ON THE INTRINSIC TIME SCALE IN THE BOUC-WEN MODEL

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The intrinsic time, other than the clock time which governs the behavior of the materials, was introduced by Valanis [13] in order to develop the endochronic, which is a theory of viscoplasticity without a yield surface. Erlicher and Point [12] have proved the thermodynamic admissibility of the Bouc–Wen model, by adopting the endochronic theory. This paper discusses the behavior of a SDOF oscillator for different measures for the intrinsic time.

Keywords: Bouc-Wen model, endochronic theory of plasticity, intrinsic time scale.

1. Introduction

The inelastic deformation processes under non-proportional cyclic loading of materials is described by the endochronic theory of plasticity, which belongs to the class of theories of functional materials with internal variables [1-5]. Based on the endochronic theory, it is possible to describe a number of peculiar features of the elastoplastic deformation of materials under loading and unloading, such as linear and nonlinear hardening, retardation of the vector and scalar properties of materials when a break in the strain path takes place, hysteresis, and stabilization of hysteresis under cyclic loading, effects of cyclic creep, etc. [6].

Also, a wide variety of hysteretic features including inelastic load-displacement law without distinct yield point, progressive loss of lateral stiffness in each loading cycle (stiffness degradation), degradation of strength when cyclically load is done to the same displacement level (strength degradation) and pinching due to slipping during force reversal, are possible to be describe with the endochronic theory [7, 8]. The constitutive equations of the endochronic theory permit to describe the non-proportional repeated variable deformation of initially isotropic materials [9-11]. Section 3 describes the behavior of a SDOF oscillator for different measures of the intrinsic time. The role of these in the behavior of the SDOF oscillator is discussed. The last Section is devoted to Conclusions

Starting from the relationship between the Bouc model and the endochronic theory, Erlicher and Point [12] have proved the thermodynamic admissibility of the Bouc–Wen model, by adopting an intrinsic time measure.

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In this paper, we discuss the role of the intrinsic time scale in the dynamic response of a SDOF oscillator. Two different measures for the intrinsic times are chosen.

The paper is organized as follows: Section 2 is devoted to the relationship between the endochronic theory and the Bouc-Wen hysteresis model, established in the spirit of Erlicher and Point [12]. By adopting the intrinsic time measures, the thermodynamic admissibility of the Bouc–Wen model is proved

2. The intrinsic time scale

The chapter titles will be numbered, if necessary, and will be written in small characters (12 pts), bold.

The presentation will be clear and concise and the symbols used therein will be specified in a symbol list (if necessary). In the paper it will be used the measurement units International System. In the paper, there will be no apparatus or installation descriptions. The intrinsic time was introduced by Valanis [13] as a non-decreasing function which depends on the strain tensor ε or the stress tensor σ. Erlicher and Point [12] defined it as

$$\mathrm{d}\vartheta = (\mathrm{d}\varepsilon : \mathrm{d}\varepsilon)^{1/2} \quad , \tag{1}$$

where the double dot product of two tensors is noted by “:” , i.e. $A:B = \delta_{il}\delta_{jk}A_{ij}B_{kl}$, with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. For $p = I$, the relation (1) is reduced to $\mathrm{d}\vartheta = \| \mathrm{d}\varepsilon \|$.

The second principle for small isothermal transformations states that the intrinsic mechanical dissipation Φ_1 has to be non-negative

$$\Phi_1 = \sigma : \dot{\varepsilon} - \dot{\Psi} \geq 0 \, , \tag{2}$$

where Ψ is the Helmoltz free energy density. If Ψ depends on a single internal variable tensor χ, the state equation is given by

$$\sigma = \frac{\partial \Psi}{\partial \varepsilon} \, . \tag{3}$$

The thermodynamic force τ associated to the internal variable χ is expressed as

$$\tau = \frac{\partial \Psi}{\partial \chi} \, . \tag{4}$$

In virtue of (3) and (4), Φ_1 becomes

$$\Phi_1 = -\frac{\partial \Psi}{\partial \chi}\frac{\mathrm{d}\chi}{\mathrm{d}t} = -\tau : \dot{\chi} \geq 0 \, , \tag{5}$$

Now, we consider the Helmholtz free energy density written as

$$\Psi = C_0/2\ \mathrm{tr}(\varepsilon)^2 + C_2/2\,\varepsilon_d : \varepsilon_d + B_0 \mathrm{tr}(\varepsilon)\mathrm{tr}(\chi) + B_2\varepsilon_d : \chi_d + \\ + D_0/2\ \mathrm{tr}(\chi)^2 + D_2/2\ \chi_d : \chi_d \tag{6}$$

where $\varepsilon_d = \varepsilon - 1/3\,\mathrm{tr}(\varepsilon)I$ and $\chi_d = \chi - 1/3\,\mathrm{tr}(\chi)I$ are the deviatoric part of the strain tensor and of the internal variable tensor χ, I is the unit tensor, and the constants verify the conditions $C_0, C_2 > 0$, $D_0, D_2 > 0$ and $B_2^2 \le C_2 D_2$ from thermodynamic considerations [1]. Using (6), the state equation (3) can be written as

$$\sigma = 1/3\,\mathrm{tr}(\sigma)I + \sigma_d = (C_0\mathrm{tr}(\varepsilon) + B_0\mathrm{tr}(\chi))I + C_2\varepsilon_d + B_2\chi_d\,. \tag{7}$$

The thermodynamic force (4) becomes

$$\tau = 1/3\,\mathrm{tr}(\tau)I + \tau_d = (B_0\mathrm{tr}(\varepsilon) + D_0\mathrm{tr}(\chi))I + B_2\varepsilon_d + D_2\chi_d\,. \tag{8}$$

In (8), $\mathrm{tr}(\tau) = 3B_0\mathrm{tr}(\varepsilon)$ is the elastic hydrostatic response. The second principle inequality (5) is rewritten

$$\Phi_1 = -\tau : \frac{\mathrm{d}\chi}{\mathrm{d}\vartheta}\frac{\mathrm{d}\vartheta}{\mathrm{dt}} \ge 0\,, \tag{9}$$

and it is satisfied if there exist a positive convex dissipation potential $\varphi\left(\frac{\mathrm{d}\vartheta}{\mathrm{dt}}\right)$ so that

$$\varphi\left(\frac{\mathrm{d}\vartheta}{\mathrm{dt}}\right) \ge 0\,,\ \varphi(0) = 0 \text{ and } \frac{\mathrm{d}\vartheta}{\mathrm{dt}} \ge 0\,. \tag{10}$$

In addition, it is easy to show that

$$\mathrm{tr}\left(\varphi\left(\frac{\mathrm{d}\vartheta}{\mathrm{dt}}\right)\right) = 0\,, \tag{11}$$

and

$$\varphi\left(\frac{\mathrm{d}\vartheta}{\mathrm{dt}}\right) = -\frac{\tau_d}{b_2}\ ,\ b_2 > 0\,. \tag{12}$$

From (8) we have

$$\tau_d = B_2\varepsilon_d + D_2\chi_d\,. \tag{13}$$

The solution of (10) for $\tau_d(0) = 0$, is

$$\tau_d = B_2\int_0^{\vartheta} \exp\left(-\frac{D_2}{b_2}(\vartheta - \vartheta'\right)\frac{\partial\varepsilon_d(\vartheta')}{\partial\vartheta'}\mathrm{d}\vartheta'\,. \tag{14}$$

From (6) we obtain

$$\sigma_d = \left(C_2 - \frac{B_2^2}{D_2}\right)\varepsilon_d + \frac{B_2}{D_2}\tau_d = \left(C_2 - \frac{B_2^2}{D_2}\right)\varepsilon_d + \frac{B_2^2}{D_2}\int_0^{\vartheta}\exp(-\frac{D_2}{b_2}(\vartheta - \vartheta')\frac{\partial\varepsilon_d(\vartheta')}{\partial\vartheta'}\mathrm{d}\vartheta'\,, \tag{15}$$

with $C_2 - \frac{B_2^2}{D_2} \ge 0$ and $\frac{B_2^2}{D_2} > 0$.

Denoting $A_0 = C_2 - \dfrac{B_2^2}{D_2}$, $A = \dfrac{B_2^2}{D_2}$, $\beta = \dfrac{D_2}{b_2}$ and $\mu(\vartheta) = A\exp(-\beta\vartheta)$, Eq.(15) becomes

$$\sigma_d = A_0\varepsilon_d + \frac{B_2}{D_2}\tau_d = A_0\varepsilon_d + \int_0^{\vartheta}\mu(\vartheta-\vartheta')\frac{\partial\varepsilon_d(\vartheta')}{\partial\vartheta'}\mathrm{d}\vartheta' . \tag{16}$$

If we are denoting

$$z = \int_0^{\vartheta}\mu(\vartheta-\vartheta')\frac{\partial\varepsilon_d(\vartheta')}{\partial\vartheta'}\mathrm{d}\vartheta' , \tag{17}$$

(16) and (17) become

$$\sigma_d = A_0\varepsilon_d + z , \quad \mathrm{d}z = A\mathrm{d}\varepsilon_d - \beta z\mathrm{d}\vartheta . \tag{18}$$

We recognize in (18) the hysteresis model proposed by Bouc [14] in the differential form

$$w(t) = A_0 u(t) + z(t) , \quad \mathrm{d}z = A\mathrm{d}u - \beta z\mathrm{d}\vartheta , \tag{19}$$

with $u(t)$ and $w(t)$ are the input and output time-dependent functions, and $A_0 \geq 0$. The function $z(t)$ represents the hysteretic auxiliary variable which describes the time history of the input variable u.

The function $\mu(\vartheta-\vartheta') \geq 0$ is continuous, bounded and positive and non-decreasing on its interval. This function is known in the literature as the hereditary kernel. In particular, the hereditary kernel has an exponential form

$$\mu(\vartheta) = A\exp(-\beta\vartheta) , \quad A,\beta > 0 . \tag{20}$$

The time function ϑ is positive and non-decreasing, and according to Bouc, may represent the total variation of u

$$\vartheta(t) = \int_0^t \left|\frac{\mathrm{d}u}{\mathrm{d}\tau}\right|\mathrm{d}\tau , \tag{21}$$

or

$$\mathrm{d}\vartheta = |\mathrm{d}u| , \text{ with } \vartheta(0) = 0 . \tag{22}$$

More general formulation for $(19)_2$ was proposed in the literature. For example, Bouc suggested the form

$$\mathrm{d}z = A\mathrm{d}u - \beta z|\mathrm{d}u| - \gamma|z|\mathrm{d}u , \quad \gamma < \beta . \tag{23}$$

Wen [16] has proposed another model, with $n > 0$

$$\mathrm{d}z = A\mathrm{d}u - (\beta\,\mathrm{sign}(z\mathrm{d}u) + \gamma)|z|^n\,\mathrm{d}u . \tag{24}$$

Baber and Wen [17] have advanced the stiffness and strength degradation model

$$\mathrm{d}z = A\mathrm{d}u - \nu(\beta\,\mathrm{sign}(z\mathrm{d}u) + \gamma)|z|^n\,\mathrm{d}u , \tag{25}$$

where ν is a positive and increasing function of the energy dissipated by the structure.

All parameters that appear in (23)-(25) for the hysteretic restoring force are controlling the scale and general shape of the hysteretic loop, while n controls the smoothness of the loop. The β and γ are describing the softening or hardening, i.e. if $\beta+\gamma$ is positive the system exhibits softening, while if $\beta+\gamma$ is negative, the system exhibits hardening, respectively. If β decreases, the width of the loop becomes large because the dissipation energy due to the hysteresis becomes larger.

These models were proposed without a thermodynamical analysis. By adopting the intrinsic time measures from the endochronic theory of plasticity, the thermodynamic admissibility of the Bouc–Wen model is proved. The conditions $A>0$ and $-\beta\le\gamma\le\beta$ are necessary and sufficient for the thermodynamic admissibility of the Bouc–Wen model [18]- [21].

In the next Section we try to understand the role of different measures for the intrinsic time in a SDOF oscillator.

3. A SDOF oscillator

Let us consider a SDOF oscillator described by a set of differential equations with hysteresis

$$\ddot{x}+\alpha kx+(1-\alpha)kz=F(t)\,, \tag{26}$$

$$\dot{z}=A\dot{x}-(\beta\,\mathrm{sign}(z\dot{x})+\gamma)\,|\,z\,|^{n}\ \vartheta\,, \tag{27}$$

where x is displacement, k is the linear stiffness coefficient and $F(t)$ is the external force. The hysteretic restoring force z is of the form of (24). The non-damping restoring force is composed by the linear restoring force αkz, and the hysteretic restoring force $(1-\alpha)kz$, where $0<\alpha<1$ is the rigidity ratio representing the relative participations of the linear and nonlinear terms. The quantity z is known as the hysteretic restoring force.

For the time function ϑ, two positive and non-decreasing functions are chosen:

1. total variation of x

$$\mathrm{d}\vartheta=|\,\mathrm{d}x\,|\,,\ \text{with}\ \vartheta(0)=0\,. \tag{28}$$

2. total variation of a variable y introduced to describe an uncertain system characterized by the parameter ς

$$\mathrm{d}\vartheta=|\,\mathrm{dy}\,|\,,\ \text{with}\ \vartheta(0)=0\,,\quad y=\varsigma x\,,\ \varsigma\in[a,b]\,,\ y(0)=0\,,\ a,b\in\mathrm{R}\,. \tag{29}$$

The hysteretic operator (27) possesses the symmetrical characteristics. For non-symmetrical characteristics, the Bouc-Wen model can be modified by introducing an additional term $\delta x\,\mathrm{sgn}\,\vartheta$ in (27) [22]

$$\dot{z}=A\dot{x}-(\beta\,\mathrm{sign}(z\dot{x})+\gamma)\,|\,z\,|^{n}\ \vartheta+\delta x\,\mathrm{sgn}\,\vartheta\,, \tag{30}$$

where δ is a non-symmetrical factor.

Example 1.

The model (26)-(28) can be generalized for a continuum system such a cable, in the problem of vibrating cable in the Stockbridge damper [23, 24]. The deformation of the cable is due to the bending moment, and the shape of the cable during deformation depends on the frequency and amplitude of the clamp motion which is not known a priori.

For this problem, a local model with the position dependent properties can be developed starting to (26) and (27). The position dependent property is supposed to be only the hysteretic restoring moment $H(s,t)$.

The flexural rigidity EI and the linear spring k are considered to be constant. The variable x from (26) is replaced by the local curvature $w''(s,t)$, with $w(s,t)$ the displacement. The intrinsic time function is $\mathrm{d}\vartheta = |\mathrm{d}w''|$. The prime denotes the derivative with respect to s, and the dot denotes the time derivative.

So, the model (26)-(28) is rewritten under the form

$$EI\dot{w}''(s,t) + H(s,t) + sF(t) = 0\,, \tag{31}$$

$$\dot{H}(s,t) = k\left(A\dot{w}'' - (\beta \operatorname{sign}(H\dot{w}'') + \gamma)\,|\,H\,|^n\ \dot{w}''\right). \tag{32}$$

Given $F(t) = f_0\omega^2\cos(\omega t)$, with f_0 and ω the amplitude and frequency of the external force, and initial conditions for $H(s,t)$ and $w''(s,t)$, $\dot{w}''(s,t)$, $w(s,t)$ can be numerically determined.

The hysteretic loops force-displacement for 12Hz and 13 Hz respectively, are plotted in Fig. 1 and Fig. 2, respectively, for $k = 30\text{N/m}$ and $EI = 2.5\text{Nm}^2$. The natural frequency is 15Hz. These loops are similar to the experimental results reported in [23].

For the non-symmetrical model (31) we have

$$\dot{H}(s,t) = k\left(A\dot{w}'' - (\beta \operatorname{sign}(H\dot{w}'') + \gamma)\,|\,H\,|^n\ \dot{w}''\right) + \delta\dot{w}''\operatorname{sgn}\dot{w}''\,. \tag{33}$$

The non-symmetrical version of the hysteretic loop for 12Hz is presented in Fig. 3 for $\delta = 10^{-7}$.

The plane phase orbits ($w, \dot{w}$) for softening hysteresis and hardening hysteresis respectively, are shown in Fig. 4.

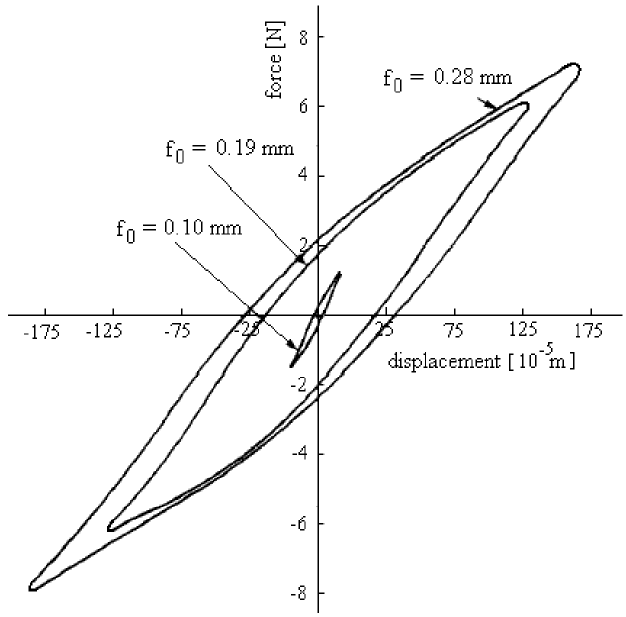

Fig. 1. Hysteresis loop for 12 Hz ($A = 0.75,\ \beta = -0.25,\ \gamma = 1$).

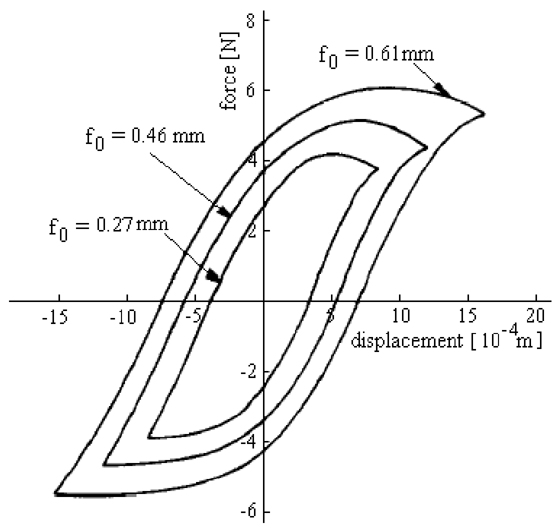

Fig.2. Hysteresis loop for 13Hz ($A = 0.9,\ \beta = 0.1,\ \gamma = 1$).

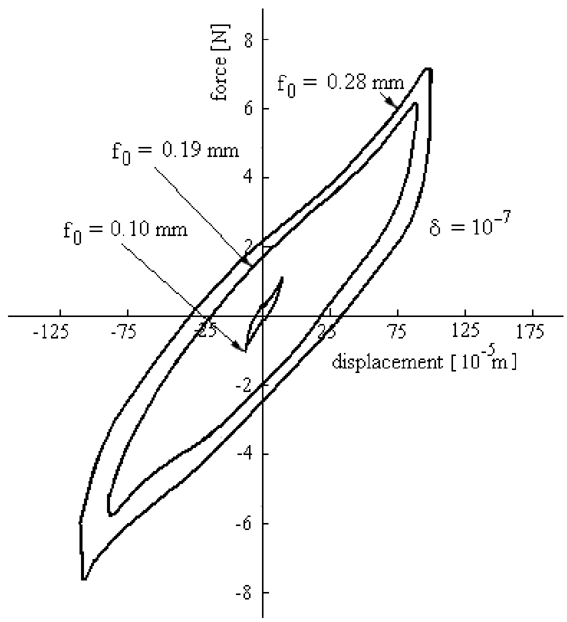

Fig. 3. Hysteresis loop for 12 Hz ($A = 0.75,\ \beta = -0.25,\ \gamma = 1,\ \delta = 10^{-7}$).

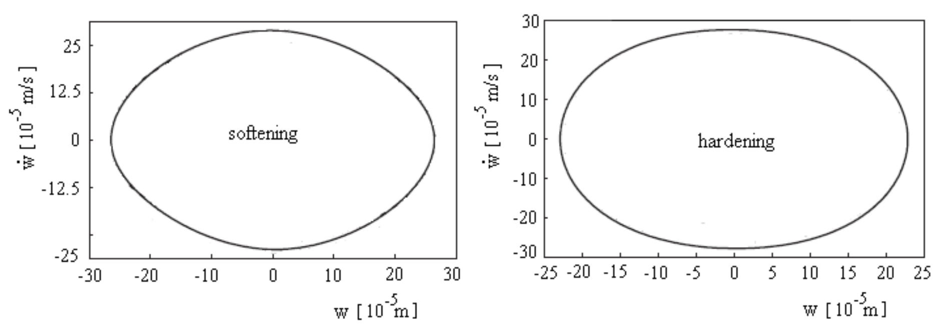

Fig. 4. The phase orbits for softening ($A = 0.75,\ \beta = -0.25,\ \gamma = 1,\ \omega = 75.4$), and hardening ($A = 0.75,\ \beta = -0.25,\ \gamma = 1,\ \omega = 75.4$) ($A = 0.9,\ \beta = -0.1,\ \gamma = 1,\ \omega = 81.6$).

Example 2.

The model (26), (27) and (29) can be applied to systems with unknown time-varying behavior. Such systems have nonlinear uncertainties with no prior knowledge of their values or bounds, and therefore the rapidly varying disturbances have to be analyzed in order to obtain the stabilization controller of the chaotic behavior via different logic systems. Usually, stable adaptive controllers are obtained by combining the back stepping and small-gain approaches. This method was used in [25] to control the chaotic motion of the double pendulum without knowledge of the parameters. Once the desired unstable trajectories to be stabilized are chosen, the control will be initialized to require the pendulum to move towards the equilibrium position.

Let us consider the equations (26) and (27) with $F(t) = f_0\omega^2\cos(\omega t)$, and the equation (29) which describes the behavior of the variable y which describes an uncertain system

$$d\vartheta = |dy|, \text{ with } \vartheta(0) = 0, \quad y = \varsigma x, \; y(0) = 0. \tag{34}$$

with a real-valued parameter $\varsigma \in [1, 3]$. The time variation of the parameter ς is plotted in Fig. 5.

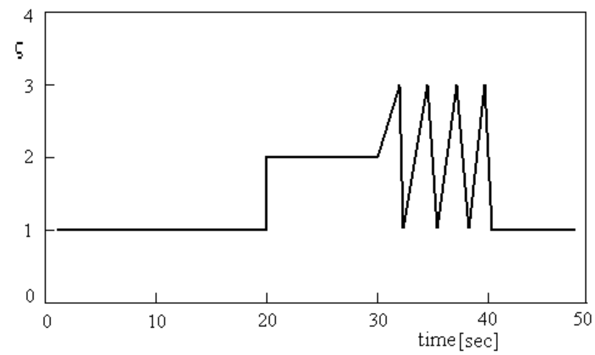

Fig. 5. The time variation of the parameter ς.

The system "does not agrees" the abrupt change of ς at $t = 20\text{s}$ and $t \in (30\text{s}, 40\text{s})$, respectively, and reacts as seen in Fig. 6.

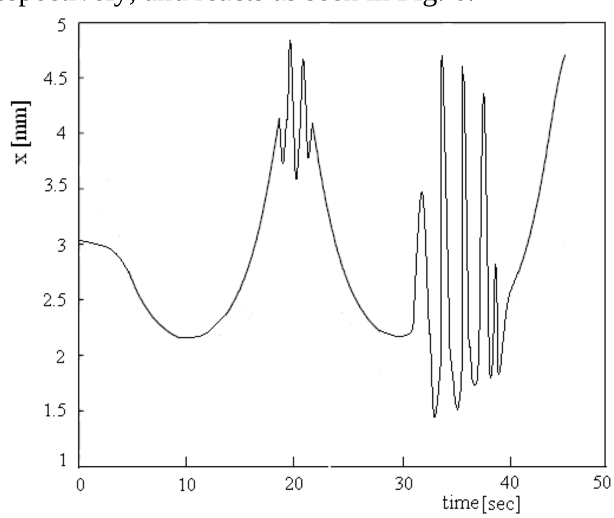

Fig. 6. The time variation of the variable $x(t)$.

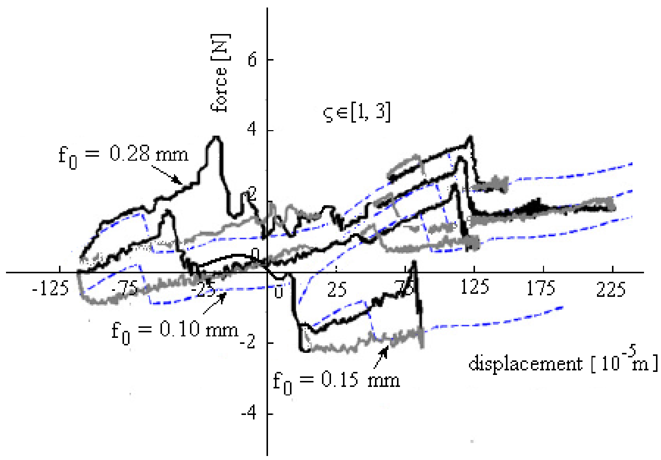

Fig. 7. Hysteretic loop for 12 Hz ($A = 0.75,\ \beta = -0.25, \gamma = 1$) and $\varsigma \in [1, 3]$.

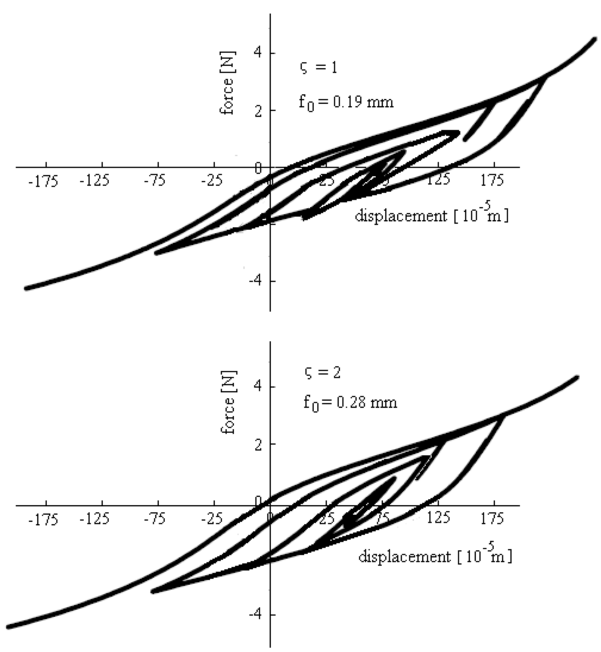

Fig. 8. Hysteresis loop for 12 Hz ($A = 0.75,\ \beta = -0.25,\ \gamma = 1$) (up $f_0 = 0.19$ mm and down $f_0 = 0.28$ mm).

The hysteretic loops force-displacement for 12Hz and $A = 0.75,\ \beta = -0.25,$ $\gamma = 1$, are plotted in Fig. 7 for $f_0 = 0.10$, 0.15 and 0.28 mm, respectively, and $\varsigma \in [1, 3]$. The allure of these curves is practically chaotic, the restoring force having an irregular evolution. For fixed values for ς, i.e. $\varsigma = 1$ and $f_0 =$ 0.28 mm, and $\varsigma = 2$, respectively, the hysteretic loops force-displacement are plotted in Fig. 8, for $f_0 = 0.19$, 12Hz and $A = 0.75$, $\beta = -0.25$, $\gamma = 1$.

Finally, other different time functions ϑ may be chosen. An interesting case may be a system with delay, where the total variation of a variable y is expressed as

$$d\vartheta = |dy|, \text{ with } \vartheta(0) = \vartheta_0, \quad y = k(r - x(t-h)), \; y(0) = y_0. \tag{35}$$

4. Conclusions

Due to nature of the hysteretic phenomenon, the dynamical systems may display complex behavior and energy dissipation properties with effect to their reliability and safety. The intrinsic time, other than the clock time, was introduced by Valanis in the frame of the theory of viscoplasticity without a yield surface in order, and used next by Erlicher and Point to prove the thermodynamic admissibility of the Bouc–Wen model.

This paper discusses the behavior of a SDOF oscillator for two measures for the intrinsic time. In the first example, a generalization for continuum cable in the Stockbridge damper is developed. The results are referred to the behavior of the non-symmetrical hysteretic loops, the softening and hardening aspects, respectively. The second example analyses the time-varying behavior with uncertainties. Here, the time varying disturbances tends to unstable motion and chaotic behavior. The intrinsic time method can be applied to a wide variety of the hysteretic systems in order to extend the understanding of the complex behavior of dynamical systems beyond the classical approaches. New aspects of the inelastic load-displacement law without distinct yield point, stiffness degradation, strength degradation and pinching are developed.

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R E F E R E N C E S

[1]. *Y.I.Kadashevich, A.B.Mosolov*, "Endochronic theory of plasticity: Fundamentals and outlook", Izv. Akad. Nauk SSSR. Mekhanika Tverd. Tela, **vol. 1**, 1989, pp. 161-168.

[2]. *A.B.Mosolov*, "Endochronic theory of plasticity", Preprint No. 353, Institute for Problems of Mechanics Academy of Sciences of the USSR, Moscow, 1988.

[3]. *A.B.Mosolov*, "Endochronic theory of plasticity. 2. Time effects under complex loading", Preprint No. 443, Institute for Problems of Mechanics, Academy of Sciences of the USSR, Moscow, 1990.

[4]. *Y.I.Kadashevich, A.B.Mosolov*, "Endochronic theory of plasticity: Current state", Probl Prochn., **vol. 6**, 1991, pp.3-12.
[5]. *N.K.Kucher*, "Endochronic theory of plasticity: Prediction of nonproportional cyclic deformation of materials", Strength of Meterials, **vol. 30**, no. 3, 1998.
[6]. *N.K.Kucher*, "A version of the endochronic theory of plasticity to describe the asymmetrical cyclic loading of materials", Strength of Meterials, **vol. 31**, no. 1, 1999.
[7]. *G.C.Foliente*, "Stochastic dynamic response of wood structural systems, PhD thesis", Virginia Polytechnic Institute and State University, Blacksburg, 1993.
[8]. *T.T.Baber, M.N.Noori*, "Modelling General Hysteresis Behaviour and Random Vibration Application", Journal of Vibration, Acoustics, Stress, and Reliability in Design. Transactions of the ASME, **vol. 108**, 1986, pp. 411-420.
[9]. *N.K.Kucher, M.V.Borodii*, "A version of the endochronic theory of plasticity for describing non-proportional cyclic deformation", lnt. J. Nonlinear Mechanics, **vol. 28**, no. 2, pp. 267-278, 1993.
[10]. *M.V.Borodii*, "Application of the endochronic theory of plasticity to simulation of nonproportional repeated variable strain-controlled loading", Probl. Prochn., **vol. 5**, 1994, pp. 3-10.
[11]. *M.V.Borodii, N.K.Kucher, V.A.Stryzhalo*, "Development of a constitutive model for biaxial low-cycle fatigue", Fatigue & Fracture of Eng. Mater. and Struct., **vol. 19**, no. 10, 1996, pp. 1169-1179.
[12]. *S.Erlicher, N.Point*, "Thermodynamic admissibility of Bouc-Wen type hysteresis models", Comptes Rendus Mecanique, **vol. 332**, 2004, pp. 51-57.
[13]. *K.C.Valanis*, "A theory of viscoplasticity without a yield surface. Part I: General theory", Arch. Mech., vol. 23, no. 4, 1971, pp. 517–533.
[14]. *R.Bouc*, "Modèle mathématique d'hystérésis", Acustica, **vol. 24**, 1971, pp. 16–25.
[15]. *R.Bouc*, "Forced vibrations of a mechanical system with hysteresis", in: Proc. 4th Conf. on Nonlinear Oscillations, Prague, Czechoslovakia, 1967.
[16]. *Y.K.Wen*, "Method for random vibration of hysteretic systems", J. Eng. Mech. Div. ASCE, **vol. 102**, 1976, pp. 249–263.
[17]. *T.T.Baber, Y.K.Wen*, "Random vibrations of hysteretic, degrading systems", J. Eng. Mech. Div. ASCE, **vol. 107**, no. 6, 1981, pp. 1069–1087.
[18]. *F.Ikhouane, J.Rodellar*, Systems with hysteresis, Analysis, Identification and Control using the Bouc-Wen Model, John Wiley& Sons, Ltd, 2007.
[19]. *N.Okuizumi, K.Kimura*, "Multiple time scale analysis of hysteretic systems subjected to harmonic excitation", Journal of Sound and Vibration, **vol. 272**, 2004, pp. 675–701.
[20]. *V.Preda,, M.F.Ionescu, V.Chiroiu, D.Dumitriu*, "A Preisach model for the analysis of the hysteretic phenomena", Revue Roumaine des Sciences Techniques – Série de Mécanique Appliquée, **vol. 55**, no. 3, 2010.
[21]. *C.W.Stammers, T.Sireteanu*, "Vibration control of machines by use of semi-active dry friction damping", Journal of Sound and Vibration, **vol. 209**, 1998, pp. 671-684.
[22]. *Wei Zhu, Dai-Hua Wang*, "Non-symmetrical Bouc-Wen model for piezoelectric ceramic actuators", Sensors and Actuators A, **vol. 181**, 2012, pp. 51-60.
[23]. *I.Pivovarov, O.G.Vinogradov*, "One application of Bouc's model for non-linear hysteresis", Journal of Sound and Vibration, **vol. 118**, no. 2, 1987, pp. 209-216.
[24]. *D.Sauter, P.Hagedorn*, "On the hysteresis of wire cables in Stockbridge dampers", International Journal of Non-Linear Mechanics, **vol. 37**, 2002, pp. 1453- 1459.
[25]. *V.Chiroiu, L.Munteanu, I.Ursu*, "On chaos control in uncertain systems"**,** CMES: Computer Modeling in Engineering & Sciences, **vol. 72**, no. 3, 2011, pp.229-246.