CONSTRUCTION OF TERNARY APPROXIMATING SUBDIVISION SCHEMES

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In this, an algorithm to produce ternary m-point (for any integer m ≥ 2) approximating subdivision scheme has been introduced that can generate the families of C^{m-1} limiting curves. The proposed scheme has developed using the uniform B-spline blending functions and its convergence is analyzed by Laurent polynomial method. It is concluded that the existing 2-point, 3-point and 5-point ternary approximating stationary subdivision scheme are either the special cases or can be generated by the proposed algorithm, after setting m= 2, 3 and 5, respectively.

Keywords: Uniform B-spline; algorithm; stationary; approximation; subdivision scheme; convergence and smoothness.

1. Introduction

Subdivision schemes have become one of the most important, significant and emerging modelling tools in computer applications, medical image processing, scientific visualization, reverse engineering and computer aided geometric designing (CAGD) etc. Because of its simple, elegant and efficient ways to create smooth curves from initial control polygon (smooth surfaces from initial control mesh) by subdividing them according to some refines rules, recursively. These refining rules take the initial polygons (meshes) to produce a sequence of finer polygons (meshes) converging to a smooth limiting curve.

The French mathematician De Rham [3] is considered as the pioneer in the field of stationary subdivision schemes, who presented a recursively corner cutting linear approximation scheme. Later on, another corner cutting piecewise linear approximation scheme was developed by Chaikin [2]. Both of these schemes generate the limit curves of C^1 continuity but the difference between them is that the curvature of De Rahm's scheme diverges and that of Chaikin is piecewise continuous. These schemes gave new direction in the field of subdivision techniques. Hassan [8] introduced a 3-point binary approximating scheme which generates C^3 limiting curve. They also presented a ternary 3-point approximating and interpolating schemes that can generate the limit curves of C^2

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and C^1 continuity, respectively. Siddiqi and Rehan [13] also developed 3-point ternary approximating scheme that generates the family of C^2 limiting curve for a particular value. They also modify this scheme, to produce the family of C^2 limiting curves, with the help of parameter [16]. Ko *et al.* [11] developed a ternary four-point approximating scheme which gives C^2 limiting curves whose approximating order is 4. Siddiqi and Younis [17] presented a new family of five-point binary approximating subdivision scheme which generates the family of C^1 limiting curves.

A binary 4-point approximating subdivision scheme that generates a C^2 limiting curve with support on [-4, 3] and is close to being interpolatory was presented by Dyn et al. [7]. Zhang et al. [18] developed another binary 4-point approximating subdivision scheme which generates a limiting curve of C^3 continuity with the help of weights. Deslauriers and Dubuc [4] suggested a binary 4-point interpolatory subdivision scheme which generates a C^1 limiting curve using the polynomial reproducing property.

Dyn et al. [6] introduced the general form of a binary 4-point interpolatory subdivision scheme, independently, with the help of a tension parameter which gives a C^1 limiting curve. Kuijt et al. [10] examined the convexity preserving properties of the 4-point interpolatory subdivision scheme of Dyn et al. [11] which also generates C^1 limit functions. Siddiqi and Younis [17] developed another ternary 3-point approximating non-stationary subdivisions scheme derived from a quadratic trigonometric B-spline basis function which generates the family of C^2 limiting curves. Hassan et al. [9] presented a ternary 4-point interpolatory subdivision scheme with a tension parameter which gives a C^2 continuous curve. Qu [12] introduced a local algorithm using the piecewise generalized conic segments for a shape preserving curve interpolation. In the following basic notion and definition related to ternary subdivision scheme are defined as

2. Preliminaries

If same mask is used in each refinement step of subdivision scheme then the scheme is called a stationary otherwise it is called non-stationary. Each stationary subdivision scheme is associated with a mask $a = \{a_i\},\ i \in Z$. The ternary subdivision schemes are the process which recursively define a sequence of control points $f^k = f_i^k, i \in Z$ by the rule of the form with mask $a = \{a_i\},\ i \in Z$

$$f_i^{k+1} = \sum_{j \in Z} a_{i-3j}\, f_j^k$$

which is formally denoted by $f^{k+1} = S\,f^k = S^k\,f^0$. A subdivision scheme is said to be uniformly convergent if for every initial data $f^0 = \{f_i\},\ i \in Z$, there is a continuous function f such that for any closed interval $[a, b]$

$$\lim_{k\to\infty} \sup_{i\in Z\cap 3^k[a,b]} \left|f_i^k - f(3^{-k}i)\right| = 0.$$

Obviously $f = S^\infty f^0$ is considered to be a limit function of the subdivision scheme S. The important subdivision schemes for applications should allow one to control the shape of the limiting curve and to be capable of generating the limiting curves of different continuity widely used in computer graphics, image processing and CAGD etc.

The ternary subdivision rules to refine the control polygon, in general form, are defined as

$$f_{3i}^{k+1} = \sum_{j\in Z} a_{3j}\, f_{i-j}^k$$
$$f_{3i+1}^{k+1} = \sum_{j\in Z} a_{3j+1}\, f_{i-j}^k$$
$$f_{3i+2}^{k+1} = \sum_{j\in Z} a_{3j+2}\, f_{i-j}^k$$

For the convergent subdivision scheme S, the corresponding mask $\{a_i\}, i \in Z$ necessarily satisfies

$$\sum_{j\in Z} a_{3j} = \sum_{j\in Z} a_{3j+1} = \sum_{j\in Z} a_{3j+2} = 1$$

Introducing a symbol called the Laurent polynomial $a(z) = \sum_{j\in Z} a_i z^i$, of a mask $\{a_i\}$, $i \in Z$ with finite support. The corresponding symbols play an efficient role to analyze the convergence and smoothness of subdivision scheme.

With the symbol, Hassan *et al.* [9] provided a sufficient and necessary condition for a uniform convergent subdivision scheme. A subdivision scheme S is uniform convergent if and only if there is an integer $L \geq 1$, such that

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_\infty < 1,$$

Subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$, where $a_1(z) = \frac{3z^2}{1+z+z^2}$ and satisfying

$$df^k = S_1 df^{k-1}, \qquad k = 1, 2, \ldots,$$

Where $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 3^k(f_{i+1}^k - f_i^k) : i \in Z\}$ and the norm $\|S\|_\infty$ of a subdivision scheme S with a mask $\{a_i\}$, $i \in Z$ is defined by

$$\|\boldsymbol{S}\|_\infty = \boldsymbol{max}\left\{\sum_{i\in Z}|\boldsymbol{a}_{3i}|, \sum_{i\in Z}|\boldsymbol{a}_{3i+1}|, \sum_{i\in Z}|\boldsymbol{a}_{3i+2}|\right\}$$

3. The Algorithm

In this section, an algorithm has been introduced to produce m-point ternary approximating subdivision scheme using the Cox-de Boor recursion relation. So, in view of Buss [1], the Cox-de Boor recursion relation can be defined as follows:

The recursion relation is the generalization to B-spline of degree k (or of order n, i.e, $k = n-1$). For this, consider T be a set of $l+1$ non-decreasing real numbers in such a way that $t_0 \leq t_1 \leq t_2, \ldots, \leq t_l$. The values t_i's, not necessarily uniformly spaced, are called knots of non-uniform spline and the set T is called knot vector. The uniform B-splines are just the special case of non-uniform B-splines in which the knots are equally spaced such that $t_{i+1} - t_i$ is a constant for $0 \leq i \leq n-1$. Note that the blending functions $N_{i,n}(t)$ of order n depend only on the knot positions and are defined by induction on $n \geq 1$ as follows.
First, for $i = 0, 1, \ldots, l-1$, let

$$N_{i,1}(t) = \begin{cases} 1 & t_i \leq t < t_{i+1} \\ 0 & otherwise, \end{cases}$$

Second for $n \geq 1$. Setting $n = k+1$, $N_{i,k+1}(t)$ is defined by the Cox-de Boor formula as,

$$N_{i,k+1}(t) = \frac{t-t_i}{t_{i+k}-t_i} N_{i,k}(t) + \frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}} N_{i,k+1}(t) \tag{1}$$

The form of above recursive formulas for the blending function immediately implies that the functions $N_{i,n}(t)$ are piecewise polynomials of degree $n-1$ and that the breaks between pieces occur at the knots t_i.
In view of above recursion formula, the Uniform B-spline blending functions $N_{i,p}(t)$ of order p over the interval $t \in [0,1]$, together with the following properties, can be defined in Eq. (2).

The blending functions should have the following properties:

- The blending functions are translates of each other, that is, $N_i(t) = N_0(t-i)$.
- The blending functions are a partition of unity, that is $\sum_i N_i(t) = 1$.
- $0 \leq N_i(t) \leq 1$ for all t.
- The functions $N_i(t)$ have continuous $p-1$ derivatives, that is, they are C^{p-1}-continuous.

$$N_{0,p}(t) = \frac{t}{p-1} N_{0,p-1}(t) + \frac{p-t}{p-1} N_{0,p-1}(t) \tag{2}$$

The mask a_i and $b_i, i = 0, 1, \ldots, m-1$ of proposed m-point ternary scheme can be calculated using the following relations

$$a_i = N_{0,m}\left((m-i) - \frac{5}{6}\right) \quad \text{and} \quad b_i = N_{0,m}\left((m-i) - \frac{1}{2}\right) \qquad i = 0, 1, 2, \ldots, m-1 \tag{3}$$

where $N_{i,m}(t)$ is a uniform B-spline basis function of degree $m-1$. In the following, some cases are being considered to produce the masks of *2*-point, *3*-

point and *4*-point ternary approximating schemes after setting $m = 2, 3$ and 4, respectively, using the recurrence relation (3)

Let us consider the case 1, the 2-point ternary approximating scheme can be obtained, after setting $m = 2$ in equation (3). The mask a_i and $b_i, i = 0, 1$ of 2-point scheme (which is also called corner cutting scheme) can be calculated from the linear B-spline basis function $N_{0,2}$. The rules to refine the control polygon are defined as:

$$f_{3i}^{k+1} = a_0\, f_i^k + a_1\, f_{i+1}^k$$
$$f_{3i+1}^{k+1} = b_0\, f_i^k + b_1\, f_{i+1}^k$$
$$f_{3i+2}^{k+1} = a_1\, f_i^k + a_0\, f_{i+1}^k$$

where $a_0 = \frac{5}{6}$, $a_1 = \frac{1}{6}$ and $b_0 = b_1 = \frac{1}{2}$.

Moreover, the obtained mask of 2-point scheme coincides with the scheme, presented by Siddiqi and Rehan in [15]. So, the proposed algorithm can be considered as the generalized form of corner cutting scheme presented in [15].

In case 2, the 3-point ternary approximating scheme can be obtained, after setting $m = 3$ in equation (3). The mask a_i and $b_i, i = 0, 1, 2$ of 3-point scheme can be calculated from the quadratic B-spline basis function $N_{0,3}$. The rules to refine the control polygon can be defined as

$$f_{3i}^{k+1} = a_0\, f_{i-1}^k + a_1\, f_i^k + a_2\, f_{i+1}^k$$
$$f_{3i+1}^{k+1} = b_0\, f_{i-1}^k + b_1\, f_i^k + b_2\, f_{i+1}^k$$
$$f_{3i+2}^{k+1} = a_2\, f_{i-1}^k + a_1\, f_i^k + a_0\, f_{i+1}^k$$

where $a_0 = \frac{25}{72}$, $a_1 = \frac{23}{36}$, $a_2 = \frac{1}{72}$, $b_1 = \frac{3}{4}$ and $b_0 = b_2 = \frac{1}{8}$

Furthermore, the obtained mask of 3-point scheme coincides with the scheme, presented by Siddiqi and Rehan in [13]. So, the proposed algorithm can be considered as the generalized form the scheme [13].

In case 3, the 4-point ternary approximating scheme can be obtained, after setting $m = 4$ in equation (3). The mask a_i and $b_i, i = 0, 1, 2, 3$ of 4-point scheme can be calculated from the cubic B-spline basis function $N_{0,4}$. The rules to refine the control polygon are defined as follows:

$$f_{3i}^{k+1} = a_0\, f_{i-1}^k + a_1\, f_i^k + a_2\, f_{i+1}^k + a_3\, f_{i+2}^k$$
$$f_{3i+1}^{k+1} = b_0\, f_{i-1}^k + b_1\, f_i^k + b_2\, f_{i+1}^k + b_3\, f_{i+2}^k$$
$$f_{3i+2}^{k+1} = a_3\, f_{i-1}^k + a_2\, f_i^k + a_1\, f_{i+1}^k + a_0\, f_{i+2}^k$$

where $a_0 = \frac{125}{1296}$, $a_1 = \frac{831}{1296}$, $a_2 = \frac{339}{1296}$, $a_3 = \frac{1}{1296}$, $b_0 = b_3 = \frac{1}{48}$ nd

$$b_1 = b_2 = \frac{23}{48}$$

The 4-point ternary approximating scheme has the ability to produce the limit curves of C^3 continuity (for proof see Theorem 4.3).

Remark: In similar manner the 5-point scheme can be obtained. The mask of this 5-point scheme coincide with the mask of scheme presented in [14]

4. Analysis of the Schemes

In this section, the smoothness and convergence of the proposed schemes have been discussed.

4.1 Theorem: The *2*-point ternary approximating subdivision scheme defined in case (1) converges and has smoothness C^1.

Proof. See [15]

4.2 Theorem: The *3*-point ternary approximating subdivision scheme defined in case (2) converges and has smoothness C^2.

Proof. See [13]

4.3 Theorem: The *4*-point ternary approximating subdivision scheme defined in case (3) converges and has smoothness C^3.

Proof. Consider the refinement equation of example (3.3) and the Laurent polynomial $a(z)$ for the mask of the scheme can be written as

$$a(z) = \frac{1}{1296}\begin{bmatrix} z^{-6} + 27\,z^{-5} + 125\,z^{-4} + 339\,z^{-3} + 621\,z^{-2} + 831\,z^{-1} \\ + 831 + 621\,z^{1} + 339\,z^{2} + 125\,z^{3} + 27\,z^{4} + z^{5} \end{bmatrix}$$

In view, to prove the smoothness of the scheme to be C^3, of Dyn and Levin [5]. We have

$$b(z) = \frac{3\,a(z)}{(1+z+z^2)^4}$$

$$= \frac{1}{144}\left[z^{-6} + 23\,z^{-5} + 23\,z^{-4} + z^{-3}\right]$$

Since, the norm of the subdivision scheme $\{S_b\}$ is

$$\|S_b\|_\infty = max\,\{\textstyle\sum_{i\in Z}|b_{3i}|\,,\ \sum_{i\in Z}|b_{3i+1}|\,,\ \sum_{i\in Z}|b_{3i+2}|\,\}$$

$$= max\left\{\frac{1}{24}, \frac{23}{48}, \frac{23}{48}\right\} < 1$$

Hence, the 4-point stationary scheme $\{S_a\}$ (introduced in case 3) is C^3.

5. Advantages of m-point Approximating Scheme

a. Order of Derivative Continuity

It can be observed from the table 1, that the order of derivative continuity of the scheme can be achieved up to the requirement after setting the different values of m in proposed algorithm.

Ternary schemes obtained from proposed algorithm

Setting $\boldsymbol{m}$	Proposed Scheme	Continuity	Coincide with
2	2- Point	C^1	Scheme [15]
3	3- Point	C^2	Scheme [13]
4	4- Point	C^3	
5	5- Point	C^4	Scheme [14]
⋮	⋮	⋮	
$\boldsymbol{m}$	$\boldsymbol{m}$- Point	C^{m-1}	

b. Special Cases

The proposed algorithm can be considered as the generalized form of the existing ternary schemes introduced by Siddiqi and Rehan in [13-16], after setting different values of m (can also be observed from the table 1). Hence the schemes, presented in [13-16], are either the special cases or can reproduce or regenerate by the proposed algorithm. The following schemes can reproduce by the proposed algorithm

- The C^1 limiting curves can be obtained after taking $m = 2$ in proposed algorithm. The obtained mask of this scheme (presented in case 1) coincides with mask of corner cutting scheme presented in [15].
- The mask of ternary 3-point stationary approximating schemes introduced in [13] and [16] (for $\mu = 0$) coincide with the mask of the scheme (presented in case 2), after setting $m=3$ in proposed algorithm.
- The coefficients of 5-point stationary approximating scheme [14] matched with the coefficients of proposed algorithm (3), after setting $m = 5$.

6. Conclusion

An algorithm to produce m-point (for any integer $m \geq 2$) ternary approximating stationary subdivision scheme has been developed that can generate the limit curves of C^{m-1} continuity. The construction of the ternary approximating schemes is associated with recurrence relation of uniform B-spline basis function which is originated from Cox-de Boor recurrence formula. It is also concluded that the schemes obtained from proposed algorithm coincide with the existing ternary approximating schemes. Thus, the proposed algorithm can be considered as the generalized form of the schemes introduced in [13, 14, 15, 16].

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