U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 3, 2014 ISSN 1223-7027

AN ALTERNATIVE TO THE THEORY OF EXTREMA

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Let $\Gamma(a)$ be a family of parametrized curves passing through $a \in D$, where D is an open subset in $\mathbb{R}^p$. In [1]÷[4] it was studied the connection between the local extremum problem and the extremum problem constrained by the family $\Gamma(a)$ for an arbitrary function $f: D \to \mathbb{R}$. In the situation when, for any function f, the two extremum problems are equivalent, $\Gamma(a)$ is called optimal family. *In [8] we emphasized sufficient conditions of optimality for a family of parametrized curves $\Gamma(a)$. In this paper we develop the ideas from [8], getting the necessary and sufficient optimality conditions for a family of parametrized curves $\Gamma(a)$.*

Keywords: Local extrema, constrained extrema, minima constrained by curves.
MSC2000: Primary 90C46; Secondary 52A40.

1. Introduction and preliminaries

Throughout this article f will refer to a function $f: D \to \mathbb{R}$, where D is an open subset in $\mathbb{R}^p$.

Let us consider the extremum problem

$$\min f(x), \text{ subject to } x \in M,$$

where M is a subset of $\mathbb{R}^p$. If M is an open set, then the extremum problem is called *unconstrained.* Otherwise, the extremum problem is called *constrained.*

The usual approach to solve this problem consists in finding sufficient and/or necessary conditions of local extremum based on some properties of the function f (e.g. convexities of class C^1, class C^2).

Another approach of solving the problem is to relate this extremum problem with a set of extremum problems for functions of type $f \circ \alpha_i$, where $\alpha_i: I_i \subset \mathbb{R} \to D$, $i \in \mathcal{I}$ is a family of parametrized curves passing through a point likely to be an extremum point. In this case, the properties of the family of parametrized curves are very important since f could be an arbitrary function ([1]÷ [4]). This approach allows the introduction the theory of extrema constrained by a Pfaff system, a generalization of both constrained and unconstrained extrema ([7], [23]÷[29]). Additionally, this mixed approach, which takes into consideration both the properties of function f and the properties of the family of parametrized curves, introduces new types of convexities for f ([5], [6]).

The purpose of the paper is to complete the results obtained in [8].

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We begin by summarizing some of the concepts already introduced elsewhere in possibly different forms.

Let D be an open subset of $\mathbb{R}^n$. For the purposes of this article, a parametrized curve α passing though a given point a $(\alpha(t_0) = a)$ is of class at least C^1, has a tangent in a, i.e. $\alpha^{(k)}(t_0) \neq 0$, for some $k \geq 1$, and its domain is a real interval I.

Definition 1.1. We say that $a \in D$ is *a minimum point for f constrained* by parametrized curve $\alpha\colon I \to \mathbb{R}^n$ passing through a if, for any $t_0 \in I$ with $\alpha(t_0) = a$, there exists a neighbourhood $I_{t_0} \subset I$ of t_0 such that $f(a) \leq f(\alpha(t)), \forall t \in I_{t_0}$.

NOTE.In the following, in order to simplify the presentation, we shall avoid using the word local in the definitions for various types of extremum points.

Definition 1.2. We say that $a \in D$ is a *minimum point for f weakly constrained* by parametrized curve $\alpha\colon I \to \mathbb{R}^n$ passing through a if it is a *right-hand minimum point* for $f \circ \alpha$, i.e for any $t_0 \in I$ with $\alpha(t_0) = a$ there exists $\varepsilon > 0$ such that $f(a) \leq f(\alpha(t)), \forall t \in [t_0, t_0 + \varepsilon)$ ([7]).

We similarly define the *maximum point constrained* by a parametrized curve and the *maximum point weakly constrained* by a parametrized curve, getting the concepts of *extremum point constrained* by a parametrized curve and of *extremum point weakly constrained extremum* by a parametrized curve. Obviously, an extremum constrained by a parametrized curve is an extremum weakly constrained by the same parametrized curve.

Let $\Gamma(a)$ be a family of parametrized curves passing through a.

Definition 1.3. We say that $a \in D$ is an *extremum point for f constrained by the family* $\Gamma(a)$ if a is an extremum point of the same kind (either minimum or maximum) for f constrained by any parametrized curve $\alpha \in \Gamma(a)$ ([1], [7]).

Definition 1.4. Similarly, we say that $a \in D$ is an *extremum point for f weakly constrained by the family* $\Gamma(a)$ if a is extremum point of the same kind (either minimum or maximum) for f weakly constrained by any parametrized curve $\alpha \in \Gamma(a)$.

The last definition, which is more general that the one before it, is useful when considering local extrema constrained by inequalities ([26]). However, in certain circumstances, the two definitions are equivalent.

Let us consider the following property of a family of parametrized curves $\Gamma(a)$:

If $\alpha \in \Gamma(a)$ and β is a parametrized curve equivalent to α, then $\beta \in \Gamma(a)$. (1)

Proposition 1.1. *Let $\Gamma(a)$ be a family that satisfies property* (1)*. Then $a \in D$ is an extremum point constrained by the family $\Gamma(a)$ if and only if it is an extremum point weakly constrained by the family $\Gamma(a)$.*

Proof. Let us assume that a is a minimum point weakly constrained by the family $\Gamma(a)$. Let $\alpha \in \Gamma(a)$, $\alpha(t_0) = a$. Since $\Gamma(a)$ satisfies condition (1), then we can assume, possibly via a change of parameter, that $t_0 = 0$. Then there exists an $\varepsilon_1 > 0$ such that $f(a) \leq f(\alpha(t))$, $\forall t \in [0, \varepsilon_1)$. Let β be the parametrized curve defined by $\beta(t) = \alpha(-t)$. According to property (1), $\beta \in \Gamma(a)$. Therefore, there exists an $\varepsilon_2 > 0$ such that $f(a) \leq f(\alpha(-t))$, $\forall t \in [0, \varepsilon_2)$, or, in other words, $f(a) \leq$

$f(\alpha(t)),\ \forall t \in [-\varepsilon_2, 0)$. Finally, we get that $f(a) \le f(\alpha(t)),\ \forall t \in (-\varepsilon_2, \varepsilon_1)$, ie a is a minimum point for f constrained by $\Gamma(a)$. □

Example 1.1. We will show that there exist families of parametrized curves for which the two concepts (constrained and weakly constrained) are not the same. In $\mathbb{R}^2$, for $a = (0,0)$, let the family $\Gamma(a)$ which consists of all the curves $\alpha_u,\ u \in \left[-\frac{\pi}{2}, \pi\right]$ defined by $\alpha_u(t) = (t\cos u, t\sin u),\ \forall t \in \mathbb{R}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x^2 + y^2, & \text{if } x \ge 0 \text{ or } y \ge 0 \\ -\left(x^2 + y^2\right), & \text{otherwise.} \end{cases} \tag{2}$$

Then a is a minimum point for f weakly constrained by $\Gamma(a)$ but is not a minimum point for f constrained by $\Gamma(a)$.

Definition 1.5. A family $\Gamma(a)$ is *optimal* if the following statement holds for any function f: if a is an extremum point for f constrained by $\Gamma(a)$, then a is a local extremum point for f.

Definition 1.6. A family $\Gamma(a)$ is *strongly optimal* if the following statement holds for any function f: if a is an extremum point for f weakly constrained by $\Gamma(a)$, then a is a local extremum point for f.

Since any extremum point for f constrained by a family $\Gamma(a)$ is an extremum point for f weakly constrained by $\Gamma(a)$, it follows that any *strongly optimal* family is also an *optimal* family. Furthermore, as a consequence of Proposition 1.1, we get that:

Proposition 1.2. *If the family* $\Gamma(a)$ *satisfies property* (1), *then* $\Gamma(a)$ *is optimal if and only if it is strongly optimal.*

2. **Main results**

In this section, we establish necessary and sufficient conditions for a family of parametrized curves to be optimal (strongly optimal).

In the following paragraphs, let $S(a)$ be a family of sequences with distinct elements converging to some $a \in D$.

Definition 2.1. A family $\Gamma(a)$ of parametrized curves passing through a is $S(a)$–*subordinate* if, for any sequence $(x_n) \in S(a)$, there exists a parametrized curve $\alpha \in \Gamma(a)$, a $t_0 \in \mathrm{dom}(\alpha)$, and a sequence of real numbers (t_k) converging to t_0, such that $\alpha(t_k)$ is a subsequence of (x_n) ([8]).

Definition 2.2. A family $\Gamma(a)$ of parametrized curves passing through a is *strongly* $S(a)$–*subordinate* if, for any sequence $(x_n) \in S(a)$, there exists a parametrized curve $\alpha \in \Gamma(a)$, a $t_0 \in \mathrm{dom}(\alpha)$, and a strictly decreasing sequence of real numbers (t_k) converging to t_0, such that $\alpha(t_k)$ is a subsequence of (x_n).

The following remark will be useful later:

Remark 2.1. (1) $a \in D$ is a local minimum point for f if and only if, for any sequence (x_n) that converges to a, there exists a subsequence (x_{n_k}) such that $f(x_{n_k}) \ge f(a),\ \forall k \in \mathbb{N}$.

(2) $a \in D$ is a minimum point for f weakly constrained by $\alpha \in \Gamma(a)$, $\alpha(t_0) = a$ if and only if, for any strictly decreasing real sequence (t_n), $t_n \to t_0$, there exists a subsequence (t_{n_k}) such that $f(\alpha(t_{n_k})) \geq f(a)$ $\forall k \in \mathbb{N}$.

Let $C(a)$ be the family of *all sequences* of distinct elements converging to a.

Theorem 2.1. *A family $\Gamma(a)$ is strongly optimal if and only if it is strongly $C(a)$–subordinate.*

Proof. Let $\Gamma(a)$ be a strongly optimal family. Let us assume that $\Gamma(a)$ is not strongly $C(a)$–subordinate. In that case, there exists a sequence of distinct elements $(x_n) \in C(a)$ such that for any $\alpha \in \Gamma(a)$ $(\alpha(t_0) = a)$ and for any strictly decreasing sequence $(t_k) \subset \mathbb{R}$ with $t_k \to t_0$, the sequence $(\alpha(t_k))$ is not a subsequence of (x_n). Let $f : D \to \mathbb{R}$,

$$f(x) = \begin{cases} -1, & \text{if } x \in \{x_1, x_2, ..., x_n, ...\} \\ \|x - a\|, & \text{otherwise.} \end{cases}$$

Evidently, a is not an extremum point for f. However, under the above assumption, we can show that a is a minimum point constrained by $\Gamma(a)$, which results in a contradiction. Indeed, let $\alpha \in \Gamma(a)$, $\alpha(t_0) = a$. Let t_n be a strictly decreasing sequence such that $t_n \to t_0$. The sequences $(\alpha(t_n))$ and (x_n) cannot have common subsequences: otherwise the sequence (t_n) would contain some subsequence (t_{n_k}) for which $\alpha(t_{n_k})$ would be a subsequence of (x_n). Consequently, for a sufficiently large n, $\alpha(t_n) \notin \{x_1, x_2, ..., x_n, ...\}$. Therefore, $(f \circ \alpha)(t_n) = \|\alpha(t_n) - a\| > 0 = (f \circ \alpha)(t_0)$. Since (t_n) is arbitrary, taking into account the above remark, it follows that t_0 is a local minimum for the function $f \circ \alpha$.

Conversely, let us assume that the family $\Gamma(a)$ is strongly $C(a)$–subordinate. Let $f\colon D \to \mathbb{R}$ be a function with a being a minimum point weakly constrained by the family $\Gamma(a)$. Let $(x_n) \in C(a)$. There exists a parametrized curve $\alpha \in \Gamma(a)$, a $t_0 \in \text{dom}(\alpha)$, a subsequence (x_{n_k}), and a strictly decreasing sequence (t_k), $t_k \to t_0$, such that $(\alpha(t_k) = x_{n_k}, \forall k \in \mathbb{N}^*$. Then $f(x_{n_k}) = f(\alpha(t_k)) \geq f(a), \forall k \in \mathbb{N}^*$. Keeping in mind the previous remark it results that a is a local minimum point for f. □

A similar proof can be given for the following theorem:

Theorem 2.2. *A family $\Gamma(a)$ is optimal if and only if it is $C(a)$–subordinate.*

Definition 2.3. *Let $\alpha\colon I \to D$ a parametrized curve passing through a. We say that $f\colon D \to \mathbb{R}$ is* continuous with respect to α in a *if for any $t_0 \in I$ such that $\alpha(t_0) = a$, $f \circ \alpha$ is continuous in t_0.*

Definition 2.4. *A family $\Gamma(a)$ of parametrized curves passing though a is called* continuity-ensuring *if, for any function $f : D \to \mathbb{R}$, the following statement is true: if f is continuous with respect to any $\alpha \in \Gamma(a)$ in a, then f is continuous in a.*

Definition 2.5. *Let $\alpha\colon I \to D$ a parametrized curve passing through a. We say that $f : D \to \mathbb{R}$ is* right continuous with respect to α in a *if for any $t_0 \in I$ such that $\alpha(t_0) = a$, $f \circ \alpha$ is right continuous in t_0.*

Definition 2.6. *A family $\Gamma(a)$ of parametrized curves passing though a is called* specially continuity-ensuring *if, for any function $f\colon D \to \mathbb{R}$, the following statement is true: if f is right-continuous with respect to any $\alpha \in \Gamma(a)$ in a, then f is continuous in a.*

Theorem 2.3. *A family* $\Gamma(a)$ *of parametrized curves is specially continuity-ensuring if and only if it is strongly* $C(a)$*–subordinate.*

Proof. Let $\Gamma(a)$ be a specially continuity-ensuring family. Let us assume that $\Gamma(a)$ is not strongly $C(a)$–subordinate. Therefore, there exists a sequence $(x_n) \in C(a)$ such that for any $\alpha \in \Gamma(a)$ $(\alpha(t_0) = a)$ and for any strictly decreasing sequence $(t_n) \subset \mathbb{R}$ with $t_n \to t_0$, the sequence $(\alpha(t_n))$ is not a subsequence of (x_n). Let $f\colon D \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in \{x_1, x_2, ..., x_n, ...\} \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $f(a) = 0$ and f is not continuous in a. However, we can show that $f \circ \alpha$ is right continuous in a for any $\alpha \in \Gamma(a)$, which contradicts the hypothesis that $\Gamma(a)$ is specially continuity-ensuring. Let $\alpha \in \Gamma(a)$ such that $\alpha(t_0) = a$ and (t_n) a strictly decreasing real sequence converging to t_0. The sequences $(\alpha(t_n))$ and (x_n) cannot have common subsequences, otherwise a subsequence (t_{n_k}) of (t_n) with the property that $(\alpha(t_{n_k}))$ is a subsequence of (x_n) would exist. Consequently, for a sufficiently large n, $\alpha(t_n) \notin \{x_1, x_2, ..., x_n, ...\}$, or $(f \circ \alpha)(t_n) = 0$, which implies that $\lim_{n\to\infty}(f \circ \alpha)(t_n) = 0$. This establishes that $f \circ \alpha$ is right-continuous in t_0.

Conversely, let $\Gamma(a)$ be a strongly $C(a)$–subordinate family. Let f be a function such that $f \circ \alpha$ is right continuous in a for any $\alpha \in \Gamma(a)$. Let $(x_n) \in C(a)$. There exists $\alpha \in \Gamma(a)$, a $t_0 \in \mathrm{dom}(\alpha)$, a subsequence (x_{n_k}), and a strictly decreasing sequence (t_k), $t_k \to t_0$ such that $\alpha(t_k) = x_{n_k}, \forall k \in \mathbb{N}^*$. The right continuity of $f \circ \alpha$ in t_0 implies that $f(x_{n_k}) \to f(a), \forall k \in \mathbb{N}^*$. Since the sequence (x_n) is arbitrarily chosen, it follows that f is continuous in a. □

A similar proof can be given for the following proposition.

Proposition 2.1. *A family* $\Gamma(a)$ *of parametrized curves is continuity- ensuring if and only if it is* $C(a)$*–subordinate.*

We can therefore state this theorem:

Theorem 2.4. *Let* $\Gamma(a)$ *be a family of parametrized curves passing through* a*. The following three statements are equivalent:*

(1) $\Gamma(a)$ *is (strongly) optimal family.*
(2) $\Gamma(a)$ *is (strongly)* $C(a)$*–subordinate family.*
(3) $\Gamma(a)$ *is (specially) continuity ensuring family.*

Examples 2.5. For any $m \in N^*$, let $\Gamma^m(a)$ be the family of all C^m parametrized curves passing through the point a and having a tangent in a. It is evident that $\Gamma^m(a)$ satisfies condition (1).

(1) It was shown in [8] that $\Gamma^m(a)$ is an optimal family. Since it also satisfies (1), and taking into account Proposition 1.1, it follows that $\Gamma^m(a)$ is a strongly optimal family.
(2) For $a = (0, 0) \in \mathbb{R}^2$, let us consider the family of parametrized curves $\Gamma^1_+(a)$ consisting in all parametrized curves $\alpha \in \Gamma^1(a)$, $\alpha(t_0) = a$, for which $\alpha'(t_0) = (u, v)$ with $u \geq 0$ or $v \geq 0$; then, $\Gamma^1_+(a)$ is an optimal family but it is not a strongly optimal family. For this purpose, we first show that $\Gamma^1_+(a)$ is $C(a)$–subordinate, which means that it is optimal. According to the previous example, $\Gamma^1(a)$ is a $C(a)$–optimal family, therefore, for any sequence $(x_n) \in C(a)$,

there exist $\alpha \in \Gamma^1(a)$, $t_0 \in dom(\alpha)$, and a real sequence (t_k), $t_k \to t_0$ such that $(\alpha(t_k))$ is a subsequence of (x_n). By changing the orientation of α, if needed, we can ensure that $\alpha \in \Gamma^1_+(a)$; this proves that $\Gamma^1_+(a)$ is $C(a)$–subordinate. Now, let $f\colon \mathbb{R}^r \to \mathbb{R}$ defined by (2). Then a is not an extremum point for f, but it is a minimum point for f weakly constrained by the family $\Gamma^1_+(a)$. This shows that $\Gamma^1_+(a)$ is not strongly optimal.

(3) For a $a \in \mathbb{R}^n$, let us denote by $\Gamma^\omega(a)$ the family of all analytical parametrized curves passing through a. Then $\Gamma^\omega(a)$ is not an optimal family ([3]).

Let $g = (g^1, ..., g^s) : D \to \mathbb{R}^s$ be a C^1-class function. We set an $a \in D$ such that $g(a) \geq 0$. Let $C_g(a)$ be the family of all sequences (x_n) of distinct elements of D with the property that $g(x_n) \geq 0$, $\forall n \in \mathbb{N}$ and $x_n \to a$. Also, let $\Gamma_g(a)$ be a family of parametrized curves α passing through a having the property that, if $\alpha(t_0) = a$, then $g(\alpha(t)) \geq 0$, for all $t \in [t_0, t_0 + \varepsilon)$.

Definition 2.7. We say that the family $\Gamma_g(a)$ is *g–optimal* if the following statement is true: if a is an extremum point for some $f\colon D \to \mathbb{R}$ weakly constrained by the family $\Gamma_g(a)$, then a is an local extremum point for f constrained by $g \geq 0$.

Let us remark that, if $g(a) > 0$, then the family $\Gamma_g(a)$ in the above definition no longer depends on g, and it is strongly optimal, while a is a free local extremum point.

Definition 2.8. The family $\Gamma_g(a)$ is called *$C_g(a)$–subordinate* if, for any sequence $(x_n) \in C_g(a)$, there exist $\alpha \in \Gamma_g(a)$, $t_0 \in \mathrm{dom}(\alpha)$, and a strictly decreasing real sequence (t_k), $t_k \to t_0$ such that $(\alpha(t_k))$ is a subsequence of (x_n).

If $g(a) > 0$, the family $\Gamma_g(a) = \Gamma(a)$ in the above definition becomes a strongly $C(a)$–subordinate family.

Definition 2.9. We say that the family $\Gamma_g(a)$ is *g–continuity-ensuring* if, for any function $f\colon D \to \mathbb{R}$ right continuous with respect to any $\alpha \in \Gamma_g(a)$, $f|_{g\geq 0}$ is continuous in a.

If $g(a) > 0$, the family $\Gamma_g(a) = \Gamma(a)$ in the above definition becomes a family specially continuity ensuring, and f is continuous in a.

In a similar fashion to Theorem 2.4, we can prove the following:

Theorem 2.6. *Let $\Gamma_g(a)$ be a family of parametrized curves as defined above. The following statements are equivalent:*

(1) *$\Gamma_g(a)$ is an optimal family.*
(2) *$\Gamma_g(a)$ is a $C_g(a)$–subordinate family.*
(3) *$\Gamma_g(a)$ is a continuity-ensuring family.*

Example 2.1. We denote by $\Gamma^m_g(a)$ the family of all parametrized curves $\alpha \in \Gamma^m(a)$, with $\alpha(t_0) = a$, such that $g(\alpha(t)) \geq 0$, for all $t \in [t_0, t_0 + \varepsilon)$. If $\mathrm{rank}\left[\frac{\partial g^i}{\partial x^j}(a)\right] = s$, then $\Gamma^m_g(a)$ is an optimal family ([8]).

The following open problems have been proposed in [8]

(1) Do there exist minimal elements with respect to inclusion in the class of all optimal families or in the class of all $C(a)$–subordinate families?

(2) Do there exist optimal families which are not $C(a)$–subordinate?

According to Theorem 2.4, the answer to the second question is negative.

The answer to the first question is positive. Indeed, let $\Gamma(a)$ be a $C(a)$–subordinate family of parametrized curves. This means that, for any sequence $(x_n) \in C(a)$, there exists a parametrized curve $\alpha \in \Gamma(a)$ satisfying the properties of Definition 2.1. This defines a relationship between the sets $C(a)$ and $\Gamma(a)$. Using the axiom of choice, we can find a function $\Phi : C(a) \to \Gamma(a)$. Let us consider $\Phi(C(a)) \subseteq \Gamma(a)$. It is obvious that $\Phi(C(a))$ is a minimal element with respect to inclusion in the class of all $C(a)$–subordinate families of parametrized curves, or, in other words, $\Phi(C(a))$ is a minimal element with respect to inclusion in the class of all optimal families.

3. **Conclusions**

In this paper we introduced and studied three concepts regarding a family of parametrized curves passing through a point a: (strongly) optimal family, (strongly) $C(a)$–subordinate and (specially) continuity ensuring. The fundamental result is Theorem 2.4, which states that these three concepts are equivalent. Using the notion of extremum point weakly constrained by a family $\Gamma(a)$, we were able to extend the previous result for a constrained extrema problem (Theorem 2.6).

For other developments of nonlinear optimization problems developed by our research team, see [9] $\div$ [22], and [30].

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