SOME REMARKABLE PROPERTIES OF THE MOMENTS OF INERTIA FOR PLANAR STRUCTURES

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The principal central moments of inertia of certain plane figures can be determined directly, by use of geometrical means only, based on some observations made by the authors concerning the central ellipses of inertia. It can be shown that the central ellipses of inertia of particular shapes such as the parallelogram, the scalene triangle, or more unambiguously the rectangle and the ellipse itself, are merely inscribed ellipses, downscaled with certain factors discovered by the authors and presented herein.

Keywords: moments of inertia, ellipse of inertia.

1. Introduction

The mechanical moments of inertia are well established and were used in engineering applications for centuries. The basic formulas can be found in classical textbooks as [1], [2] or [3] for moments of inertia of the cross-sections see ref. [4]. The use of modern computers can be considered as a rapid solution for the moments of inertia of complex shapes and apparently this topic is closed for any research. However, many recent researches contradict this idea. Brlek et al. in ref. [5] investigate the mass center and moments of inertia of discrete sets of material points placed in a grid. Orășanu in ref. [6] proposes original mathematical models of reduction of the elementary bodies to systems of material points with the same inertia properties. Petrescu et al. [7] investigate the moment of inertia of a flywheel. Ostanin and Sperl in ref. [8] referring to the unstable rotation around the direction close to the body's second principal axis, featuring a well-known tennis-racket (also known as Garriott-Dzhanibekov) effect consisting of a series of seemingly spontaneous 180 degrees flips, are investigating the control of the nonspherical tensor of inertia (TOI) as a means to optimize and stabilize the attitude control of a spacecraft. Jauch et al. in ref. [9] present a parametric study, in which different parameters of a flywheel are varied in order to find the maxima in energy density and specific energy. Eremeyev and Elishakoff [10] discuss the classic rotary inertia notion and extend it for

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microstructured beams introducing new microinertia parameters as an additional dynamic response to microstructure changes**.** Pellecchia et al. present in ref. [11] a general methodology to evaluate the mass moments of two-dimensional domains and axisymmetric solids made of functionally graded materials. Szántó et al. in ref. [12] presents a method for the simultaneous measurement of the moment of inertia and braking torque of the rotor of electric motors.

2. Moments of inertia of planar structures

The mechanical moments of inertia and product of inertia about central axes (axes with origin in the mass center) for common planar structures, assumed to be made of homogeneous material of mass density ρ and of constant thickness h, can be deduced from their definitions [1] , [2]:

$$J_{xC} = \rho h \iint_{(A)} y^2 dA\,; J_{yC} = \rho h \iint_{(A)} x^2 dA\,; J_{xyC} = \rho h \iint_{(A)} xy dA. \quad (1)$$

The surface integral covers the area A of the planar structure. The principal moments of inertia represent the extreme (maximum and minimum) values of the moments of inertia obtained for a particular inclination θ of the principal axes. About these axes, the product of inertia J_{xy} is cancelled, the maximum is denoted by J_1 and the minimum J_2. The orientation of the principal axes relative to the initial axes is given by [2]:

$$\tan(2\theta) = \frac{2J_{xyC}}{J_{yC} - J_{xC}} \quad (2)$$

About the obtained principal axes, it can be defined the ellipse of inertia:

$$J_1 x^2 + J_2 y^2 = C^2 \quad (3)$$

in which C is a constant, to be determined in each case. Leaving aside the common factor ρh required for the mechanical moments of inertia, it will be obtained the central moments of inertia of the cross-sections, which are commonly used in the Strength of materials studies. We mention in the following the central moments of inertia for some widely used planar structures, with A the area of the respective planar structure:

a) Ellipse of semi-axes a along Ox and b along Oy (Fig. 1 a):

$$J_x = \frac{\pi a b^3}{4} = A\frac{b^2}{4};\quad J_y = \frac{\pi a^3 b}{4} = A\frac{a^2}{4}\,; J_{xy} = 0 \quad (4)$$

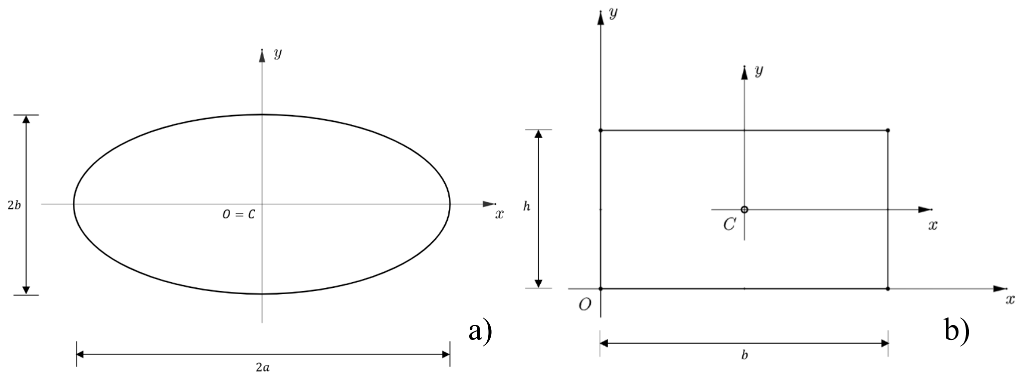


Fig. 1 Ellipse centered in the *xOy* frame origin (a), Rectangle in first quadrant (b)

b) Ellipse of semi-axes *a* along *Ox* and *b* along *Oy* (Fig. 1 a):

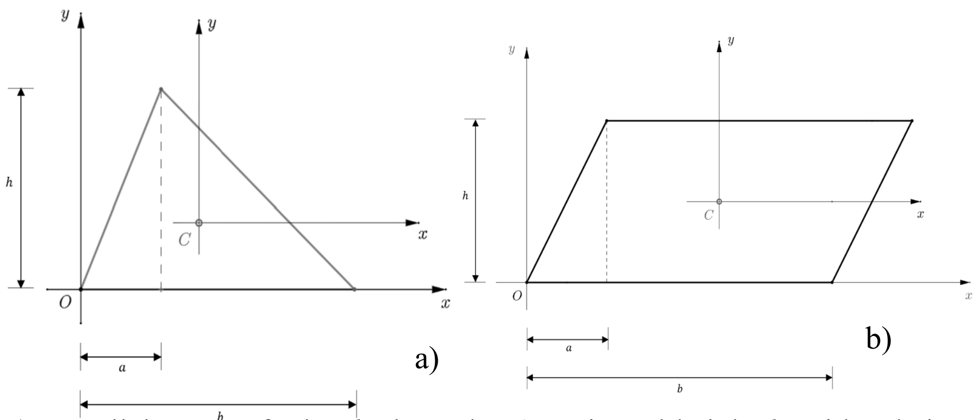
$$J_x = \frac{\pi a b^3}{4} = A\frac{b^2}{4}; \quad J_y = \frac{\pi a^3 b}{4} = A\frac{a^2}{4}; J_{xy} = 0 \tag{5}$$

c) Rectangle of edges *b* along Ox and *h* along Oy (Fig. 1 b):

$$J_{xC} = \frac{bh^3}{12} = A\frac{h^2}{12}; \ J_{yC} = \frac{hb^3}{12} = A\frac{b^2}{12}; \ J_{xyC} = 0 \tag{6}$$

d) Arbitrary scalene triangle of base *b* along *Ox*, height *h* along *Oy* and *a* is the projection of the upper corner on the horizontal base (Fig. 2 a):

$$J_{xC} = \frac{bh^3}{36} = A\frac{h^2}{18}; \ J_{yC} = bh\frac{b^2 - a\,(b-a)}{36} = A\frac{b^2 - ab + a^2}{18};$$
$$J_{xyC} = \frac{(2a-b)bh^2}{72} = A\frac{(2a-b)h}{36} \tag{7}$$



e) Parallelogram of edge *b* along the *Ox* axis and height *h*, with *a* being the

Fig. 2 Arbitrary scalene triangle in first quadrant (a) , parallelogram in first quadrant (b) horizontal projection of the inclined edge of the parallelogram (Fig. 2 b): Being less presented in textbooks, we deduce the moments of inertia for this figure:

a. Moment of inertia about *Cx* axis is obtained using elementary strips of length *b* and height *dy*:

$$J_{xC} = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy = \frac{bh^3}{12} = \frac{Ah^2}{12} \tag{8}$$

b. Moment of inertia about *Cy* can be obtained in an original way, using the same elementary strips b.dy which can be considered to be an infinitesimal rectangle with moment of inertia about its center: $dJ_y = \frac{b^3 dy}{12}$. Since the center of this strip is moved relative to the Cy axis of the parallelogram by a distance $d = \frac{a}{h} y$, one can use the Steiner's theorem to transfer the moment of inertia of the strip to the parallel axis *Cy* of the parallelogram:

$$J_{yC} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{b^3}{12} + b \left(\frac{a}{h} y \right)^2 \right) dy = \frac{hb}{12}(b^2 + a^2) = \frac{A}{12}(b^2 + a^2) \tag{9}$$

c. The product of inertia about the central axes, can be obtained using the same elementary strips b.dy, which can be considered to be an infinitesimal rectangle with product of inertia about its center $dJ_{xy} = 0$. Use is made of the Steiner's theorem to transfer the product of inertia of the strip to the central axes of the parallelogram:

$$J_{xyC} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{ab}{h} y^2 dy = \frac{abh^2}{12} = \frac{Aha}{12} \tag{10}$$

3. Properties of scaled ellipses of inertia

In order to calculate the central and principal axial moments of inertia of a planar lamina, one has to evaluate the following expressions [1,2]:

$$J_{max,C} = J_{1,C} = \frac{J_{xC} + J_{yC}}{2} + \frac{1}{2}\sqrt{\left(J_{yC} - J_{xC}\right)^2 + 4J_{xyC}^2} \tag{11}$$

$$J_{min,C} = J_{2,C} = \frac{J_{xC} + I_{yC}}{2} - \frac{1}{2}\sqrt{\left(I_{yC} - J_{xC}\right)^2 + 4J_{xyC}^2} \tag{12}$$

which lead to the central and principal radii of gyration given by:

$$i_{1,C} = \sqrt{\frac{J_{1,C}}{A}}\ ;\ i_{2,C} = \sqrt{\frac{J_{2,C}}{A}} \tag{13}$$

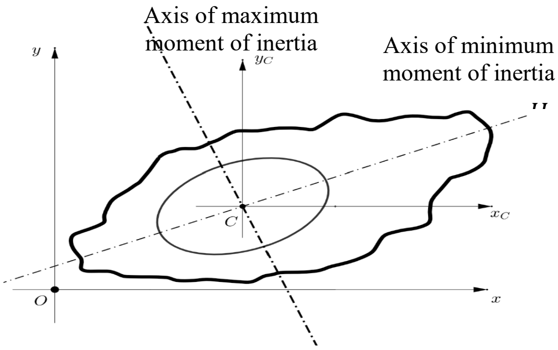


Fig. 3 Typical ellipse of inertia layout for an arbitrary plane lamina

These resulting values can be used to determine the central principal ellipse of inertia (Fig. 3):

$$\frac{x_{(II)}^2}{i_{2,C}^2} + \frac{y_{(I)}^2}{i_{1,C}^2} = 1 \tag{14}$$

As stated in [1], *the ellipse of inertia follows the geometry of the figure*, in perfect agreement with the fact that its major axis is taken along the secondary principal axis of inertia, its minor axis being conversely taken along the principal axis of inertia. Let us make a step forward and analyze the possibility of finding geometric similarities between the ellipses of inertia of certain planar laminae and their proper shapes.

a) The ellipse.

Unsurprisingly, the ellipse of inertia of this particular shape is nothing but a downscaled ellipse completely similar to that of the contour of the lamina.

a) Lemma. Scaling down the area of the given ellipse by the factor $K = 4$, (linear scaling factor of $k = \sqrt{K} = 2$), it can be obtained the inertia ellipse and its moments of inertia.

Proof. Since the ellipse of inertia of an elliptic lamina is an ellipse with axes downscaled by $k = 2$, we can write the following scaling equalities between the semi-axes of the lamina and those of the ellipse of inertia:

$$i_1 = \frac{a}{k}\ ;\ \ i_2 = \frac{b}{k}. \tag{15}$$

Substituting these semiaxes in (13), with $A = \pi ab$, can be obtained $J_{1,C}$ and $J_{2,C}$:

$$J_{1C} = i_1^2 A = \frac{a^2}{4}\pi ab \; ; \; J_{2C} = i_2^2 A = \frac{b^2}{4}\pi ab \; , \tag{16}$$

in complete agreement with the integrals (5), proving thus this lemma.

Using this lemma can be directly expressed the principal central moments of inertia of an ellipse without calculating the double integrals in (1).

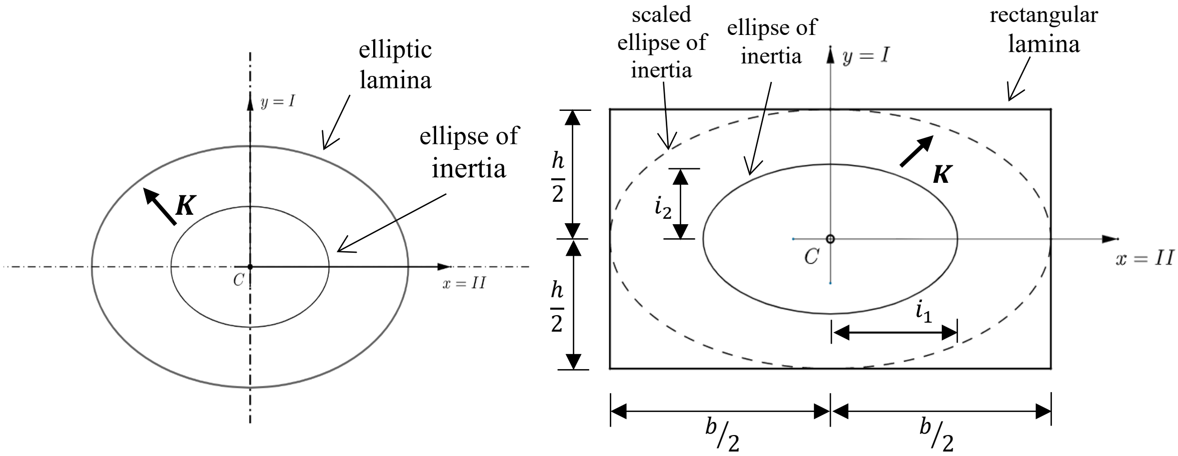


Fig. 4 Ellipse of inertia of an elliptic planar lamina (a). Ellipse of inertia of a rectangular lamina (b)

b) The rectangle.

Since any rectangle possesses an associated central ellipse tangent to the midpoints of its sides, such a geometric shape is another candidate for our method.

b) Lemma. Scaling down the area of the given rectangle by factor $K = 3$, (linear scaling factor of $k = \sqrt{K} = \sqrt{3}$), it can be obtained the inertia ellipse and its moments of inertia.

Proof. By analogy with the ellipse, we scale down each half- side of the rectangle:

$$i_1 = \frac{b}{2k} \; ; \; i_2 = \frac{h}{2k} \tag{17}$$

and by inserting these semiaxes in the general relations (13), and using $A = bh$, it can be immediately obtained:

$$J_{1C} = \frac{b^2}{4K} bh = \frac{b^3 h}{12} \; ; \; J_{2C} = \frac{h^2}{4K} bh = \frac{h^3 b}{12} \tag{18}$$

which are identical to the results (6) obtained by integration.

c) The scalene triangle.

In case of an scalene triangle, the ellipse of inertia from Mechanics proves out to be similar to the Steiner inellipse also called midpoint ellipse of the corresponding triangle (Fig. 5). The area A of a triangle can be expressed by the triangle's sides only (denoted s_1, s_2, s_3) using the Heron's formula:

$$A = \sqrt{p(p-s_1)(p-s_2)(p-s_3)} \tag{19}$$

in which $p = \frac{1}{2}(s_1 + s_2 + s_3)$ is the semi-perimeter of the triangle.

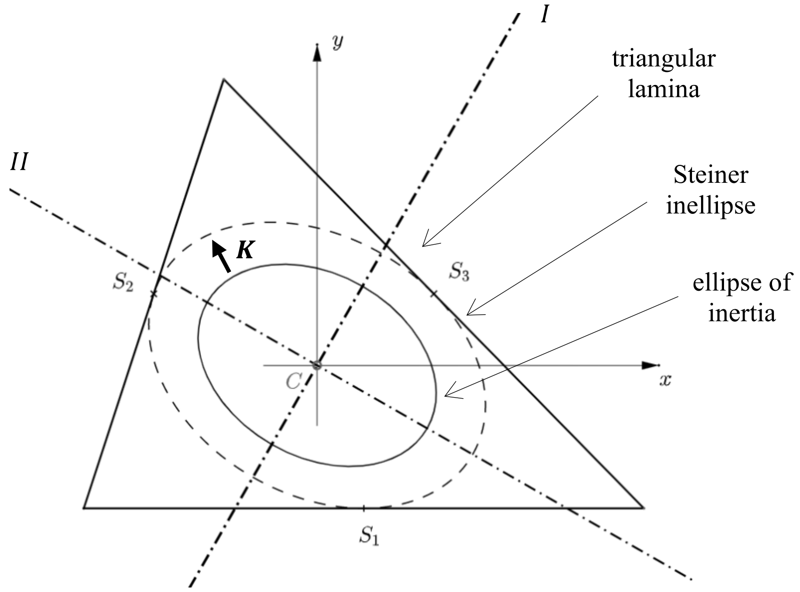


Fig. 5 Ellipse of inertia of a triangular lamina

The midpoint ellipse (centered in C, tangent to the midpoints of the triangle's sides) is known to have semi-axes [13]:

$$a_1 = \frac{1}{6}\sqrt{s_1^2 + s_2^2 + s_3^2 + 2 \cdot Z} \tag{20}$$

and

$$b_1 = \frac{1}{6}\sqrt{s_1^2 + s_2^2 + s_3^2 - 2 \cdot Z} \tag{21}$$

The term Z is equal to:

$$Z = \sqrt{s_1^4 + s_2^4 + s_3^4 - (s_1 s_2)^2 - (s_2 s_3)^2 - (s_1 s_3)^2} \tag{22}$$

c) Lemma By scaling down the semiaxes of the midpoint ellipse by a linear scaling factor of $k = \sqrt{2}$ it can be obtained the inertia ellipse and the central principal moments of inertia of the triangle. In fact, injecting in (11) and (12) the central moments of inertia of the triangle (Fig. 5), one gets the central and principal moments of inertia of a triangle:

$$J_{1,2C} = \frac{A}{36}\left[(a^2 + b^2 + h^2 - ab) \pm \sqrt{(a^2 + b^2 + h^2 - ab)^2 - 3b^2h^2}\right] \quad (23)$$

It remains to be proven that the midpoint ellipse leads by the given scaling, the same central principal moments of inertia.

Proof. Equating the congruence between the scaled ellipse of inertia and the Steiner inellipse, we get:

$$i_1 = \frac{a_1}{k} \; ; \; i_2 = \frac{b_1}{k} \quad (24)$$

Subsequently,

$$K \cdot \frac{J_{1C}}{A} = a_1^2 \; ; \; K \cdot \frac{J_{2C}}{A} = b_1^2 \quad (25)$$

Substituting (20) and (21) in (25), the central principal moments of inertia J_{1C} and J_{2C} are obtained:

$$J_{1,2C} = \frac{A}{72}(s_1^2 + s_2^2 + s_3^2 \pm 2 \cdot Z). \quad (26)$$

In order to prove that the resulting formula (28) is confirming the classical expression (23), one has to use the identities:

$$s_1^2 = b^2; \; s_2^2 = a^2 + h^2; \; s_3^2 = a^2 + b^2 + h^2 - 2ab. \quad (27)$$

Obviously $s_1^2 + s_2^2 + s_3^2 = 2(a^2 + b^2 + h^2 - ab)$ and

$$Z^2 = 4(a^2 + b^2 + h^2 - ab)^2 - 3[b^2(a^2 + h^2) + (a^2 + b^2 + h^2)(a^2 + b^2 + h^2 - 2ab)] = (a^2 + b^2 + h^2 - ab)^2 - 3b^2h^2$$

which proves that the scaled midpoint ellipse represents indeed the principal moments of inertia of the triangle. Thus, the above deduced expressions (28) for the principal and central axial moments of inertia for a triangle provide a the more practical approach based on the triangle's edges.

d) The parallelogram

The parallelogram may be perceived as a plane compound figure comprised of a middle rectangle and two right triangles. All of these shapes have been proven to admit an inscribed ellipse similar to their ellipse of inertia. Thus, it is not surprising that the parallelogram's principal and central axial moments of inertia may be determined by the proposed geometric method as well.

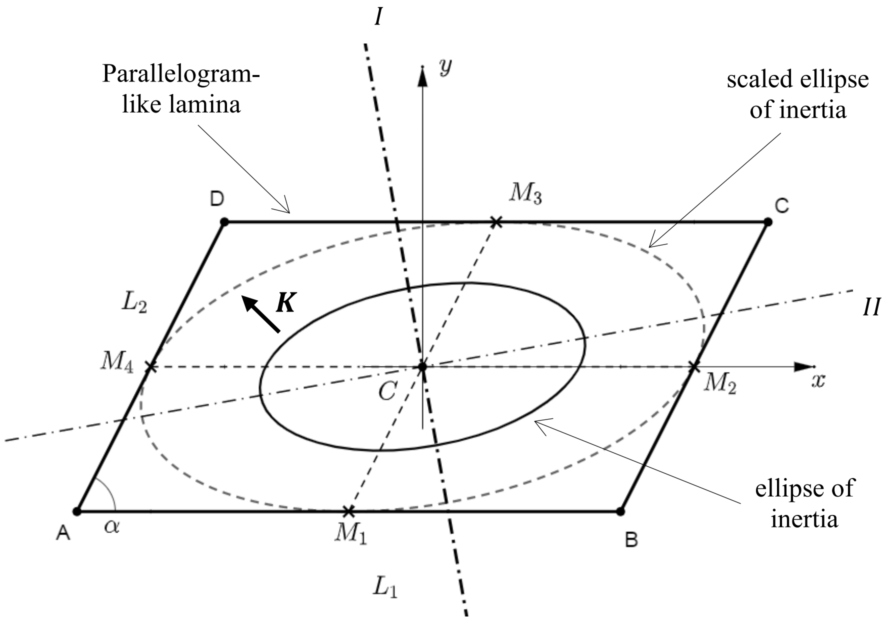


Fig. 6 Ellipse of inertia of o parallelogram planar lamina

c) Lemma The ellipse of inertia follows the obliquity of the parallelogram, and when scaled by an area factor of $K = 3$, (linear scaling factor of $k = \sqrt{K} = \sqrt{3}$), it becomes tangent to the midpoints of its sides providing the moments of inertia of the parallelogram.

Proof. We can take advantage of this tangency by setting up Apollonius' theorem ($a_e^2 + b_e^2 = c^2 + d^2$, with a_e, b_e – semi-major and semi-minor axes of the ellipse, c, d – any two conjugate semidiameters) in a convenient form. Since by definition the pairs of opposite sides in a parallelogram are respectively equal and parallel, the segments M_4M_2 and M_1M_3 – equal to the sides L_1 and L_2 of the parallelogram – shall be our conjugate diameters for the scaled ellipse of inertia defined by major axis $(k \cdot 2i_1)$ and minor axis $(k \cdot 2i_2)$.

Providing these substitutions into Apollonius' theorem, we have:

$$(k \cdot i_1)^2 + (k \cdot i_2)^2 = \left(\frac{L_1}{2}\right)^2 + \left(\frac{L_2}{2}\right)^2 \quad (28)$$

In terms of J_{1C} and J_{2C}:

$$K \cdot \frac{J_{1C}}{A} + K \cdot \frac{J_{2C}}{A} = \frac{L_1^2}{4} + \frac{L_2^2}{4} \quad (29)$$

hence obtaining a relationship for the sum of the principal and central axial moments of inertia:

$$J_{1C} + J_{2C} = \frac{A}{12} \cdot (L_1^2 + L_2^2) \quad (30)$$

in which we have taken into account that $K = 3$.

On the other hand, the area of our scaled ellipse of inertia may be expressed in terms of the conjugate semi-diameters and the angle between them. But the latter in merely α, the defining scalene angle of the parallelogram: $\sphericalangle\,(M_4M_2, M_1M_3) = \sphericalangle\,(L_1, L_2) = \alpha$. Equating this lesser known form with the typical area of the ellipse, we get:

$$\pi\,(k \cdot i_1)\,(k \cdot i_2) = \pi \frac{L_1}{2}\frac{L_2}{2}\sin\alpha \tag{31}$$

In terms of J_{1C} and J_{2C}, taking into account that $L_1L_2 \sin\alpha$ represents the area A of our parallelogram, we have:

$$K \cdot \frac{\sqrt{J_{1C}\,J_{2C}}}{A} = \frac{A}{4} \tag{32}$$

thus obtaining an expression for the product of the principal and central axial moments of inertia:

$$J_{1C}\,J_{2C} = \frac{A^4}{144} \tag{33}$$

We may now put together eq. (30) and (33) into a system of equations which reduces to a quadratic equation of the form $x^2 - Sx + P = 0$, written according to Vièta's formulae for the sum and product of the roots. Solving, yields:

$$J_{1,2C} = \frac{A}{24} \cdot (L_1^2 + L_2^2) \pm \frac{A}{24} \cdot \sqrt{[(L_1^2 + L_2^2)^2 - 4A^2]} \tag{34}$$

which offers the possibility for determining the principal and central axial moments of inertia of a parallelogram directly in terms of its geometric defining parameters: L_1, L_2 and α.

The deduced formula (34) must confirm the principal moments of inertia obtained by the classical algorithm, inserting (7), (8) and (9) into (10) and (11), with previous notations (Fig. 2b):

$$J_{1,2C} = \frac{A}{24}(a^2 + b^2 + h^2) \pm \frac{A}{24}\sqrt{a^4 + b^4 + h^4 + 2a^2b^2 - 2b^2h^2 + 2a^2h^2} \tag{35}$$

We may perceive the above algorithm providing formula (34) as a purely geometrical alternative, in order to obtain expressions for J_{1C} and J_{2C}. Indeed, since $L_1^2 = b^2$; $L_2^2 = a^2 + h^2; A = bh$ once injected in the proposed formula (34) lead to:

$$J_{1,2C} = \frac{A}{24} \cdot (a^2 + b^2 + h^2) - \frac{A}{24} \cdot \sqrt{a^4 + b^4 + h^4 + 2a^2b^2 - 2b^2h^2 + 2a^2h^2} \tag{36}$$

This result proves the validity of the purely geometric proposed approach.
Based on the above lemmas, a general theorem can be stated:

Theorem. For geometrical planar structures admitting an ellipse tangent to all sides, it can be found a downscaling factor k allowing to deduce the central principal moments of inertia from the semiaxes of this ellipse (see Table 1 below). The proof was given by the presented lemmas for some of the most useful planar structures.

Table 1.

Scaling factors for the Steiner's inellipse providing the principal inertia ellipse

Planar structure	Inscribed ellipse	
	Linear scaling factor $k = \sqrt{K}$	Area scaling factor K
ellipse	2	4
rectangle	$\sqrt{3}$	3
triangle	$\sqrt{2}$	2
parallelogram	$\sqrt{3}$	3

6. Conclusions

The central and principal axial moments of inertia of plane figures commonly used in Mechanics and Engineering may be determined directly, on the basis of geometrical parameters and a scaling factor deduced in each case in the present work.

The condition that the plane figures allow inscribed ellipses tangent to their perimeter must be satisfied. The area of the planar surface, the scaling factor between the ellipse of inertia and the inscribed ellipse, along with the specific geometric properties of the latter provide an original mathematical alternative for the classic double integral formulae established in handbooks.

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