COMPACTNESS OF THE COMPLEX GREEN OPERATOR IN A STEIN MANIFOLD

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Let X be a Stein manifold of dimension n and let Ω be a bounded pseudoconvex domain with smooth boundary $b\Omega$ in X. If $1 \leq q \leq n-2,\ n \geq 3$ and if $b\Omega$ satisfies both (P_q) and (P_{n-q-1}), then the Green operator G_q is a compact operator (and so is G_{n-q-1}). Moreover, we show that the compactness in the $\bar{\partial}$-Neumann problem on locally convexiable domains, yield the corresponding characterization of compactness of the complex Green operator(s) on these domains.

Keywords: $\bar{\partial}$ and $\bar{\partial}$-Neumann operators, pseudoconvex domains, Stein manifold.

1. Introduction and main results

On $b\Omega$, $\bar{\partial}$ induces the tangential Cauchy-Riemann operator $\bar{\partial}_b$. The $\bar{\partial}_b$ operator is not only important in several complex variables, it is also important in the theory of partial differential operators. Let $\bar{\partial}_b^*$ be the L_2-adjoint of $\bar{\partial}_b$, and $\Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b$ be the Kohn Laplacian. If $0 \leq q \leq n-1$, $\Box_b$ is invertible (on $(\ker(\bar{\partial}_b)^\perp$ when $q = 0$, and on $\ker(\bar{\partial}_b^*)^\perp$ when $q = n-1$ with inverse G_q. G_q is the complex Green operator.

The phenomenon of symmetric requirements at levels q and $(n-1-q)$ was pointed out by ([1], p.289). He associates to a $(0,q)$-form u on $b\Omega$ and $(0, n-1-q)$ -form $\tilde{u}$ (obtained through a modified Hodge-$*$ construction) such that $\|u\| \approx \|\tilde{u}\|$, $\bar{\partial}_b\tilde{u} = (-1)^q\widetilde{(\bar{\partial}_b^* u)}$ and $\bar{\partial}_b^*\tilde{u} = (-1)^q\widetilde{(\bar{\partial}_b u)}$, modulo terms that are $O(\|u\|)$. Consequently, a compactness estimate holds for $(0,q)$-forms if and only if the corresponding estimate holds for $(0, n-1-q)$-forms. In view of the characterization of compactness on convex domains [2], such a symmetry between form levels is absent in the $\bar{\partial}$-Neumann problem.

The $\bar{\partial}_b$ complex on the boundary of a complex manifold was first formulated by Kohn-Rossi [3] to study the boundary values of holomorphic functions and holomorphic extensions. In [4], Catlin introduced a weakened version of complex Hessian blow up condition and instead requires only that there exist plurisubharmonic functions with arbitrarily large complex Hessians. He calls this condition property (P) and its natural generalization to $(0,q)$-forms, called(P_q), is now a well-known sufficient condition for compactness of the $\bar{\partial}$-Neumann operator (see [5, 6]).

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When $b\Omega$ is the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, the operators $\bar{\partial}_b$, hence $\bar{\partial}_b^*$ and $\Box_b$, have closed range in $L^2(b\Omega)$was shown in [6, 7, 8], this property has been established in [9] for compact pseudoconvex orientable CR-submanifolds of hypersurface type of dimension at least five. In [9], Raich and Straube showed that if the boundary $b\Omega$ of a smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$satisfies (P_q) and (P_{n-1-q}), then G_q is a compact operator on $L^2_{p,q}(b\Omega)$.

The goal of this article is to generalize this result to the case when $b\Omega$ is a boundary of a bounded smooth pseudoconvex domain in a Stein manifold. More precisely, if $1 \leq q \leq n-2,\ n \geq 3$ and if $b\Omega$ satisfies both (P_q) and (P_{n-1-q}), we prove that G_q is a compact operator (and so is G_{n-1-q}). Our methods involve $\bar{\partial}$-techniques follow [10], a jump formula in the spirit of Shaw [6], and a detailed study of compactness of the $\bar{\partial}$-Neumann operator N on the annulus between two pseudoconvex domains. Moreover, we also show that compactness of G_q implies compactness of $N_q$on (p,q)-forms on Ω. Finally, if $b\Omega$ is locally convexifiable then (P_q) and (P_{n-1-q}) is equivalent to compactness of G_q (see [11] as well).

Theorem 1.1. Let Ω be a bounded pseudoconvex domain with smooth boundary in a Stein manifold X of dimension n and let $1 \leq q \leq n-2,\ n \geq 3$. If $b\Omega$ satisfies both (P_q) and (P_{n-1-q}), then G_q and G_{n-1-q} are compact operators on $L^2_{p,q}(b\Omega)$ and $L^2_{p,n-1-q}(b\Omega)$, respectively.

The proof of Theorem 1.1 can be obtained in several steps. First, we prove the compactness estimates of the $\bar{\partial}$-Neumann problem on an annulus between two pseudoconvex domains in a Stein manifold. Second, a compactness of the $\bar{\partial}$-Neumann operator on such domains. Third, we prove compactness of the canonical solution operators for $\bar{\partial}_b$ on the same annulus. Finally, we prove the existence and compactness of the complex Green operator.

Theorem 1.1 and the results of Raich-Straube [10] and Fu-Straube [11, 12] immediately allow us to characterize compactness of the complex Green operator on smooth bounded locally convexifiable domains.

Theorem 1.2. Let Ω be a smooth bounded locally convexiable domain in a Stein manifold X of dimension n and let $1 \leq q \leq n-2,\ n \geq 3$. Then, the following statements are equivalent:

(i) The complex Green operator $G_q$is compact.

(ii) Both $G_q$and G_{n-1-q} are compact.

(iii) The $\bar{\partial}_b$-Neumann operators N_q and N_{n-1-q} are compact.

(iv) $b\Omega$ satisfies both (P_q) and (P_{n-1-q}).

(v) $b\Omega$ does not contain (germs of) complex varieties of dimension q nor of dimension $(n-1-q)$.
In fact, on a locally convexifiable domain, compactness of N_q is equivalent to each of (iv) and (v), at level q (see [11, 12]). By Theorem 1.4 in [10], (ii) implies (iii). Also, (iii), (iv), and (v) are equivalent on these domains, and by Theorem 1.1, they imply (ii). (i) and (ii) are equivalent by the symmetry in the form levels for $\bar{\partial}_b$.

2. Basic Properties

Let X be a complex manifold of dimension n with a Hermitian metric σ. Let $\Omega \subset X$ be an open submanifold with smooth boundary $b\Omega$ and defining function ρ so that $|\partial\rho| = 1$ on $b\Omega$. Denote by $L_1, \dots, L_n$, a C^∞ special boundary coordinate chart in a small neighborhood U of some point $z_0 \in b\Omega$, i.e., $L_i \in T^{1,0}$on $U \cap \bar{\Omega}$ with L_i tangential for $1 \le i \le n-1$ and $< L_i,\ L_j >= \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Note that $L_i(\rho) = 0$ for $1 \le i \le n-1$ and $L_n(\rho) = 1$. Denote $\bar{L}_1, \dots, \bar{L}_n$, the conjugate of $L_1, \dots, L_n$, respectively; these form an orthonormal basis of $T^{1,0}$ on U. The dual basis of (1,0) forms are $\omega^1, \dots, \omega^n = \sqrt{2}\,\partial\rho$. Set φ_{ij} and ρ_{ij} to be the coefficients of $\partial\bar{\partial}\varphi$ and $\partial\bar{\partial}\rho$, respectively. That is $\partial\bar{\partial}\varphi = \sum_{j,k=1}^{n} \varphi_{jk}\ \omega^j \wedge \bar{\omega}^k$ and $\partial\bar{\partial}\rho = \sum_{j,k=1}^{n} \rho_{jk}\ \omega^j \wedge \bar{\omega}^k$. For two (p,q)-forms $f = \sum_{I,J} f_{I,J}\,\omega^I \wedge \bar{\omega}^J$ and $g = \sum_{I,J} g_{I,J}\,\omega^I \wedge \bar{\omega}^J$, where $0 \le p,q \le n$, $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are multiindices and $\omega^I = \omega^1 \wedge \dots \wedge \omega^p$, $\bar{\omega}^J = \bar{\omega}^1 \wedge \dots \wedge \bar{\omega}^q$, one defines $(f,g) = \sum_{I,J} f_{I,J}\,\bar{g}_{I,J}$. Let $A_{z,(p,q)}$be the space of (p,q)-forms at z equipped with the standard Hermitian metric and let $C^\infty_{\mathrm{p,q}}(\Omega)$ be the space of complex-valued differential forms of class $C^\infty$and of type (p,q) on Ω. The Cauchy-Riemann operator $\bar{\partial}: C^\infty_{\mathrm{p,q-1}}(\Omega) \longrightarrow C^\infty_{\mathrm{p,q}}(\Omega)$ is defined by

$$\bar{\partial} f = \sum_{I,K}\sum_{j=1}^{n} \bar{L}_k f_{I,J}\ \bar{\omega}^k \wedge \omega^I \wedge \bar{\omega}^J + \cdots, \tag{2.1}$$

where the dots refer to terms of order zero in f. Let $D_{p,q}(U)$ be the space of (p,q)-forms f on U such that

$$f_{I,J} = 0 \text{ on } b\Omega \text{ when } n \in J. \tag{2.2}$$

Thus, for forms $f \in D\,(U)$,

$$\vartheta f = (-1)^{p-1} \sum_{I,K}\sum_{j=1}^{n} \delta_j^{\varphi} f_{I,jK}\omega^I \wedge \bar{\omega}^K + \cdots, \tag{2.3}$$

where $\delta_j^{\varphi} = e^{\varphi} L_j(e^{-\varphi})$ and the dots indicate terms in which no $f_{I,J}$ and $f_{I,jK}$are differentiated and which do not involve φ. We use $L^2_{\mathrm{p,q}}(\Omega, \varphi)$ to denote the space of (p,q)-forms with coefficients in the space of square integrable functions $L^2(\Omega)$ with respect to the weighted function $e^{-\varphi}$. For a real function φ in class C^2, the weighted L^2_φ-inner product and norm is defined by

$$< f, g >_{\varphi} = \int_{\Omega} (f, g)\, e^{-\varphi} dV \quad and \quad \|f\|_{\varphi}^{2} = < f, f >_{\varphi},$$

where dV is the volume element induced by the Hermitian metric. Let $\bar{\partial}: L^2_{\mathrm{p,q}}(\Omega) \longrightarrow l^2_{\mathrm{p,q+1}}(\Omega)$ be the maximal closure of the Cauchy-Riemann operator and $\bar{\partial}^*$ be its Hilbert space adjoint. The space of the harmonic (p, q)-forms is defined by $\aleph_{p,q}(\Omega) = \{u \in D_{p,q}(\Omega): \bar{\partial} u = \bar{\partial}^* u = 0\}$

The $\bar{\partial}$-Neumann operator $N_q: L^2_{\mathrm{p,q}}(\Omega) \longrightarrow l^2_{\mathrm{p,q}}(\Omega)$ is the inverse of the restriction of $\Box_q$ to $(\aleph_{p,q}(\Omega))^{\perp}$, where $\Box_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is the complex Laplacian operator. For nonnegative integer k, one defines the Sobolev space

$$\mathrm{W}^k_{\mathrm{p,q}}(\Omega) = \left\{f \in L^2_{\mathrm{p,q}}(\Omega): \|f\|_{W^k(\Omega)} < +\infty\right\},$$

where the Sobolev norm of order k is defined by

$$\|f\|^2_{W^k(\Omega)} = \int_{\Omega} \sum_{|\alpha| \le k} |D^{\alpha} f|^2 \, dV$$

Where $D^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots\ldots \left(\frac{\partial}{\partial x_{2n}}\right)^{\alpha_{2n}}$, for $= (\alpha_1, \ldots, \alpha_{2n})$, $|\alpha| = \sum \alpha_j$, $x_1, \ldots, x_{2n}$ are real coordinates for Ω.

Definition 2.1. A compactness estimate is said to hold for the $\bar{\partial}$-Neumann problem on Ω if for every $M > 0$ there is a constant $C_M > 0$ such that the estimate

$$\|u\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega)} \le MQ(u, u) + C_M \|u\|^2_{\mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega)}$$

is valid for all $u \in dom\, \bar{\partial} \cap \mathrm{dom}\, \bar{\partial}^* \subset \mathrm{L}^2_{\mathrm{p,q}}(\Omega)$. Here $Q(u, u)$ refers to the form $Q(u, u) = \|u\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega)} + \left\|\bar{\partial} u\right\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega)} + \left\|\bar{\partial}^* u\right\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega)}$ and $\|u\|_{\mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega)}$ refers to the Sobolev norm of order -1 for forms on Ω.

Definition 2.2. The boundary $b\Omega$ satisfies (P_q) if for every positive number M, there exists $U_M \subset b\Omega$, $\lambda_M \in C^2(U_M)$ so that for all $z \in U_M$ and $v \in A_{z,(p,q)}$,

(1) $0 \le \lambda_M \le 1$,

(2) $\sum_{j,k=1}^{n} \lambda_{jk}(z)\ (z, v) \ge M|v(z)|^2$, for all $z \in b\Omega$, where $\lambda_{jk}(z), j, k = 1,2, \ldots, n$ is defined by $\partial\bar{\partial}\lambda(z) = \sum_{j,k=1}^{n} \lambda_{jk}(z)\ \omega^j \wedge \bar{\omega}^k$.

Definition 2.3. For any $u \in \mathrm{L}^2_{\mathrm{p,q-1}}(\mathrm{b}\Omega)$, if for some $\alpha \in \mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$, we have

$$\int_{\mathrm{b}\Omega} u \wedge \bar{\partial} f = (-1)^{p+q} \int_{\mathrm{b}\Omega} \alpha \wedge f,$$

for every $f \in C^{\infty}_{n-p,n-1-q}(X)$, then u is said to be in dom$\bar{\partial}_b$ and $\bar{\partial}_b u = f$.

The $\bar{\partial}_b$ operator is a closed, densely defined, linear operator from $\mathrm{L}^2_{\mathrm{p,q-1}}(\mathrm{b}\Omega)$ to $\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$, where $0 \le p \le n$, $1 \le q \le n-1$.

Definition 2.4. Dom $\bar{\partial}_b^*$ is the subset of $\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$ composed of all forms f for which there exists a constant $C > 0$ such that

$$\left|< f, \bar{\partial}_b u >_{\mathrm{b}\Omega}\right| \le C\, \|u\|_{b\Omega},$$

for all $u \in dom\ \bar{\partial}_b$. For all $f \in dom\ \bar{\partial}_b^*$, let $\bar{\partial}_b^* f$ be the unique form in $L^2_{p,q}(b\Omega)$ satisfying

$$< \bar{\partial}_b^* f, u >_{b\Omega} = < f, \bar{\partial}_b u >_{b\Omega},$$

for all $u \in$ dom $\bar{\partial}_b$. Let $\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b : dom\ \Box_b \longrightarrow L^2_{p,q}(b\Omega)$ be the $\bar{\partial}_b$-Laplacian operator defined on $dom\ \Box_b = \{u \in L^2_{p,q}(b\Omega) : u \in dom\ \bar{\partial}_b \cap dom\ \bar{\partial}_b^*;\ \bar{\partial}_b \in dom\ \bar{\partial}_b^*\ and\ \bar{\partial}_b^* \in dom\ \bar{\partial}_b\}$. The space of harmonic forms $\aleph^b_{p,q}(b\Omega)$ is denoted by

$$\aleph^b_{p,q}(b\Omega) = \{u \in D_{p,q}(\Omega) : \bar{\partial}_b u = \bar{\partial}_b^* u = 0\}.$$

Following [13]; Proposition 1.3, the $\bar{\partial}_b$-Laplacian operator is a closed, densely defined self-adjoint operator. One defines the complex Green operator

$$G_q : L^2_{p,q}(b\Omega) \longrightarrow dom\ \Box_b$$

as follows: If $\alpha \in \aleph^b_{p,q}(\Omega)$, set $G_q \alpha = 0$. If $\alpha \in Rang\ \Box_q$ define $G_q \alpha = \beta$, where β is the unique solution of $\Box_b \beta = \alpha$ with $\beta \perp \aleph^b_{p,q}(\Omega)$, and we extend G_q by linearity. It is easy to see that G_q is a bounded operator.

3. Proof of Theorem 1.1

This section deals with the main result of this article. The proof of Theorem 1.1 can be obtained in several steps.

3.1. Compactness estimates for the $\bar{\partial}$ -Neumann problem

Lemma 3.1. Let X be a complex manifold of dimension $n \geq 3$. Let Ω and Ω_1 are two bounded pseudoconvex domains such that $\Omega \subset \Omega_1 \subset X$. We call $\Omega^+ = \Omega_1 \backslash \bar{\Omega}$ an 'annulus'. Assume the outer boundary of Ω^+ satisfies property (P_q), and the inner boundary satisfies property (P_{n-q-1}). Then, the compactness estimates for (p, q)-forms, $0 < q < n-1$, holds for the $\bar{\partial}$-Neumann problem on Ω^+.

Proof. By a partition of unity argument, we need to prove this lemma for supported in a small neighborhood of the boundary since Q is elliptic in the interior. Let u be supported in a small neighborhood U of $b\Omega_1$ and let ρ be a defining function of Ω_1. Let $z_0 \in b\Omega_1$ and let M be a positive constant. Since Ω^+ satisfies (P_q) at $b\Omega_1$, there is a plurisubharmonic function $\lambda^1 \in C^\infty(\bar{\Omega}_1)$ with $0 \leq \lambda^1 \leq 1$, such that for all $z \in b\Omega_1$,

$$\sum_{j,k=1}^{n} \lambda^1_{jk}(z)\ t_j \bar{t}_k \geq M|t|^2 . \tag{3.1}$$

By continuity of the second derivative of λ^1, there exists a neighborhood U (dependent of M) of z_0 such that $\lambda^1 \in C^\infty(U)$ and (3.1) holds for all $z \in U \cap \bar{\Omega}_1$. Let Au denotes the sum in (2.1), then one obtains

$$\|\mathrm{A}u\|_{\varphi}^{2}=\sum_{I,J}\sum_{j=1}^{n}\left\|\bar{L}_j u_{I,J}\right\|_{\varphi}^{2}-\sum_{I,K}\sum_{j,k=1}^{n}<\bar{L}_k u_{I,jK}\,,\bar{L}_j\bar{u}_{I,kK}>_{\varphi}. \tag{3.2}$$

Let Bu denote the sum in (2.3), one obtains

$$\|\mathrm{B}u\|_{\varphi}^{2}=(-1)^{p-1}\sum_{I,K}\sum_{j=1}^{n}\delta_j^{\varphi}u_{I,jK}\omega^I\wedge\bar{\omega}^K+(-1)^{p-1}\sum_{I,K}\sum_{j=1}^{n}(\bar{L}_j\varphi)u_{I,jK}\omega^I\wedge\bar{\omega}^K. \tag{3.3}$$

Since Au and Bu differ from $\bar{\partial}$ and $\bar{\partial}^*$ by terms of order 0 which do not depend on φ, it follows from (3.2) and (3.3) and by direct calculation that

$$\frac{1}{18}\sum_{I,K}\sum_{j,k=1}^{n}\left(\lambda_{jk}u_{I,jK}\,,\bar{u}_{I,kK}\right)_{\varphi}+\int_{U\cap b\Omega}\sum_{I,K}\sum_{j,k=1}^{n}\rho_{jk}u_{I,jK}\;\bar{u}_{I,kK}\,e^{-\varphi}dS+$$

$$\frac{1}{2}\sum_{I,K}\sum_{j=1}^{n}\left\|\bar{L}_j u_{I,J}\right\|_{\varphi}^{2}\le C\,\|u\|_{\varphi}^{2}+2\left\|\bar{\partial}u\right\|_{\varphi}^{2}+4\left\|\bar{\partial}^*u\right\|_{\varphi}^{2}. \tag{3.4}$$

Since $\varphi=\chi(\lambda^1)\in C^{\infty}(\bar{\Omega}_1)$ satisfies $\frac{1}{2}\le e^{-\varphi}\le 1$, it follows from (3.4) that

$$\frac{1}{36}\sum_{I,K}\sum_{j,k=1}^{n}\left(\lambda_{jk}u_{I,jK}\,,\bar{u}_{I,kK}\right)_{\varphi}+\frac{1}{2}\int_{U\cap b\Omega_1}\sum_{I,K}\sum_{j,k=1}^{n}\rho_{jk}u_{I,jK}\;\bar{u}_{I,kK}\,e^{-\varphi}dS$$

$$+\frac{1}{4}\sum_{I,J}\sum_{j=1}^{n}\left\|\bar{L}_j u_{I,J}\right\|_{\varphi}^{2}\le C\,\|u\|_{\varphi}^{2}+2\left\|\bar{\partial}u\right\|_{\varphi}^{2}+4\left\|\bar{\partial}^*u\right\|_{\varphi}^{2}. \tag{3.5}$$

Since Ω^+ is pseudoconvex domains at $b\Omega_1$, it follows from (3.5) that

$$\frac{M}{18}\int_{U\cap\Omega_1}|u|^2dV\le C\,\|u\|^2+2\left\|\bar{\partial}u\right\|^2+4\left\|\bar{\partial}^*u\right\|^2.$$

Let $S_{\delta_1}=\{z\in X\colon -\delta_1\le\rho(z)\le 0\}$. Since $b\Omega_1$ is compact, by a finite covering $\{U_v\}_{v=1}^{m}$ of $b\Omega_1$ by neighborhoods U_v as above, there exists a positive number δ_1 (depend on M) such that

$$M\int_{S_{\delta_1}}|u|^2dV\le C\left(Q(u,u)+\|u\|_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)}^2\right). \tag{3.6}$$

Since $b\Omega$ satisfies property (P_q) at $b\Omega$, there is a plurisubharmonic function $\lambda^2\in C^{\infty}(\bar{\Omega})$ with $0\le\lambda^2\le 1$, such that for all $z\in U\cap\bar{\Omega}$,

$$\sum_{j,k=1}^{n}\lambda_{jk}^2(z)\,t_j\bar{t}_k\ge M|t|^2\,. \tag{3.7}$$

For every$(t_n,\dots,t_n)\in\mathbb{C}^n$. By continuity of the second derivative of λ^2, there exists a neighborhood U (dependent of M) of z_0 such that $\lambda^2\in C^{\infty}(U)$ and (3.7) holds for all $z\in U\cap\bar{\Omega}$. Let $\lambda=-\lambda^2$ and let $\varphi=\chi(\lambda^2)\in C^{\infty}(\bar{\Omega})$. Notice that $-1\le\lambda\le 0$ and $\varphi=\chi(\lambda)=\frac{1}{6}e^{\lambda}$. Thus $\frac{1}{6}e^{-\lambda}\le e^{-\varphi}\le e^{-\lambda}$. Hence (3.4) implies

$$\frac{1}{108}\sum_{I,K}\sum_{j,k=1}^{n}\left(\lambda_{jk}u_{I,jK}\,,\bar{u}_{I,kK}\right)_{\lambda}+\frac{1}{6}\int_{U\cap b\Omega}\sum_{I,K}\sum_{j,k=1}^{n}\rho_{jk}u_{I,jK}\ \bar{u}_{I,kK}\,e^{-\lambda}dS$$

$$+\frac{1}{12}\sum_{I,J}\sum_{j=1}^{n}\left\|\bar{L}_{j}u_{I,J}\right\|_{\lambda}^{2}\leq C'\|u\|_{\lambda}^{2}+2\left\|\,\bar{\partial}u\right\|_{\lambda}^{2}+4\left\|\bar{\partial}^{*}u\right\|_{\lambda}^{2}. \quad (3.8)$$

Set $\|\bar{L}\,u\,\|_{\lambda}^{2}=\sum_{I,J}\sum_{j=1}^{n}\left\|\bar{L}_{j}u_{I,J}\right\|_{\lambda}^{2}+\|u\|_{\lambda}^{2}.$

Then we get that if $j<n$

$$\left\|\bar{L}_{j}u_{I,J}\right\|_{\lambda}^{2}=\left\|\delta_{j}^{\lambda}u_{I,jK}\right\|_{\lambda}^{2}+<\lambda_{jj}u_{I,J}\,,\bar{u}_{I,J}>_{\lambda}+\int_{U\cap b\Omega}\rho_{jj}\left|u_{I,J}\right|^{2}e^{-\lambda}dS$$

$$=O(\|\bar{L}\,u\,\|_{\lambda}\|u\,\|_{\lambda}).$$

Thus, for $\alpha>0$, it follows that

$$\sum_{I,J}\sum_{j=1}^{n}\left\|\bar{L}_{j}u_{I,J}\right\|_{\lambda}^{2}$$

$$\geq\frac{1}{1+\alpha}\left\{\sum_{I,K}\sum_{j=1}^{n-1}\left\|\delta_{j}^{\lambda}u_{I,jK}\right\|_{\lambda}^{2}-\int_{U\cap\Omega}\sum_{I,J}\sum_{j=1}^{n-1}\lambda_{jj}\left|u_{I,J}\right|^{2}e^{-\lambda}\,dV\right\}-C_{\alpha}\|u\|_{\lambda}^{2}$$

$$-\frac{\alpha}{1+\alpha}\sum_{I,J}\sum_{j=1}^{n-1}\left\|\bar{L}_{j}u_{I,J}\right\|_{\lambda}^{2}$$

$$-\frac{1}{1+\alpha}\int_{U\cap b\Omega}\sum_{I,J}\sum_{j=1}^{n-1}\rho_{jj}\left|u_{I,J}\right|^{2}e^{-\lambda}\,dS \quad (3.9)$$

From (3.8) and (3.9) and by taking $\alpha=8$ i.e., $12(1+\alpha)=108$, one obtains

$$\frac{1}{108}\left\{\int_{U\cap\Omega}\left(\sum_{I,K}\sum_{j,k=1}^{n}\left(\lambda_{jk}u_{I,jK}\,,\bar{u}_{I,kK}\right)_{\lambda}-\sum_{I,J}\sum_{j=1}^{n}\lambda_{jj}\left|u_{I,J}\right|^{2}e^{-\lambda}\right)dV\right\}$$

$$+\frac{1}{6}\left\{\int_{U\cap b\Omega}\left(\sum_{I,K}\sum_{j,k=1}^{n}\left(\rho_{jk}u_{I,jK}\,,\bar{u}_{I,kK}\right)_{\lambda}-\sum_{I,J}\sum_{j=1}^{n}\rho_{jj}\left|u_{I,J}\right|^{2}e^{-\lambda}\right)dS\right\}$$

$$+\frac{1}{108}\left\{\sum_{I,J}\sum_{j=1}^{n}\left\|\delta_{j}^{\lambda}u_{I,J}\right\|_{\lambda}^{2}dS-\sum_{I,J}\sum_{j=1}^{n}\left\|\bar{L}u_{I,J}\right\|_{\lambda}^{2}\right\}$$

$$\leq C'^{\|u\|_{\lambda}^{2}}+2\left\|\,\bar{\partial}u\right\|_{\lambda}^{2}+4\left\|\bar{\partial}^{*}u\right\|_{\lambda}^{2} \quad (3.10)$$

In the second line of (3.10), the integrand (without the weight factor) is therefore

$$\sum_{I,K}\sum_{j,k=1}^{n}\rho_{jk}u_{I,jK}\,\bar{u}_{I,kK}-\sum_{I,K}\sum_{j=1}^{n}\rho_{jj}\left|u_{I,J}\right|^2$$
$$=\sum_{I,K}\sum_{j,k=1}^{n}\left(\rho_{jk}-\frac{1}{q}\left(\sum_{k=1}^{n-1}\rho_{kk}\right)\delta_{jk}\right)u_{I,jK}\,\bar{u}_{I,kK} \qquad (3.11)$$

This is because every $\left|u_{I,J}\right|^2$ can be written in precisely q ways as $\left|u_{I,jK}\right|^2$. Note that the Hessian of ρ is negative semi definite on the complex tangent space at points of $b\Omega \subset b\Omega^+$. As a result, the second line in (3.10) is nonnegative: the right hand side equals at least $|u|^2$ times the sum of the smallest q eigenvalues of the Hermitian matrix

$$\left(\rho_{jk}-\frac{1}{q}\left(\sum_{k=1}^{n-1}\rho_{kk}\right)\delta_{jk}\right)_{j,k=1}^{n-1}.$$

Such a sum equals minus the trace of $\sum_{k=1}^{n-1}\rho_{kk}$ plus a sum of q eigenvalues of $\left(\rho_{jk}\right)_{j,k=1}^{n-1}$, hence is at least equal to the negative of the sum of the largest $(n-1-q)$ eigenvalues of $\left(\rho_{jk}\right)_{j,k=1}^{n-1}$, and so is nonnegative. For each $z \in b\Omega$, we may diagonalize $\left(\rho_{jk}\right)_{j,k=1}^{n-1}$ under a unitary transformation and the positive semi-definiteness is invariant under such transformation. Thus

$$\left(\rho_{jk}-\frac{1}{q}\left(\sum_{k=1}^{n-1}\rho_{kk}\right)\delta_{jk}\right)_{j,k=1}^{n-1}.$$

is positive semidefinite in $U \cap b\Omega$. Observe that as in (3.11), the integrand (without the weight factor $e^{-\varphi}$) in the first line in (3.10) is then

$$\sum_{I,K}\sum_{j,k=1}^{n}\lambda_{jk}u_{I,jK}\,\bar{u}_{I,kK}-\sum_{I,K}\sum_{j=1}^{n}\lambda_{jj}\left|u_{I,J}\right|^2$$
$$=\sum_{I,K}\sum_{j,k=1}^{n}\left(\lambda_{jk}-\frac{1}{q}\left(\sum_{k=1}^{n-1}\lambda_{kk}\right)\delta_{jk}\right)u_{I,jK}\,\bar{u}_{I,kK}$$

These terms can be estimated by the right hand side of (3.10) plus $C_\lambda\left\|e^{-\lambda/2}u\right\|^2_{W^{-1}_{p,q}(\Omega^+)}$ as in [10]. Thus, estimate (3.10) remains valid when the sums in the first line are restricted so that no normal components of u appear, and the right hand side is augmented by $C_\lambda\left\|e^{-\lambda/2}u\right\|^2_{W^{-1}_{p,q}(\Omega^+)}$. Omitting the nonnegative second and third lines from (3.11) (in its modified form), we obtain for u supported in a special boundary chart:

$$\int_{U\cap\Omega}\sum_{I,K}\sum_{j,k=1}^{n}\left(\lambda_{jk}-\frac{1}{q}\left(\sum_{k=1}^{n-1}\lambda_{kk}\right)\delta_{jk}\right)u_{I,jK}\,\bar{u}_{I,kK}e^{-\lambda}dV$$
$$\leq C\left(\|u\|_{\lambda}^{2}+\|\bar{\partial}u\|_{\lambda}^{2}+\|\bar{\partial}^{*}u\|_{\lambda}^{2}\right)+C_{\lambda}\|e^{-\lambda/2}u\|_{W_{p,q}^{-1}(\Omega^{+})}^{2}. \quad (3.12)$$

We use (3.12) with $\lambda=-\mu_M$ near the support of u, where μ_M satisfies (1) and (2) in Definition 2.2 of (P_{n-1-q}). At a point, the integrand on the left hand side of (3.12) (without the exponential factor) is at least as big as $|u|^2$times the sum of the smallest q eigenvalues of the Hermitian matrix

$$\left((-\mu_M)_{jk}-\frac{1}{q}\left(\sum_{k=1}^{n-1}(\mu_M)_{kk}\right)\delta_{jk}\right)_{j,k=1}^{n-1}.$$

Such a sum equals minus the trace of $\sum_{k=1}^{n-1}(\mu_M)_{kk}$ plus a sum of q eigenvalues of $\left((\mu_M)_{jk}\right)_{j,k=1}^{n-1}$, hence is at least equal to the negative of the sum of the largest $(n-1-q)$ eigenvalues of $\left((\mu_M)_{jk}\right)_{j,k=1}^{n-1}$, which in turn is at least equal to the sum of the smallest $(n-1-q)$ eigenvalues of $\left((\mu_M)_{jk}\right)_{j,k=1}^{n-1}$, (by the Schur majorization theorem ([14], Theorem 4.3.26). That is, the sum is at least equal to the sum of the smallest $(n-1-q)$ eigenvalues of $\left(\frac{\partial^2\mu_M}{\partial z_j\partial\bar{z}_k}\right)_{j,k=1}^{n-1}$, so is at least equal to M. After absorbing the term $C\|u\|_{L_{p,q}^{2}(\Omega)}^{2}$and rescaling M, it follows that

$$M\int_{U\cap\Omega_1}|u|^2dV\leq C\left(Q(u,u)+\|u\|_{L_{p,q}^{2}(\Omega^{+})}^{2}\right)+C_M\|u\|_{W_{p,q}^{-1}(\Omega^{+})}^{2}.$$

Let $S_{\delta_2}=\{z\in X:-\delta_2\leq\rho(z)\leq 0\}$. Since $b\Omega$ is compact, by a finite covering $\{U_\nu\}_{\nu=1}^{m}$ of $b\Omega$ by neighborhoods U_ν as above, there exists a positive number δ_2 (depend on M) such that

$$M\int_{S_{\delta_2}}|u|^2dV\leq C\left(Q(u,u)+\|u\|_{L_{p,q}^{2}(\Omega^{+})}^{2}\right)+C_M\|u\|_{W_{p,q}^{-1}(\Omega^{+})}^{2}. \quad (3.13)$$

Let $S_\delta=S_{\delta_1}\cup S_{\delta_2}$, where $\delta=\min\{\delta_1,\delta_2\}$. Then, by (3.6) and (3.13), one obtains

$$M\int_{S_\delta}|u|^2dV\leq C_1\left(Q(u,u)+\|u\|_{L_{p,q}^{2}(\Omega^{+})}^{2}\right)+C'_M\|u\|_{W_{p,q}^{-1}(\Omega^{+})}^{2}. \quad (3.14)$$

Now, we estimate the integral over $\Omega^{+}\backslash S_d$. Choose $\gamma_\delta\in C_0^\infty(\Omega^{+})$ so that $\gamma_\delta(z)=1$ whenever $\rho(z)\leq-\delta$ and $z\in\Omega^{+}\backslash S_d$. For a constant s still to be determined we have the inequality

$$\|\gamma_\delta u\|_{L_{p,q}^{2}(\Omega^{+})}^{2}\leq s\|\gamma_\delta u\|_{W_{p,q}^{1}(\Omega^{+})}^{2}+\frac{1}{s}\|\gamma_\delta u\|_{W_{p,q}^{-1}(\Omega^{+})}^{2}. \quad (3.15)$$

On the other hand, since Q is elliptic, by Gärding's inequality, there is a constant C_2 depending only on the diameter of the domain Ω^+ such that

$$\begin{aligned}\|\gamma_\delta u\|^2_{W^1_{p,q}(\Omega^+)} &\le C_2\left(Q(\gamma_\delta u,\gamma_\delta u)+\|\gamma_\delta u\|^2_{L^2_{p,q}(\Omega^+)}\right)\\ &\le 2C_2(\left\|\gamma_\delta(\bar\partial u)\right\|^2_{L^2_{p,q}(\Omega^+)}+\left\|\gamma_\delta(\bar\partial^* u)\right\|^2_{L^2_{p,q}(\Omega^+)}\\ &+\left\|[\gamma_\delta,\bar\partial]u\right\|^2_{L^2_{p,q}(\Omega^+)}+\left\|[\gamma_\delta,\bar\partial^*]u\right\|^2_{L^2_{p,q}(\Omega^+)}+\|\gamma_\delta u\|^2_{L^2_{p,q}(\Omega^+)}\\ &\le \left\|\bar\partial u\right\|^2_{L^2_{p,q}(\Omega^+)}+\left\|\bar\partial^* u\right\|^2_{L^2_{p,q}(\Omega^+)}+C_\delta\|u\|^2_{L^2_{p,q}(\Omega^+)}. \qquad (3.16)\end{aligned}$$

Since the sum of the commutator terms is bounded by $C_\delta\|u\|^2$ for some constant C_3 dependent of δ. From (3.15) and (3.16), for a suitable choice of s small, we get

$$\begin{aligned}\|\gamma_\delta u\|^2_{L^2_{p,q}(\Omega^+)} &\le 2C_2 s\left(Q(u,u)+\|u\|^2_{L^2_{p,q}(\Omega^+)}\right)\\ &+2C_2C_3 s\|u\|^2_{L^2_{p,q}(\Omega^+)}+\frac{1}{s}\|u\|^2_{W^{-1}_{p,q}(\Omega^+)}. \qquad (3.17)\end{aligned}$$

By combining (3.14) and (3.17), one obtains

$$\begin{aligned}M\|u\|^2_{L^2_{p,q}(\Omega^+)} &\le \int_{S_\delta}|u|^2 dV+M\|\gamma_\delta u\|^2_{L^2_{p,q}(\Omega^+)}\\ &\le (C_1+2C_2 sM)Q(u,u)+(C_1+2C_2 sM+2C_2C_3 sM)\|u\|^2_{L^2_{p,q}(\Omega^+)}\\ &+\left(C'_M+\frac{M}{s}\right)\|u\|^2_{W^{-1}_{p,q}(\Omega^+)}.\end{aligned}$$

Now, we choose small s and large M so that $\frac{C_1}{M}+2C_2 s+2C_2C_3 s<\frac{1}{2}$ and so that $\frac{C_1}{M}+2C_2 s+2C_2C_3 s<\frac{\varepsilon}{2}$. Then, one obtains the compactness estimate

$$\|u\|^2_{L^2_{p,q}(\Omega^+)} \le \varepsilon Q(u,u)+C_\varepsilon\|u\|^2_{W^{-1}_{p,q}(\Omega^+)}, \qquad (3.18)$$

where $C_\varepsilon=2\left(\frac{C'_M}{M}+\frac{1}{s}\right)$. Thus, the proof follows

3.2. Compactness of the $\bar\partial$-Neumann operator

An immediate consequence of the basic estimate (3.18) is the following result whose proof can be found in Hörmander [15]. The closed range property of $\bar\partial$ is observed in this section by combining the compactness estimate with results in Hörmander [15].

Lemma 3.2. Let X be a complex manifold of dimension n. Let Ω and Ω_1 are two bounded pseudoconvex domains such that $\bar\Omega\subset\Omega_1\subset X$. We call $\Omega^+=\Omega_1\backslash\bar\Omega$ an 'annulus'. Assume the outer boundary of Ω^+ satisfies property (P_q), and the inner boundary satisfies property (P_{n-1-q}). Then, for $1\le q\le n-2,\ n\ge 3$, we have

(i) The space of harmonic forms $\aleph_{p,q}(\Omega^+)$ is finite dimensional.

(ii) The operator $\bar\partial$ has closed range in $L^2_{p,q}(\Omega^+)$ and $L^2_{p,q+1}(\Omega^+)$.

(iii) The operator $\bar\partial$ has closed range in $L^2_{p,q}(\Omega^+)$ and $L^2_{p,q-1}(\Omega^+)$.

(iv) The $\bar{\partial}$-Neumann operators $N_q^{\Omega^+}$ is compact from $L^2_{p,q}(\Omega^+)$ to itself.

(v) The canonical solution operators to $\bar{\partial}$ given by $\bar{\partial}^* N_q^{\Omega^+} : L^2_{p,q}(\Omega^+) \longrightarrow L^2_{p,q-1}(\Omega^+)$ and $N_{q+1}^{\Omega^+}\bar{\partial}^* : L^2_{p,q+1}(\Omega^+) \longrightarrow L^2_{p,q}(\Omega^+)$ are compact.

(vi) The embedding of the space dom $\bar{\partial} \cap$ dom $\bar{\partial}^*$, provided with the graph norm $\|u\|^2_{L^2_{p,q}(\Omega)} + \|\bar{\partial}u\|^2_{L^2_{p,q}(\Omega)} + \|\bar{\partial}^* u\|^2_{L^2_{p,q}(\Omega)}$ into $L^2_{p,q}(\Omega^+)$ is compact.

Proof. Inequality (3.18) implies that, from every sequence $\{U_\nu\}_{\nu=1}^m$ in dom $\bar{\partial} \cap$ dom $\bar{\partial}^*_{\lambda_M}$ with $\|u_\nu\|_{\lambda_M}$ bounded and $\bar{\partial}u_\nu \longrightarrow 0$, $\bar{\partial}^*_{\lambda_M}u_\nu \longrightarrow 0$, one can extract a subsequence which converges in (weighted) $L^2_{p,q}(\Omega^+)$. It suffices to find a subsequence which converges in $W^{-1}_{p,q}(\Omega^+)$ (using that $L^2_{p,q}(\Omega^+) \longrightarrow W^{-1}_{p,q}(\Omega^+)$ is compact); (3.18) implies that such a subsequence is Cauchy (hence convergent) In $L^2_{p,q}(\Omega^+)$. General Hilbert space theory (Hörmander [15]; Theorems 1.1.3 and 1.1.2) now gives that $\aleph_{p,q}(\Omega^+)$ is finite dimensional and that $\bar{\partial} : L^2_{p,q}(\Omega^+) \longrightarrow L^2_{p,q+1}(\Omega^+)$ and $\bar{\partial}^*_{\lambda_M} : L^2_{p,q}(\Omega^+) \longrightarrow L^2_{p,q-1}(\Omega^+)$ have closed range. Therefore, we have the estimate

$$\|u\|^2_{L^2_{p,q}(\Omega^+)} \leq \varepsilon Q(u,u) + C_\varepsilon \|H_q u\|^2_{L^2_{p,q}(\Omega^+)}, \tag{3.19}$$

for $u \in$ dom $\bar{\partial} \cap$dom $\bar{\partial}^*$. This estimate implies the existence of $N_q$as a bounded operator on $L^2_{p,q}(\Omega^+)$ that inverts $\Box_q$ on $\aleph_{p,q}(\Omega^+)$. Moreover, the range of $\bar{\partial} : L^2_{p,q}(\Omega^+) \longrightarrow L^2_{p,q+1}(\Omega^+)$ has finite codimension in ker $\bar{\partial} \subset L^2_{p,q}(\Omega^+)$, because $\aleph_{p,q}(\Omega^+)$is finite dimensional). But the (unweighted) orthogonal complement of this range in ker $\bar{\partial} \subset L^2_{p,q}(\Omega^+)$ equals $\aleph_{p,q}(\Omega^+)$, which is therefore finite dimensional as well.

To see the compactness of $N_q^{\Omega^+}$, it suffices to show compactness on $\aleph_{p,q}(\Omega^+)$ (since $N_q^{\Omega^+}$ is zero on $\aleph_{p,q}(\Omega^+)$). When $u \in \aleph^\perp_{p,q}(\Omega^+)$, we have from (3.19) (since $N_q^{\Omega^+}u \in \aleph^\perp_{p,q}(\Omega^+)$)

$$\begin{aligned}\|N_q^{\Omega^+}f\|^2_{L^2_{p,q}(\Omega^+)} &\leq \|\bar{\partial}N_q^{\Omega^+}f\|^2_{L^2_{p,q}(\Omega^+)} + \|\bar{\partial}^*N_q^{\Omega^+}f\|^2_{L^2_{p,q}(\Omega^+)} \\ &= \|(\bar{\partial}^*N_{q+1}^{\Omega^+})^*f\|^2_{L^2_{p,q}(\Omega^+)} + \|\bar{\partial}^*N_q^{\Omega^+}f\|^2_{L^2_{p,q}(\Omega^+)}. \end{aligned} \tag{3,20}$$

Thus, we only need to show that both $\bar{\partial}^*N_q^{\Omega^+}$ and $\bar{\partial}^*N_{q+1}^{\Omega^+}$ are compact. Now $\bar{\partial}^*N_{q+1}^{\Omega^+}$ gives the weighted norm minimizing solution to $\bar{\partial}u = f$ when $f \in Im\,\bar{\partial} \subset L^2_{p,q+1}(\Omega^+)$. For such f, (3.18) therefore implies (with constants independent of M)

$$\begin{aligned}\|\bar{\partial}^*N_{q+1}^{\Omega^+}f\|^2_{L^2_{p,q}(\Omega^+)} &\leq \|\bar{\partial}^*_{\lambda_M}N_{\lambda_M,q+1}^{\Omega^+}f\|^2_{L^2_{p,q}(\Omega^+)} \\ &\leq C\|\bar{\partial}^*_{\lambda_M}N_{\lambda_M,q+1}^{\Omega^+}f\|^2_{\lambda_M}\end{aligned}$$

$$\leq \frac{C}{M}\|f\|^2_{\lambda_M} + C_M\left\|\bar{\partial}^*_{\lambda_M} N^{\Omega^+}_{\lambda_M,q+1} f\right\|^2_{\mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega^+)}$$
$$\leq \frac{C}{M}\|f\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)} + C_M\left\|\bar{\partial}^*_{\lambda_M} N^{\Omega^+}_{\lambda_M,q+1} f\right\|^2_{\mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega^+)}. \quad (3.21)$$

Because C is independent of M and $\bar{\partial}^*_{\lambda_M} N^{\Omega^+}_{\lambda_M,q+1} : \mathrm{L}^2_{\mathrm{p,q+1}}(\Omega^+) \longrightarrow \mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega^+)$ is compact ($\mathrm{L}^2(\Omega^+)$ imbed compactly into $\mathrm{W}^{-1}(\Omega^+)$), (3.21) implies that $\bar{\partial}^* N_{q+1}$ is compact on Im $\bar{\partial}$ ([16], Lemma 2.1, [17], Proposition V.2.3). But on the orthogonal complement of Im $\bar{\partial}$, $\bar{\partial}^* N^{\Omega^+}_{q+1} = 0$, and so $\bar{\partial}^* N^{\Omega^+}_{q+1}$ is compact from $\mathrm{L}^2_{\mathrm{p,q+1}}(\Omega^+)$ to $\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)$. To estimate $\bar{\partial}^* N_q$, we cannot invoke (3.18) directly (because $\bar{\partial}^* N_q$ is a $(q-1)$-form), and an additional step is needed. We have (again for $f \in Im\,\bar{\partial} \subset \mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)$)

$$\left\|\bar{\partial}^* N^{\Omega^+}_{q+1} f\right\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)} = < \bar{\partial}\bar{\partial}^*_{\lambda_M} N^{\Omega^+}_{\lambda_M,q} f, N^{\Omega^+}_{\lambda_M,q} f >_{\lambda_M}$$
$$= < f, N^{\Omega^+}_{\lambda_M,q} f >_{\lambda_M}$$
$$\leq \frac{2C}{M}\|f\|^2_{\lambda_M} + \frac{M}{2C}\left\|N^{\Omega^+}_{\lambda_M,q} f\right\|^2_{\lambda_M}$$
$$\leq \frac{2C}{M}\|f\|^2_{\lambda_M} + \frac{1}{2}\left\|\bar{\partial}^*_{\lambda_M} N^{\Omega^+}_{\lambda_M,q} f\right\|^2_{\lambda_M} + C_M\left\|N^{\Omega^+}_{\lambda_M,q+1} f\right\|^2_{\mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega^+)}.$$

Here we have used that $\bar{\partial} f = 0$ and that $f \perp \aleph_{p,q}(\Omega^+)$ (since $f \in Im\,\bar{\partial} \subset \mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)$) in the equality in the second line, the inequality $|ab| \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$, and (3.18) for the last estimate. The middle term in the last line can now be absorbed, and combining the resulting estimate with

$$\left\|\bar{\partial}^* N_q f\right\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)} \leq \left\|\bar{\partial}^*_{\lambda_M} N^{\Omega^+}_{\lambda_M,q} f\right\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)}$$

gives an analogue of (3.21). The rest of the argument is the same as above. (vi) follows from (3.18) as in Lemma 1.1 in [12].

Now, we establish the global regularity for N. From the estimate (3.18) we can derive a priori estimates for N in the Sobolev k -space.

Corollary 3.3. A compactness estimate (3.18) implies boundedness of the $\bar{\partial}$-Neumann operator N_q in $W^k_{p,q}(\Omega^+)$ for any $k > 0$.

Proof. By a standard fact of elliptic regularization, one sees that the global regularity for the $\bar{\partial}$-Neumann operator holds if

$$\|u\|^2_{\mathrm{W}^{\mathrm{k}}_{\mathrm{p,q}}(\Omega^+)} \leq \|\Box u\|^2_{\mathrm{W}^{\mathrm{k}}_{\mathrm{p,q}}(\Omega^+)}, \quad (3.22)$$

for any $u \in C^\infty_{\mathrm{p,q}}(\Omega^+) \cap$ dom $\Box$, and for any positive integer k. Moreover, since the operator $\Box$, it is non-characteristic with respect to the boundary. Hence

$$\|u\|^2_{\mathrm{W}^{\mathrm{k}}_{\mathrm{p,q}}(\Omega^+)} \leq \|u\|^2_{\mathrm{W}^{\mathrm{k-2}}_{\mathrm{p,q}}(\Omega^+)} + \left\|\Lambda^{k-1} D u\right\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)}, \quad (3.23)$$

where Λ is the tangential differential operator of order k. By (3.18) we have

$$\|D\Lambda^{-1} u\|^2_{\mathrm{L}^2_{\mathrm{p,q}}(\Omega^+)} \leq Q(u,u) + C\|u\|^2_{\mathrm{W}^{-1}_{\mathrm{p,q}}(\Omega^+)}.$$

In fact, it follows by the non-characteristic with respect to the boundary of $\bar{L}_n$; the operator D can be understood as D_r or Λ.
Now we estimate the last term of (3.23), we have

$$\begin{aligned}
\|\Lambda^{k-1}Du\|^2_{L^2_{p,q}(\Omega^+)} &\le \|D\Lambda^{-1}\Lambda^k u\|^2_{L^2_{p,q}(\Omega^+)} + C\|u\|^2_{W^{k-1}_{p,q}(\Omega^+)}\\
&\le Q(\Lambda^k u, \Lambda^k u) + C\|u\|^2_{W^{k-1}_{p,q}(\Omega^+)}\\
&\le <\Lambda^k \Box u, \Lambda^k u>_{L^2_{p,q}(\Omega^+)} + \|[\bar{\partial}, \Lambda^k]u\|^2_{L^2_{p,q}(\Omega^+)} + \|[\bar{\partial}^*, \Lambda^k]u\|^2_{L^2_{p,q}(\Omega^+)}\\
&+\|[\bar{\partial}^*, [\bar{\partial}, \Lambda^k]]u\|^2_{L^2_{p,q}(\Omega^+)} + \|[\bar{\partial}, [\bar{\partial}^*, \Lambda^k]]u\|^2_{L^2_{p,q}(\Omega^+)} + C\|u\|^2_{W^{k-1}_{p,q}(\Omega^+)}\\
&\le \|\Lambda^k \Box u\|^2_{L^2_{p,q}(\Omega^+)} + \|\Lambda^{k-1}Du\|^2_{L^2_{p,q}(\Omega^+)} + \|\Lambda^{k-2}D^2 u\|^2_{L^2_{p,q}(\Omega^+)} + C\|u\|^2_{W^{k-1}_{p,q}(\Omega^+)}\\
&\le \|\Box u\|^2_{W^k_{p,q}(\Omega^+)} + \|\Lambda^{k-1}Du\|^2_{L^2_{p,q}(\Omega^+)} + C\|u\|^2_{W^{k-1}_{p,q}(\Omega^+)},
\end{aligned}$$

where the second inequality follows by (3.18). Then the term $\|\Lambda^{k-1}Du\|^2_{L^2_{p,q}(\Omega^+)}$ can be absorbed by the left-hand side term. By induction method, one obtains the estimate (3.22).

3.3. Compactness of the canonical solution operator

In this subsection, we produce a compact solution operator for $\bar{\partial}_b$ on the annulus between two pseudoconvex domains in a Stein manifold. To do so, we follow Shaw ([6]) in representing a $\bar{\partial}_b$-closed form u on the boundary as the difference of two $\bar{\partial}$-closed forms, α^- on Ω and α^+ on the complement: $= \alpha^+ - \alpha^-$. Then, roughly speaking, property (P_q) lets us solve the equation $\bar{\partial}_b\beta^- = \alpha^-$on Ω, with suitable compactness estimates, while property (P_{n-1-q}) let's us do the same for $\bar{\partial}_b\beta^+ = \alpha^+$ on an appropriate 'annular' region surrounding $\bar{\Omega}$.

Lemma 3.4. Let X be a Stein manifold of dimension n and let Ω be a bounded pseudoconvex domain with smooth boundary in X . If $b\Omega$ satisfies both (P_q) and (P_{n-1-q}), then, for every $1 \le q \le n-2,\ n \ge 3$, there exists a compact solution operator $S: L^2_{p,q}(b\Omega) \cap \ker(\bar{\partial}_b) \to L^2_{p,q-1}(b\Omega)$
such that $\bar{\partial}_b S = I$.

Proof. By embedding X into $\mathbb{C}^{2n+1}$, we can pullback a ball containing the image of $b\Omega$ to obtain a strictly pseudoconvex set B such that $\bar{\Omega} \subset B$. Let $\Omega^+ = B\backslash\bar{\Omega}$. In [18] a Martinelli-Bochner-Koppelman type kernel constructed for Stein manifolds, and in [19] the transformation induced by this kernel satisfies a jump formula. As a result, there exists an integral kernel $K_q(\zeta, z)$ of type (p, q) at z and $(n-p, n-1-q)$ at ζ satisfying a Martinelli-Bochner-Koppelman formula such that one can define

$$\int_M K_q(\zeta, z) \wedge \alpha(\zeta) = \begin{cases} \alpha^+(z) & if \quad z \in \Omega^+ \\ \alpha^-(z) & if \quad z \in \Omega, \end{cases}$$

Where $\alpha^+(z) = K^+\alpha(z) \quad if \ z \in \Omega^+$ and $\alpha^-(z) = K^-\alpha(z) \quad if \ z \in \Omega$ if $z \in \Omega$ (see [18]; Section 2.3). Let $\alpha \in C^\infty_{p,q}(b\Omega) \cap \ker \bar{\partial}$ and let k be a fixed nonnegative integer. From [20], Lemma 9.3.5, there exist $\bar{\partial}$-closed forms $\alpha^+(z) \in C^k_{p,q}(\bar{\Omega}^+) \subset W^k_{p,q}(\Omega^+)$ and $\alpha^-(z) \in C^k_{p,q}(\bar{\Omega}) \subset W^k_{p,q}(\Omega)$such that we have the decomposition

$$\alpha = \alpha^+ - \alpha^- \quad on \ b\Omega.$$

Moreover, we also have the estimates:

$$\|\alpha^+\|_{W^{-1/2}(\Omega^+)} \leq C\|\alpha\|_{L^2(b\Omega)}, \tag{3.24}$$

$$\|\alpha^-\|_{W^{-1/2}(\Omega)} \leq C\|\alpha\|_{L^2(b\Omega)}. \tag{3.25}$$

Since Ω is pseudoconvex, one defines $u^- = \bar{\partial}^* N^\Omega_q \alpha^-$, where N^Ω_q denotes the $\bar{\partial}$-Neumann operator for the domain Ω. By using Theorem 6.1.4 in [20], it follows that $u^- \in W^{1/2}_{p,q-1}(\Omega)$, $\bar{\partial}u^- = \alpha^-$ and

$$\|Du^-\|_{W^{-1/2}(\Omega_\delta)} \leq \|\alpha^-\|_{W^{-1/2}(\Omega_\delta)} \leq C\|\alpha\|_{L^2(b\Omega)},$$

for some constant C in dependent of α. Restricting u^- to the boundary we have $\bar{\partial}_b u^- = \tau\alpha^-$ on $b\Omega$ and using the trace theorem for Sobolev spaces, one obtains

$$\|u^-\|_{L^2(b\Omega)} \leq C\|Du^-\|_{W^{-\frac{1}{2}}(\Omega_\delta)} \leq C\|\alpha\|_{L^2(b\Omega)}. \tag{3.26}$$

Similarly on Ω^+, one defines $u^+ = \bar{\partial}^* N^{\Omega^+}_q \alpha^+$, where $N^{\Omega^+}_q$ denotes the $\bar{\partial}$-Neumann operator for the domain Ω^+ and $u^+ \in W^{1/2}_{p,q-1}(\Omega^+)$, $u^+ \in W^{1/2}_{p,q-1}(\Omega^+)$, $\bar{\partial}_b u^+ = \alpha^+$and $u^+$is one derivative smoother than α^+ in the interior of Ω^+. Also, by using Theorem 6.1.4 in [20], it follows that

$$\|Du^+\|_{W^{-1/2}(\Omega^+_\delta)} \leq C\|\alpha^+\|_{W^{-1/2}(\Omega_\delta)} \leq C\|\alpha\|_{L^2(b\Omega)},$$

for some constant C in dependent of. Restricting u^+ to $b\Omega^+$ we have $\bar{\partial}_b u^+ = \tau\alpha^+$on Ω^+ and by using the trace theorem for Sobolev spaces, one obtains

$$\|u^+\|_{L^2(b\Omega)} \leq C\|Du^+\|_{W^{-\frac{1}{2}}(\Omega_\delta)} \leq C\|\alpha\|_{L^2(b\Omega)}, \tag{3.27}$$

for some constant C independent of α . Letting

$$u = u^+ - u^- \quad on \ b\Omega.$$

Then $\bar{\partial}_b u = \tau\alpha$ on $b\Omega$. We also have from (3.26) and (3.27),

$$\|u\|_{L^2(b\Omega)} \leq C\|\alpha\|_{L^2(b\Omega)}, \tag{3.28}$$

where C is independent of α. (3.28) was derived for $\alpha \in C^\infty_{p,q}(b\Omega)$. But $C^\infty_{p,q}(b\Omega) \cap \ker \bar{\partial}_b$ is dense in $\ker \bar{\partial}_b$ ([20], Lemma 9.3.8). In view of (3.24) and (3.25), (3.28) then implies that α maps bounded sets in $\ker \bar{\partial}_b \subset L^2_{p,q}(b\Omega)$ into relatively compact sets in $L^2_{p,q}(b\Omega)$. Both $N^{\Omega^+}_q$ and N^Ω_q are compact (Ω^+ satisfies the assumptions in Lemma 3.1). Thus $\bar{\partial}^* N^{\Omega^+}_q$ and $\bar{\partial}^* N^\Omega_q$ are compact in $W^{1/2}_{p,q}(\Omega^+)$ and $W^{1/2}_{p,q}(\Omega)$, respectively (again from [21]). The embedding $W^{1/2}(\Omega^+) \longrightarrow L^2(\Omega^+)$ and $W^{1/2}(\Omega) \longrightarrow L^2(\Omega)$ are also compact. Then, α is compact on $ker\ \bar{\partial}_b$, hence on $L^2_{p,q}(b\Omega)$.

3.4. Existence and Compactness of the Complex Green Operator

In this subsection, we must show that G_q is compact and by the symmetry between form levels, G_{n-1-q} is then compact as well. From (3.28), the range of $\bar{\partial}_b$, denoted by $Rang\ \bar{\partial}_b$, is closed in every degree. Then, we have ker $\bar{\partial}_b$=Rang $\bar{\partial}_b^*$ and the following orthogonal decomposition:

$$\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega) = ker\ \bar{\partial}_b \oplus Rang\ \bar{\partial}_b^* = Rang\ \bar{\partial}_b \oplus Rang\ \bar{\partial}_b^*.$$

Repeating the arguments of Theorem 8.4.10 in Chen-Shaw [20], one can prove that for every $\alpha \in dom\ \bar{\partial}_b \cap \mathrm{dom}\ \bar{\partial}_b^*$,

$$\|\alpha\|^2 \le C\left(\left\|\bar{\partial}_b\alpha\right\|^2 + \left\|\bar{\partial}_b^*\alpha\right\|^2\right) = C < \Box_b\alpha, \alpha > \le\ C\|\Box_b\alpha\|\|\alpha\|,$$

i.e.,

$$\|\alpha\|\ \le C\|\Box_b\alpha\|. \tag{3.29}$$

Since $\Box_b$ is a linear closed densely defined operator, then, from Theorem 1.1.1 in [15], Rang $\Box_b$ is closed. Thus, from (1.1.1) in [15] and the fact that $\Box_b$ is self adjoint, we have the Hodge decomposition

$$\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega) = rang\ \Box_b \oplus \aleph^b_{p,q}(b\Omega\,) = \bar{\partial}_b\bar{\partial}_b^* dom\ \Box_b \oplus \bar{\partial}_b^*\bar{\partial}_b dom\ \Box_b.$$

Since $\Box_b$ is one to one on dom$\Box_b$ from (3.29), then there exists a unique bounded inverse operator $\quad G_q\colon rang\ \Box_b \longrightarrow dom\ \Box_b \cap\ (\aleph^b_{p,q}(b\Omega\,))^\perp$ such that $G_q\Box_b\alpha = \alpha$ on dom$\Box_b$. We can write $\Box_b G_q = I$ on $dom\ \Box_b \cap (\aleph^b_{p,q}(b\Omega\,))^\perp$. From the definition of G_q, we extend G_q to $\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$ one obtains $\Box_b G_q = I$ on $\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$. For $u \in \mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$, we have the Hodge decomposition

$$u = \bar{\partial}_b\bar{\partial}_b^* G_q u + \bar{\partial}_b^*\bar{\partial}_b G_q u.$$

([20], Theorem 9.4.2). In particular, if $\bar{\partial}_b u = 0$, $\bar{\partial}_b^* G_q u$ gives the canonical solution to the equation $\bar{\partial}_b\alpha = u$. G_q can be expressed in terms of these canonical solution operators at levels q and $q + 1$ and their adjoints ([22], p. 1577):

$$G_q = \left(\bar{\partial}_b^* G_q\right)^*\left(\bar{\partial}_b^* G_q\right) + \left(\bar{\partial}_b^* G_{q+1}\right)\left(\bar{\partial}_b^* G_{q+1}\right)^*.$$

This formula is analogous to the corresponding formula for N_q ([5, 12]). Thus, compactness of G_q is equivalent to compactness of both $\bar{\partial}_b^* G_q$ and $\bar{\partial}_b^* G_{q+1}$.

From Lemma 3.4, the canonical solution operator $\bar{\partial}_b^* G_q$ is compact on $\mathrm{L}^2_{\mathrm{p,q}}(\mathrm{b}\Omega)$. We now consider$\bar{\partial}_b^* G_{q+1}$. $b\Omega$ also satisfies (P_{q+1}) (because $(P_q) \Longrightarrow (P_{q+1})$). By assuming that $2q \le (n-2)$ and since $q \le (n-2-q)$, thus $b\Omega$ also satisfies $\left(P_{n-1-(q+1)}\right) = (P_{n-2-q})$. Consequently, the previous case applies (with q replaced by $(q+1)$), and $\bar{\partial}_b^* G_{q+1}$ is compact. Since we may assume without loss of generality that $q \le (n-1-q)$, i.e. $2q \le (n-1)$, in proving Lemma 3.4, the only case left to consider is $2q = (n-1)$. We argue as follows:$(\bar{\partial}_b^* G_q)^*$, the canonical solution operator to $\bar{\partial}_b^*$, is compact because $\bar{\partial}_b^* G_q$ is. Because $q - 1 = n - 1 - (q+1)$, the symmetry yields a compact solution operator for $\bar{\partial}_b$ (as an operator from (p,q)-forms to $(p,q+1)$-forms). Thus, the canonical solution operator $\bar{\partial}_b^* G_{q+1}$ is compact. Thus, the proof of Theorem 1.1 follows.

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