq I^{\alpha}\phi(x) \tag{26}$$

where

$$I^{\alpha} = ADE, \tag{27}$$

is called the Bernstein polynomials operational matrix of fractional integration.

6. The product operational matrix

It is always necessary to evaluate the product of $\phi(x)$ and $\phi(x)^T$, which is called the product matrix for the Bernstein polynomials basis. The operational matrices for the product $\hat{C}$ is given by

$$c^T\phi(x)\phi(x)^T \simeq \phi(x)^T\hat{C}, \tag{28}$$

where $\hat{C}$ is an $(m+1)\times(m+1)$ matrix. So from (9) we have

$$\begin{aligned} c^T\phi(x)\phi(x)^T &= c^T\phi(x)\left(T_m(x)^T A^T\right) = [c^T\phi(x), x(c^T\phi_m(x)), \dots, x^m(c^T\phi_m(x))]A^T \\ &= \left[\sum_{i=0}^{n} c_i B_{i,m}(x), \sum_{i=0}^{n} c_i x B_{i,m}(x), \dots, \sum_{i=0}^{n} c_i x^m B_{i,m}(x)\right]. \end{aligned} \tag{29}$$

Now, we approximate all functions $x^k B_{i,n}(x)$ in terms of $\{B_{i,m}\}_{i=0}^{m}$ for $i,k = 0,1,\dots,m$. By (14), we have

$$x^k B_{i,m}(x) \simeq e_{k,i}^T\phi_m(x). \tag{30}$$

that $e_{k,i} = [e_{k,i}^0, e_{k,i}^1, \dots, e_{k,i}^m]^T$, then we obtain the components of the vector of $e_{k,i}$

$$\begin{aligned} e_{k,i} &= Q^{-1}\left(\int_0^1 x^k B_{i,m}(x)\phi(x)dx\right) \\ &= Q^{-1}\left[\int_0^1 x^k B_{i,m}(x)B_{0,m}(x)dx, \int_0^1 x^k B_{i,m}(x)B_{1,m}(x)dx, \dots, x^k B_{i,m}(x)B_{m,m}(x)dx\right]^T \\ &= \frac{Q^{-1}}{2m+k+1}\left[\frac{\binom{m}{0}}{\binom{2m+k}{i+k}}, \frac{\binom{m}{1}}{\binom{2m+k}{i+k+1}}, \dots, \frac{\binom{m}{m}}{\binom{2m+k}{i+k+m}}\right]^T \quad i,k = 0,1,\dots,m. \end{aligned} \tag{31}$$

Thus we obtain finally

$$\begin{aligned}\sum_{i=0}^{n} c_i x^k B_{i,m}(x) &= \sum_{i=0}^{n} c_i \left(\sum_{j=0}^{n} e_{k,i}^{j} B_{j,m}(x)\right) = \sum_{j=0}^{n} B_{j,m}(x) \left(\sum_{i=0}^{n} c_i e_{k,i}^{j}\right)\\ &= \phi_m(x)^T \left[\sum_{i=0}^{n} c_i e_{k,i}^{0}, \sum_{i=0}^{n} c_i e_{k,i}^{1}, \ldots, \sum_{i=0}^{n} c_i e_{k,i}^{m}\right]^T\\ &= \phi_m(x)^T \left[e_{k,0}, e_{k,1}, \ldots, e_{k,m}\right] c = \phi_m(x)^T V_{k+1} c, \qquad (32)\end{aligned}$$

where $V_{k+1}(k = 0, 1 \ldots, m)$ is an $(m+1) \times (m+1)$ matrix that has vectors $e_{k,i}(i = 0, 1, \ldots, m)$ given, for each columns. If we choose an $(m+1)(m+1)$ matrix $\bar{C} = [V_1 c, V_1 c, \ldots, V_{m+1} c]$, from (29) and (32) we can write

$$c^T \phi(x)\phi(x)^T \simeq \phi(x)^T \bar{C} A^T \qquad (33)$$

and therefore we obtain the operational matrix of product, $\hat{C} = \bar{C} A^T$.

7. **Solving fractional optimal control problems**

Consider the following fractional optimal control problem

$$J = \frac{1}{2} \int_0^1 [q(t)x^2(t) + r(t)u^2(t)]dt, \qquad (34)$$

$${}_0^c D_t^{\alpha} x(t) = a(t)x(t) + b(t)u(t), \qquad (35)$$

$$x(0) = x_0.$$

We expand the fractional derivative of the state variable by the Bernstein basis ϕ:

$${}_0^c D_t^{\alpha} x(t) \simeq C^T \phi(t), \qquad (36)$$

$$u(t) = U^T \phi(t), \qquad (37)$$

where

$$C^T = [c_0, c_1, \ldots, c_m], \qquad (38)$$

$$U^T = [u_0, u_1, \ldots, u_m], \qquad (39)$$

are unknowns. Using (9) and (27), $x(t)$ can be represented as

$$x(t) = {}_0I_t^{\alpha} {}_0^c D_t^{\alpha} x(t) + x(0) \simeq (C^T I^{\alpha} + d^T)\phi, \qquad (40)$$

where I^{α} is the fractional operational matrix of integration of order α and

$$d^T = [x_0, 0, \ldots, 0]. \qquad (41)$$

Also using (14) and (18) we approximate functions $a(t), b(t), q(t), r(t)$ by the Bernstein basis as

$$a(t) \simeq A^T \phi, \qquad b(t) \simeq B^T \phi, \qquad (42)$$

$$q(t) \simeq Q^T \phi, \qquad r(t) \simeq R^T \phi, \qquad (43)$$

where

$$A^T = [a_0, a_1, \ldots, a_m], \qquad B^T = [b_0, b_1, \ldots, b_m], \tag{44}$$

$$Q^T = [q_0, q_1, \ldots, q_m], \qquad R^T = [r_0, r_1, \ldots, r_m], \tag{45}$$

and using (18) we obtain coefficients a_i, b_i, q_i, r_i and using Eqs.(37), (40) and (43), the performance index J can be approximated as

$$\begin{aligned} J \simeq J[C,U] &= \frac{1}{2}\int_0^1 [(Q^T\phi(t))((C^T I^\alpha + d^T)\phi(t)\phi(t)^T(C^T I^\alpha + d^T)^T) \\ &+ (R^T\phi(t))(U^T\phi(t)\phi(t)^T U)dt]. \end{aligned} \tag{46}$$

Using Eqs. (36), (37), (40) and (42) the dynamical system (35) can also be approximated as

$$C^T\phi - A^T\phi\phi^T(C^T I^\alpha + d^T)^T - B^T\phi\phi^T U = 0. \tag{47}$$

Now using Eq(28) we have

$$A^T\phi\phi^T \simeq \phi^T\hat{A}^T, \tag{48}$$

$$B^T\phi\phi^T \simeq \phi^T\hat{B}^T, \tag{49}$$

and using Eqs. (48) and (49) in (47) we obtain

$$C^T\phi - \phi^T\hat{A}^T(C^T I^\alpha + d^T)^T - \phi^T\hat{B}^T U = 0 \tag{50}$$

or

$$(C^T - (C^T I^\alpha + d^T)\hat{A} - U\hat{B})\phi = 0. \tag{51}$$

Finally using (53) we convert the dynamical system (35) to the linear system of algebraic equations

$$(C^T - (C^T I^\alpha + d^T)\hat{A} - U\hat{B}) = 0. \tag{52}$$

Let

$$J^*[C,U,\lambda] = J[C,U] + [C^T - (C^T I^\alpha + d^T)\hat{A} - U\hat{B}]\lambda, \tag{53}$$

where $\lambda = (\lambda_0\,\lambda_1\,\cdots\,\lambda_m)^T$, is the unknown Lagrange multiplier. Now the necessary conditions for the extremum are

$$\frac{\partial J^*}{\partial C} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0. \tag{54}$$

By determining C, U we can determine the approximate values of $u(t)$ and $x(t)$ from (37) and (40), respectively. The method we presented here is based on Rietz direct method for solving variational problems [7].

8. **Illustrative examples**

Example 8.1. *Consider the following time invariant problem*

$$J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt \tag{55}$$

subject to the system dynamics

$${}_0^cD_t^\alpha x(t) = -x(t) + u(t), \tag{56}$$

with initial condition $x(0) = 1$.

Our aim is to find $u(t)$ which minimizes the performance index J. For this problem we have the exact solution in the case when $\alpha = 1$ given by

$$x(t) = cosh(\sqrt{2}t) + \beta sinh(\sqrt{2}t),$$
$$u(t) = (1 + \sqrt{2}\beta)cosh(\sqrt{2}t) + (\sqrt{2} + \beta)sinh(\sqrt{2}t),$$

where

$$\beta = -\frac{cosh(\sqrt{2}) + \sqrt{2}sinh(\sqrt{2})}{\sqrt{2}cosh(\sqrt{2}) + sinh(\sqrt{2})} \simeq -0.98 \tag{57}$$

Using (36), (37) we approximate ${}_0^cD_t^\alpha x(t)$ and $u(t)$. In Table 1, we give the absolute error of $x(t)$ when α is demonstrated by the Bernstein operational matrices. In Figs. 1 and 2, the state variable $x(t)$ and the control variable $u(t)$ are plotted for $\alpha = 0.8$ and different values of with $m = 3, 4, 5$. It is obvious that with increase in the number of the Bernstein basis, the approximate values of $x(t)$ and $u(t)$ converge to the exact solutions. Figs. 3 and 4 demonstrate the approximation of $x(t)$ and $u(t)$ for different values of α together with the exact solution for $\alpha = 1$ when $m = 3$.

Table 1

Absolute error of $x(t)$ in Example 1 when $\alpha = 1$.

t	m=3	m=4	m=5
0.0	-0.00123	-0.0000899	-0.00000625
0.1	0.000341	0.0000477	0.0000134
0.2	0.000508	0.0000325	0.0000212
0.3	0.000112	0.00000774	0.0000324
0.4	-0.000287	0.0000213	0.0000473
0.5	-0.000397	0.0000643	0.0000620
0.6	-0.000150	0.000103	0.0000749
0.7	0.000293	0.0001121	0.0000888
0.8	0.000629	0.0000914	0.0001077
0.9	0.000371	0.0000941	0.0001312

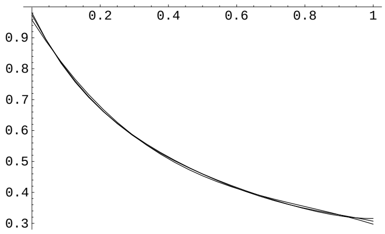

Fig. 1 Approximate solutions for $x(t)$

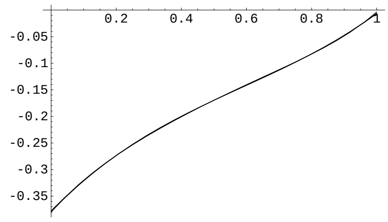

Fig. 2 Approximate solutions for $x(t)$.

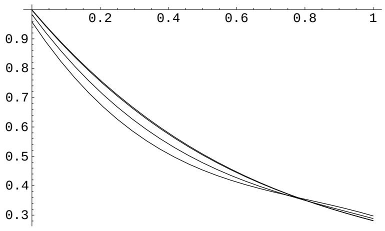

Fig.3 Approximate solutions of $x(t)$ for $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution.

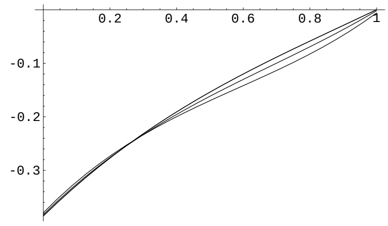

Fig.4 Approximate solutions of $u(t)$ for $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution.

Example 8.2. *This example considers a time varying fractional optimal control problem. We find the control $u(t)$ which minimizes the performance index J given in Example 8.1 subject to the following dynamical system:*

$$ {}_0^cD_t^\alpha x(t) = tx(t) + u(t), \tag{58} $$

$$ x(0) = 1. \tag{59} $$

In Figs. 5 and 6, the state variable $x(t)$ and the control variable $u(t)$ are plotted for $\alpha = 0.8$ and different values of m. It is obvious that with increase in the number of the Bernstein basis, the approximate values of $x(t)$ and $u(t)$ converge to the exact solutions. Figs. 7 and 8 demonstrate the approximation of $x(t)$ and $u(t)$ for different values of α together with the exact solution for $\alpha = 1$ when $m = 5$.

9. Conclusion

In the present work we developed an efficient and accurate method for solving a class of fractional optimal control problems. The Bernstein polynomials operational matrices of fractional integration, product matrix and coefficient matrix $\hat{C}$ were derived for constrained optimization and used to reduce the main problem to the problem of solving a system of algebraic equations. A general procedure of forming these matrices was given. Illustrative examples were presented to demonstrate the validity and applicability of the new method. *Mathematica* was used for computations in this paper.

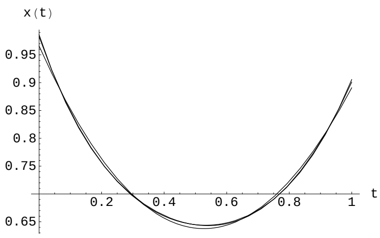

Fig. 5 Approximate solutions for $x(t)$

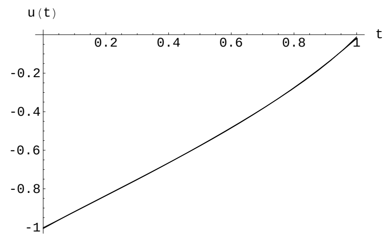

Fig. 6 Approximate solutions for $x(t)$.

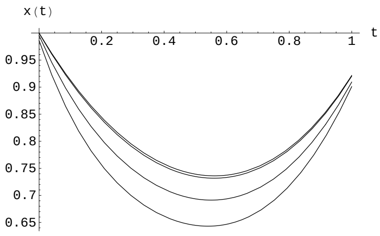

Fig. 7 Approximate solutions of $x(t)$ for $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution.

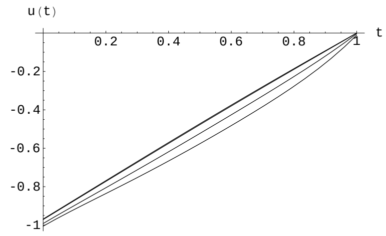

Fig. 8 Approximate solutions of $u(t)$ for $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution.

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