DUALS OF A BANACH ALGEBRA AS DUAL BANACH ALGEBRAS

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Let $\mathfrak{A}$ be a Banach algebra. We investigate $(2n)$-th $(n \geq 1)$ dual of $\mathfrak{A}$ as a dual Banach algebra. We show that, if $\mathfrak{A}^{(2n-4)}$ is Arens regular for some$(n \geq 2)$,then the weak amenability of $\mathfrak{A}^{(2n)}$ implies the weak amenability of $\mathfrak{A}^{**}$.

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1. Introduction

Throughout this paper $\mathfrak{A}$ is a Banach algebra. This algebra is called a *dual Banach algebra* if $\mathfrak{A} = E^*$ for a closed submodule E of $\mathfrak{A}^*$. This concept was introduced by V.Runde in [14]. For example the class of dual Banach algebras includes all W^*-algebras, all algebras $M(G)$ for locally compact groups G, all algebras $L(E)$ for reflexive Banach spaces E and all biduals of Arens regular Banach algebras.

For a Banach $\mathfrak{A}$-bimodule $X,$ *a derivation* from $\mathfrak{A}$ into X is a bounded linear map $D : \mathfrak{A} \to X$ satisfying

$$D(ab) = a.D(b) + D(a).b \ (a, b \in \mathfrak{A}).$$

This derivation is called *inner* if there is $x \in X$ such that

$$D(a) = a.x - x.a \ (a \in \mathfrak{A}).$$

The dual space X^* of X can be made into a Banach $\mathfrak{A}$-bimodule as well via

$$< a.f, x >=< f, xa >, < f.a, x >=< f, ax > \ \ (a \in \mathfrak{A}, f \in X^*, x \in X)$$

A Banach algebra $\mathfrak{A}$ is said to be *amenable* if every derivation $D : \mathfrak{A} \to X^*$ is inner, for every Banach $\mathfrak{A}$-bimodul X. $\mathfrak{A}$ is called *weakly amenable* if every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is inner ([2] and [12]).

The second dual $\mathfrak{A}^{**}$ is a Banach algebra with the *first [or second] Arnes product* $\square$ *[or* $\Diamond$*]* which are given by following formulas

$$< F\square G, f >=< F, G.f >$$

$$< G.f, a >=< G, f.a >$$

$$< F\Diamond G, f >=< G, f.F >$$

$$< f.F, a >=< F, a.f > \ \ (F, G \in \mathfrak{A}^{**}, f \in \mathfrak{A}^*, a \in \mathfrak{A}).$$

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The Banach algebra $\mathfrak{A}$ is called *Arens regular* if two products $\Box$ and $\Diamond$ coincide. We refer to Arens' original paper [1] and the survey paper [5].

As known, neither the amenability of $\mathfrak{A}$ implies that of $\mathfrak{A}^{**}$, nor the weak amenability of $\mathfrak{A}$ implies that of $\mathfrak{A}^{**}$, see [10,6]. By Gourdeau theorem the amenability of $\mathfrak{A}^{**}$ implies the amenability of $\mathfrak{A}$ (see[11]). But in generally weak amenability of $\mathfrak{A}^{**}$ dose not imply the weak amenability of $\mathfrak{A}$. This problem was considered by few outhors. We mention these results in following.

Proposition 1.1. *Let $\mathfrak{A}$ be a Banach algebra. In each of the following cases, the weak amenability of $\mathfrak{A}^{**}$ implies weak amenability of $\mathfrak{A}$.*

(1) *$\mathfrak{A}$ is left ideal in $\mathfrak{A}^{**}$,*
(2) *$\mathfrak{A}$ is right ideal in $\mathfrak{A}^{**}$ and $\mathfrak{A}^{**}\mathfrak{A} = \mathfrak{A}^{**}$,*
(3) *$\mathfrak{A}$ is dual Banach algebra,*
(4) *$\mathfrak{A}$ and the map $\varphi : \mathfrak{A} \times \mathfrak{A}^* \longrightarrow \mathfrak{A}^*$ by $\varphi(a, f) = a.f$ are Arens regular,*
(5) *$\mathfrak{A}$ is Arens regular and every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is weakly compact,*
(6) *The second adjoint of each derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ satisfies $D''(\mathfrak{A}^{**}) \subseteq WAP(\mathfrak{A})$.*

Proof. See [4,6,8,9,10]. ■

In section (2) we investigate $(2n)$-th $(n \geq 1)$ dual of $\mathfrak{A}$ as a dual Banach algebra, and we study the relations between Arens regularity of $\mathfrak{A}^{(2n-2)}$ and dual Banach algebra $\mathfrak{A}^{(2n)} (n \geq 1)$. In section (3) we extend the condition (3) of above proposition to $\mathfrak{A}^{**}$, $\mathfrak{A}^{(4)}$, ... and $\mathfrak{A}^{(2n)}$ as dual Banach algebras. Our work is similar to following results of Medghalchi and Yazdanpanah.

Proposition 1.2. *Let $\mathfrak{A}$ be an Arens regular Banach algebra such that $\mathfrak{A}^{(4)}$ is weakly amenable and each derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is weakly compact. Then $\mathfrak{A}$ is weakly amenable (see corollary (3.15) in [13]).*

Proposition 1.3. *Let $\mathfrak{A}$ be a* completely Arens regular *Banach algebra (i.e. $\mathfrak{A}^{(2n)}$ is Arens regular for every $n \in \mathbb{N}$) such that $\mathfrak{A}^{(2n)}$ is weakly amenable for some $n \in \mathbb{N}$, and each derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is weakly compact. Then $\mathfrak{A}$ is weakly amenable (see corollary (3.17) in [13]).*

A part of our corollary (3.3) is similar to proposition (1.2), but our proof will be shorter, because we use the concept of dual Banach algebra. Also a part of our corollary (3.5) is similar to proposition (1.3), but we assume the Arens regularity only for $A^{(2n-4)}$ for some $(n \geq 2)$ instead of complete Arens regularity. Finally we have some interesting results in commutative Banach algebras at the end of section (3).

2. $\mathfrak{A}^{(2n)}$ as a dual Banach algebra

Definition 2.1. *The Banach algebra $\mathfrak{A}$ is a* dual *Banach algebra if there is a closed submodule E of $\mathfrak{A}^*$ such that $\mathfrak{A} = E^*$. the space E is the* predual *of $\mathfrak{A}$ [14].*

Lemma 2.2. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A} = E^*$ as a Banach space for some Banach space E. Then $\mathfrak{A}$ is a dual Banach algebra (with predual E) if and only if the multiplication in $\mathfrak{A}$ is weak* separately continuous (see [14] and [3]).*

Using above lemma for $\mathfrak{A}^{**}$ and $\mathfrak{A}^{(2n)}$ $(n \geq 1)$ we have the following results (see also example (4) in [14]).

Corollary 2.3. *Let $\mathfrak{A}$ be a Banach algebra. Then $\mathfrak{A}^{**} = (\mathfrak{A}^*)^*$ is a dual Banach algebra (with perdual $\mathfrak{A}^*$) if and only if $\mathfrak{A}$ is Arens regular.*

Corollary 2.4. *Let $\mathfrak{A}$ be a Banach algebra and $(n \geq 1)$. Then $\mathfrak{A}^{(2n)}$ is a dual Banach algebra if and only if $\mathfrak{A}^{(2n-2)}$ is Arens regular.*

It is easy to check that if $\mathfrak{A}^{(2n)}$ for some $(n \geq 1)$ with one of the Arens products is Arens regular then $\mathfrak{A}^{(2n-2)}, ..., \mathfrak{A}^{**}$ and $\mathfrak{A}$ are Arens regular, so there is only one Arens product in each of the algebras $\mathfrak{A}^{(2n)}, ..., \mathfrak{A}^4$ and $\mathfrak{A}^{**}$. So we have

Proposition 2.5. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{(2n)}$ is a dual Banach algebra for some $(n \geq 1)$. Then $\mathfrak{A}^{(2n-2)}, ..., \mathfrak{A}^{(4)}$ and $\mathfrak{A}^{**}$ are dual Banach algebras.*

We can show above results in the following diagram, in which (AR) refers to Arens regularity and (DA) denots the dual Banach algebra. The symbols $(\rightarrow)$ and $(\leftrightarrow)$ show conclution and equivalency.

$$\mathfrak{A}^{(2n-2)}(AR) \rightarrow \mathfrak{A}^{(2n-4)}(AR) \rightarrow ... \rightarrow \mathfrak{A}^{**}(AR) \rightarrow \mathfrak{A}(AR)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$\mathfrak{A}^{(2n)}(DA) \rightarrow \mathfrak{A}^{(2n-2)}(DA) \rightarrow ... \rightarrow \mathfrak{A}^{(4)}(DA) \rightarrow \mathfrak{A}^{**}(DA).$$

Now Let $\mathfrak{A}$ be a commutative Banach algebra. We know that $\mathfrak{A}$ is Arens regular if and only if $\mathfrak{A}^{**}$ is commutative. Also $\mathfrak{A}$ is Arens regular if and if $\mathfrak{A}^{(2n)}$ is Arens regular, for every $n \in \mathbb{N}$ (see [7]). So we have

Proposition 2.6. *Let $\mathfrak{A}$ be a commutative Banach algebra, then $\mathfrak{A}^{(2n)}$ (with one of Arens products) $(n \geq 1)$ is dual Banach algebra if and only if $\mathfrak{A}^{**}$ is dual Banach algebra.*

Proof. $\mathfrak{A}^{(2n)} = (\mathfrak{A}^{2n-2})^{**}$ is dual if and only if $\mathfrak{A}^{(2n-2)}$ is Arens regular. Also $\mathfrak{A}^{(2n-2)}$ is Arens regular if and only if $\mathfrak{A}$ is Arens regular, and $\mathfrak{A}$ is Arens regular if and only if $\mathfrak{A}^{**}$ is dual. ■

So we have the following diagram in the commutative case:

$$\mathfrak{A}^{(2n)}(AR) \leftrightarrow \mathfrak{A}^{(2n-2)}(AR) \leftrightarrow ... \leftrightarrow \mathfrak{A}^{**}(AR) \leftrightarrow \mathfrak{A}(AR)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$\mathfrak{A}^{(2n+2)}(DA) \leftrightarrow \mathfrak{A}^{(2n)}(DA) \leftrightarrow ... \leftrightarrow \mathfrak{A}^{(4)}(DA) \leftrightarrow \mathfrak{A}^{**}(DA)$$

3. When weak amenability of $\mathfrak{A}^{(2n)}$ implies that of $\mathfrak{A}$?

We use the following result of Ghahramani and Laali in [9].

Theorem 3.1. *Let $\mathfrak{A}$ be a dual Banach algebra. If $\mathfrak{A}^{**}$ is weakly amenable, then $\mathfrak{A}$ is weakly amenable.*

First we consider $\mathfrak{A}^{**}$ as a dual Banach algebra with its first Arens product and apply theorem (3.1) for $\mathfrak{A}^{**}$. This will be similar to proposition (1.2), but our proof is shorter.

Proposition 3.2. *If $\mathfrak{A}$ be an Arens regular Banach algebra such that $\mathfrak{A}^{(4)}$ is weakly amenable, then $\mathfrak{A}^{**}$ is weakly amenable.*

Proof. Since $\mathfrak{A}$ is Arens regular, then $\mathfrak{A}^{**} = (\mathfrak{A}^*)^*$ is a dual Banach algebra by corollary (2.3). So $\mathfrak{A}^{**}$ is weakly amenable by theorem (3.1). ■

Corollary 3.3. *Let $\mathfrak{A}$ be an Arens regular Banach algebra with one of the following conditions*

(1) *every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is weakly compact,*
(2) *the map $\varphi : \mathfrak{A} \times \mathfrak{A}^* \to \mathfrak{A}^*(\varphi(a, f) = a.f)$ is Arens regular.*

If $\mathfrak{A}^{(4)}$ is weakly amenable, then $\mathfrak{A}$ is weakly amenable.

Proof. $\mathfrak{A}^{**}$ is weakly amenable by proposition (3.2) and then $\mathfrak{A}$ is weakly amenable by conditions (4) and (5) of proposition (1.1). ■

Now we apply theorem (3.1) for dual Banach algebra $A^{(2n-2)}(n \geq 2)$ with its first Arens product. We obtain the following result that is the general form of proposition (3.2).

Proposition 3.4. *If $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{(2n-4)}$ is Arens regular and $\mathfrak{A}^{(2n)}$ is weakly amenable for some $(n \geq 2)$, then $\mathfrak{A}^{**}$ is weakly amenable.*

Proof. Since $\mathfrak{A}^{(2n-4)}$ is Arens regular, then $\mathfrak{A}^{(2n-2)} = \left(\mathfrak{A}^{(2n-4)}\right)^{**}$ is a dual Banach algebra by corollary (2.4), so $\mathfrak{A}^{(2n-2)}$ is weakly amenable by theorem (3.1). Because $\mathfrak{A}^{(2n-2)}$ is dual Banach algebra then $\mathfrak{A}^{(2n-4)}$ is dual Banach algebra by proposition (2.5), and again we apply theorem (3.1) for $\mathfrak{A}^{(2n-4)}$ and we conclude that $\mathfrak{A}^{(2n-4)}$ is weakly amenable. Similarly $\mathfrak{A}^{(2n-6)}$, $\mathfrak{A}^{(2n-8)}$, ... and $\mathfrak{A}^{**}$ are dual Banach algebras by proposition (2.5). By frequent applying theorem (3.1) for these dual Banach algebras we conclude that they are weakly amenable. ■

Corollary 3.5. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{(2n-4)}$ is Arens regular for some $(n \geq 2)$, with one of the following conditions*

(1) *every derivation* $D : \mathfrak{A} \to \mathfrak{A}^*$ *is weakly compact,*

(2) *the map* $\varphi : \mathfrak{A} \times \mathfrak{A}^* \to \mathfrak{A}^*(\varphi(a, f) = a.f)$ *is Arens regular.*

If $\mathfrak{A}^{(2n)}$ *is weakly amenable, then* $\mathfrak{A}$ *is weakly amenable.*

Proof. $\mathfrak{A}^{**}$ is weakly amenable by proposition (3.4) and then $\mathfrak{A}$ is weakly amenable by conditions (4) and (5) of proposition (1.1). ■

In the commutative Banach algebras we have the following result

Proposition 3.6. *Let* $\mathfrak{A}$ *be a commutative and Arens regular Banach algebra, with one of the following conditions*

(1) *every derivation* $D : \mathfrak{A} \to \mathfrak{A}^*$ *is weakly compact,*

(2) *the map* $\varphi : \mathfrak{A} \times \mathfrak{A}^* \to \mathfrak{A}^*(\varphi(a, f) = a.f)$ *is Arens regular.*

If $\mathfrak{A}^{(2n)}$ *is weakly amenable for some* $n \geq 1$*, then* $\mathfrak{A}$ *is weakly amenable.*

Proof. We know that in commutative Banach algebra $\mathfrak{A}$, the Arens regularity of $\mathfrak{A}$ is equivalant to Arens regularity of each $\mathfrak{A}^{(2n)}(n \geq 1)$. So in particular $\mathfrak{A}^{(2n-4)}$ is Arens regular, and the assertion is proved by corollary (3.5). ■

Corollary 3.7. *Let* $\mathfrak{A}$ *be a dual Banach algebra such that* $\mathfrak{A}^{(2n-4)}$ *is Arens regular for some* $n \geq 2$*. If* $\mathfrak{A}^{(2n)}$ *is weakly amenable then* $\mathfrak{A}$ *is weakly amenable.*

Proof. This is a consequence of proposition (3.4) and theorem (3.1). ■

Corollary 3.8. *Let* $\mathfrak{A}$ *be a commutative, Arens regular and dual Banach algebra. If* $\mathfrak{A}^{(2n)}$ *is weakly amenable for some* $n \geq 1$*, then* $\mathfrak{A}$ *is weakly amenable.*

Proof. In commutative Banach algebra $\mathfrak{A}$, the Arens regularity of $\mathfrak{A}$ is equivalent to Arens regularity of $\mathfrak{A}^{(2n-4)}$. Hence $\mathfrak{A}^{**}$ is weakly amenable by proposition (3.4), and $\mathfrak{A}$ will be weakly amenable by theorem (3.1). ■

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