SET-VALUED OPTIMIZATION PROBLEMS UNDER CONE CONVEXITY

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In this paper, we study set-valued optimization problems, set-valued fractional programming problems, set-valued D. C. optimization problems, set-valued semi-infinite programming problems, and set-valued minimax programming problems via generalized cone convexity assumptions. We establish the existing results of optimality conditions of scalar and vector optimization problems in the setting of set-valued maps.

Keywords: convex cone, set-valued map, contingent epiderivative, weak subdifferential, set-valued optimization problem, approximate quasi efficient solutions, optimality conditions.

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1. Introduction

Set-valued optimization problem which is being a new branch of optimization problem attracts the attention of researchers to an increasing extent in the last few years. Set-valued optimization problem makes a bridge between different areas in optimization theory. It plays an important role in multiobjective programming problems, functional analysis, statistics, the theory of decision making, game theory, and approximation theory.

Antczak and Pitea [1] studied parametric approach to multitime multiobjective fractional variational problems under (F, ρ)-convexity. Pitea and Postolache [23] developed duality theorems for a new class of multitime multiobjective variational problems in 2012. They also [24, 25] established the necessary and sufficient efficiency conditions for minimization of vectors of curvilinear functionals on the second order jet bundle. Borwein [4] introduced the notion of cone convexity for set-valued maps. Cone convexity has a vital role in developing the existence theorem of efficient points of set-valued optimization problems. The concept of contingent derivative of set-valued maps was introduced by Aubin [3]. It is an extension of the concept of Frechet differentiability to the set-valued case. Corley [6] established the existence results and developed Lagrangian duality theory for set-valued maximization problems via contingent derivative. Jahn and Rauh [21] introduced another notion of differentiability of set-valued maps viz. the notion of contingent epiderivative, which is an extension of the concept of directional derivative to the set-valued case. Sheng and Liu [26] investigated the KKT conditions of set-valued optimization problems via generalized contingent epiderivative and preinvexity assumptions. The second-order KKT necessary and sufficient conditions of set-valued optimization problems have been established by Zhu et al. [27] via the second-order contingent derivative.

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Optimization problems involving difference of convex mappings are called D. C. optimization problems. The sufficient optimality conditions of generalized D. C. multiobjective optimization problems have been studied by Guo et al. [19] via the notion of subdifferential.

Semi-infinite and minimax programming problems are some classes of optimization problems. Mishra and Jaiswal [22] have established the sufficient optimality conditions and Mond-Weir type duality theorems of nondifferentiable multiobjective semi-infinite programming problems via generalized convexity assumptions.

Das and Nahak [7–18] introduced the notion of ρ-cone convexity of set-valued maps. They establish the sufficient KKT conditions and develop the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions.

In this paper, we study set-valued optimization problems, set-valued fractional programming problems, set-valued D. C. optimization problems, set-valued semiinfinite programming problems, and set-valued minimax programming problems via ρ-cone convexity assumptions.

2. Definition and Preliminaries

Let Y be a real normed space and K be a nonempty subset of Y. Then K is called a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Furthermore, K is called non-trivial if $K \neq \{\theta_Y\}$, proper if $K \neq Y$, pointed if $K \cap (-K) = \{\theta_Y\}$, solid if $\text{int}(K) \neq \emptyset$, closed if $\overline{K} = K$, and convex if $\lambda K + (1 - \lambda)K \subseteq K$, for all $\lambda \in [0, 1]$, where $\text{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and θ_Y is the zero element of Y.

Let K be a solid pointed convex cone in Y. There are two types of cone-orderings in Y with respect to K. For any two elements $y_1, y_2 \in Y$, we have

$$y_1 \leq y_2 \text{ if } y_2 - y_1 \in K$$

and

$$y_1 < y_2 \text{ if } y_2 - y_1 \in \text{int}(K).$$

The following notions of minimality are mainly used with respect to a solid pointed convex cone K in a real normed space Y.

Definition 2.1. *Let B be a nonempty subset of a real normed space Y. Then minimal and weakly minimal points of B are defined as*

(i) *$y' \in B$ is a minimal point of B if there is no $y \in B \setminus \{y'\}$, such that $y \leq y'$.*
(ii) *$y' \in B$ is a weakly minimal point of B if there is no $y \in B$, such that $y < y'$.*

We recall the notions of contingent cone and second-order contingent set in a real normed space.

Definition 2.2. *[2,3] Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, and $y' \in \overline{B}$. The contingent cone to B at y' is denoted by $T(B, y')$ and is defined as follows:*

An element $y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y.$$

Definition 2.3. *[2, 3, 5] Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, $y' \in \overline{B}$, and $u \in Y$. The second-order contingent set to B at y' in the direction u is denoted by $T^2(B, y', u)$ and defined as*

An element $y \in T^2(B, y', u)$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n u + \frac{1}{2}\lambda_n{}^2 y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$, $\{t'_n\}$ in $\mathbb{R}$, with $t_n, t'_n > 0$, $t_n \to \infty$, $t'_n \to \infty$, $\frac{t'_n}{t_n} \to 2$, and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to u \text{ and } t'_n(t_n(y'_n - y') - u) \to y.$$

Let X, Y be real normed spaces, 2^Y be the set of all subsets of Y, and K be a solid pointed convex cone in Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y, i.e., $F(x) \subseteq Y$, for all $x \in X$. The effective domain, image, graph, and epigraph of F are defined by

$$\operatorname{dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$F(A) = \bigcup_{x \in A} F(x), \text{ for any } \emptyset \neq A \subseteq X,$$

$$\operatorname{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\},$$

and

$$\operatorname{epi}(F) = \{(x, y) \in X \times Y : y \in F(x) + K\}.$$

Let A be a nonempty subset of X, $x' \in A$, $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \operatorname{dom}(F)$, and $y' \in F(x')$. Jahn and Rauh [21] introduced the notion of contingent epiderivative of set-valued maps which plays a vital role in various aspects of set-valued optimization problems.

Definition 2.4. *[21] A single-valued map $D_\uparrow F(x', y') : X \to Y$ whose epigraph coincides with the contingent cone to the epigraph of F at (x', y'), i.e.,*

$$\operatorname{epi}(D_\uparrow F(x', y')) = T(\operatorname{epi}(F), (x', y')),$$

is said to be the contingent epiderivative of F at (x', y').

Jahn et al. [20] introduced the notion of second-order contingent epiderivative of set-valued maps which also has a fundamental role in set-valued optimization problems.

Definition 2.5. *[20] A single-valued map $D_\uparrow^2 F(x', y', u, v) : X \to Y$ whose epigraph coincides with the second-order contingent set to the epigraph of F at $(x', y') \in \operatorname{gr}(F)$ in a direction $(u, v) \in X \times Y$, i.e.,*

$$\operatorname{epi}(D_\uparrow^2 F(x', y', u, v)) = T^2(\operatorname{epi}(F), (x', y'), (u, v)),$$

is said to be the second-order contingent epiderivative of F at (x', y') in the direction (u, v).

We now turn our attention to the notion of cone convexity of set-valued maps, introduced by Borwein [4].

Definition 2.6. *[4] Let A be a nonempty convex subset of a real normed space X. A set-valued map $F : X \to 2^Y$, with $A \subseteq \operatorname{dom}(F)$, is called K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

3. Set-Valued Fractional Programming Problems

Let X be a real normed space and A be a nonempty subset of X. Let $F : X \to 2^{\mathbb{R}^m}$, $G : X \to 2^{\mathbb{R}^m}$, and $H : X \to 2^{\mathbb{R}^k}$ be set-valued maps, with

$$A \subseteq \mathrm{dom}(F) \cap \mathrm{dom}(G) \cap \mathrm{dom}(H).$$

Let $F = (F_1, F_2, ..., F_m)$, $G = (G_1, G_2, ..., G_m)$, and $H = (H_1, H_2, ..., H_k)$, where the set-valued maps $F_i : X \to 2^{\mathbb{R}}$, $G_i : X \to 2^{\mathbb{R}}$; $i = 1, 2, ..., m$, and $H_j : X \to 2^{\mathbb{R}}$; $j = 1, 2, ..., k$, are defined by

$$\mathrm{dom}(F_i) = \mathrm{dom}(F), \mathrm{dom}(G_i) = \mathrm{dom}(G), \text{ and } \mathrm{dom}(H_j) = \mathrm{dom}(H),$$

$$x \in A, y = (y_1, y_2, ..., y_m) \in F(x) \implies y_i \in F_i(x), \forall i = 1, 2, ..., m,$$

$$z = (z_1, z_2, ..., z_m) \in G(x) \implies z_i \in G_i(x), \forall i = 1, 2, ..., m,$$

and

$$w = (w_1, w_2, ..., w_k) \in H(x) \implies w_j \in H_j(x), \forall j = 1, 2, ..., k.$$

Assume that $F_i(x) \subseteq \mathbb{R}_+$ and $G_i(x) \subseteq \mathrm{int}(\mathbb{R}_+), \forall i = 1, 2, ..., m$ and $x \in A$. Let $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_m) \in \mathbb{R}^m_+$. Define elements $\frac{y}{z}, \lambda' z \in \mathbb{R}^m$ and a subset $\lambda' G(x)$ of $\mathbb{R}^m$ by

$$\frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, ..., \frac{y_m}{z_m} \right)$$

$$\lambda' z = (\lambda'_1 z_1, \lambda'_2 z_2, ..., \lambda'_m z_m),$$

and

$$\lambda' G(x) = \{\lambda' z : z \in G(x)\}.$$

For $x \in A$, define a subset $\frac{F(x)}{G(x)}$ of $\mathbb{R}^m$ by

$$\frac{F(x)}{G(x)} = \Big\{ \frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, ..., \frac{y_m}{z_m} \right) : y = (y_1, y_2, ..., y_m) \in F(x),$$

$$z = (z_1, z_2, ..., z_m) \in G(x) \Big\}.$$

Consider a set-valued fractional programming problem (FP):

$$\begin{aligned} \underset{x \in A}{\text{minimize}} \quad & \frac{F(x)}{G(x)} \\ \text{subject to} \quad & H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset. \end{aligned} \tag{FP}$$

The feasible set of the problem (FP) is given by

$$S_{FP} = \{x \in A : H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset\}.$$

Definition 3.1. *A point $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$, with $x' \in S_{FP}$, $y' \in F(x')$, and $z' \in G(x')$, is called a minimizer of the problem* (FP) *if for all $(x, \frac{y}{z}) \in X \times \mathbb{R}^m$, with $x \in S_{FP}$, $y \in F(x)$, and $z \in G(x)$,*

$$\frac{y}{z} - \frac{y'}{z'} \notin (-\mathbb{R}^m_+) \setminus \{\boldsymbol{0}_{\mathbb{R}^m}\}.$$

Definition 3.2. *A point $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$, with $x' \in S_{FP}$, $y' \in F(x')$, and $z' \in G(x')$, is called a weak minimizer of the problem* (FP) *if for all $(x, \frac{y}{z}) \in X \times \mathbb{R}^m$, with $x \in S_{FP}$, $y \in F(x)$, and $z \in G(x)$,*

$$\frac{y}{z} - \frac{y'}{z'} \notin (-\mathrm{int}(\mathbb{R}^m_+)).$$

Definition 3.3. *Let A be a nonempty convex subset of a real normed space X, $e \in \text{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then F is said to be ρ-K-convex with respect to e on A if there exists $\rho \in \mathbb{R}$, such that*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho\lambda(1 - \lambda)\|x_1 - x_2\|^2 e + K,$$

$$\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

We construct an example of ρ-cone convex set-valued map, which is not cone convex.

Example 3.1. *Consider a set-valued map $F : [-1, 1] \subseteq \mathbb{R} \to 2^{\mathbb{R}^2}$ defined by*

$$F(t) = \begin{cases} \{(x - 2t^2, x^2 - 2t^2) : x \geq 0\}, & \text{if } 0 \leq t \leq 1, \\ \{(x - 2t^2, x - 2t^2) : x \leq 0\}, & \text{if } -1 \leq t < 0. \end{cases}$$

We prove that F is not $\mathbb{R}^2_+$-convex on $[-1, 1]$ but is (-2)-$\mathbb{R}^2_+$-convex with respect to $e = \mathbf{1}_{\mathbb{R}^2} = (1, 1)$ on $[-1, 1]$. □

Let $\lambda' \in \mathbb{R}^m_+$ and $G : X \to 2^{\mathbb{R}^m}$ be a set-valued map. Define a set-valued map $(-\lambda'G) : X \to 2^{\mathbb{R}^m}$ by

$$(-\lambda'G)(x) = -\lambda'G(x), \forall x \in \text{dom}(G).$$

We establish the sufficient optimality conditions of the problem (FP) by using the notion of contingent epiderivable and ρ-cone convexity assumptions.

Theorem 3.1. *(Sufficient optimality conditions) Let A be a nonempty convex subset of a real normed space X, x' be an element of the feasible set S_{FP} of the problem (FP), $y' \in F(x')$, $z' \in G(x')$, $\lambda' \in \frac{F(x')}{G(x')}$, $w' \in H(x') \cap (-L)$, and $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$. Assume that F, $-\lambda'G$ are ρ_1-$\mathbb{R}^m_+$-convex, ρ_2-$\mathbb{R}^m_+$-convex, respectively, with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A. Let F, $-\lambda'G$ be contingent epiderivable at (x', y'), $(x', -\lambda'z')$, respectively and H be contingent epiderivable at (x', w'). Suppose that there exists $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^* \neq \mathbf{0}_{\mathbb{R}^m}$, and*

$$(\rho_1 + \rho_2)\langle y^*, \mathbf{1}_{\mathbb{R}^m} \rangle + \rho_2 \langle z^*, \mathbf{1}_{\mathbb{R}^k} \rangle \geq 0, \tag{3.1}$$

such that

$$\begin{aligned} &\langle y^*, D_\uparrow F(x', y')(x - x') + D_\uparrow(-\lambda'G)(x', -\lambda'z')(x - x') \rangle \\ &+ \langle z^*, D_\uparrow H(x', w')(x - x') \rangle \geq 0, \forall x \in A, \end{aligned} \tag{3.2}$$

$$y' - \lambda'z' = \mathbf{0}_{\mathbb{R}^m}, \tag{3.3}$$

and

$$\langle z^*, w' \rangle = 0. \tag{3.4}$$

Then $(x', \frac{y'}{z'})$ is a weak minimizer of the problem (FP).

4. Optimization Problems with Difference of Set-Valued Maps

Let X, Y, Z be real normed spaces and A be a nonempty convex subset of X. Let K and L be solid pointed convex cones in Y and Z, respectively. Suppose that $F_1 : X \to 2^Y$, $F_2 : X \to 2^Y$, $G_1 : X \to 2^Z$, and $G_2 : X \to 2^Z$ are set-valued maps with

$$A \subseteq \text{dom}(F_1) \cap \text{dom}(F_2) \cap \text{dom}(G_1) \cap \text{dom}(G_2).$$

Consider an optimization problem (DP) with the difference of set-valued maps:

$$\begin{aligned} &\underset{x \in A}{\text{minimize}} && F_1(x) - F_2(x) \\ &\text{subject to} && (G_1(x) - G_2(x)) \cap (-L) \neq \emptyset. \end{aligned} \tag{DP}$$

Here, the feasible set S_{DP} of the problem (DP) is defined by

$$S_{DP} = \{x \in A : (G_1(x) - G_2(x)) \cap (-L) \neq \emptyset\}.$$

Definition 4.1. *A point $(x', y_1' - y_2') \in X \times Y$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, is called a minimizer of the problem* (DP) *if for all $(x, y_1 - y_2) \in X \times Y$, with $x \in S_{DP}$, $y_1 \in F_1(x)$, and $y_2 \in F_2(x)$,*

$$(y_1 - y_2) - (y_1' - y_2') \notin (-K) \setminus \{\theta_Y\}.$$

Definition 4.2. *A point $(x', y_1' - y_2') \in X \times Y$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, is called a weak minimizer of the problem* (DP) *if for all $(x, y_1 - y_2) \in X \times Y$, with $x \in S_{DP}$, $y_1 \in F_1(x)$, and $y_2 \in F_2(x)$,*

$$(y_1 - y_2) - (y_1' - y_2') \notin (-\text{int}(K)).$$

Let $y^* \in Y^*$, $z^* \in Z^*$, $x' \in A$, $y_2' \in F_2(x')$, $z_2' \in G_2(x')$, $T_1 \in \partial_s F_2(x'; y_2')$ and $T_2 \in \partial_s G_2(x'; z_2')$. Define maps $y^* T_1 + z^* T_2 : X \to \mathbb{R}$ and $y^* F_1 + z^* G_1 : X \to 2^{\mathbb{R}}$ by

$$(y^* T_1 + z^* T_2)(x) = \langle y^*, T_1(x) \rangle + \langle z^*, T_2(x) \rangle, x \in X$$

and

$$(y^* F_1 + z^* G_1)(x) = \langle y^*, F_1(x) \rangle + \langle z^*, G_1(x) \rangle, x \in \text{dom}(F_1) \cap \text{dom}(G_1).$$

We establish the necessary KKT conditions of the set-valued D. C. optimization problem (DP) by using ρ-cone convexity assumptions.

Theorem 4.1. *Let A be a nonempty convex subset of a real normed space X, $e_2 \in \text{int}(K)$, $e_2' \in \text{int}(L)$, and $\rho_2, \rho_2' \in \mathbb{R}$. Let $(x', y_1' - y_2')$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, be a weak minimizer of the problem* (DP) *and there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with*

$$z_1' - z_2' \in (-L).$$

Suppose that $F_1 : X \to 2^Y$, $G_1 : X \to 2^Z$ are K-convex, L-convex, respectively, on A. Also, suppose that $F_2 : X \to 2^Y$, $G_2 : X \to 2^Z$ are strongly ρ_2-K-convex, strongly ρ_2'-L-convex, with respect to e_2, e_2', respectively, on A. Assume that $\partial_s F_2(x'; y_2') \neq \emptyset$ and $\partial_s G_2(x'; z_2') \neq \emptyset$. Then there exists $(\theta_{Y^*}, \theta_{Z^*}) \neq (y^*, z^*) \in K^+ \times L^+$, such that

$$y^* T_1 + z^* T_2 \in \partial(y^* F_1 + z^* G_1)(x'; \langle y^*, y_1' \rangle + \langle z^*, z_1' \rangle),$$
$$\forall T_1 \in \partial_s F_2(x'; y_2') \text{ and } T_2 \in \partial_s G_2(x'; z_2')$$

and

$$\langle z^*, z_1' - z_2' \rangle = 0.$$

We establish the sufficient KKT conditions of the set-valued D. C. optimization problem (DP) by using ρ-cone convexity assumptions.

Theorem 4.2. *Let A be a nonempty convex subset of a real normed space X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \text{int}(K)$, and $e_2' \in \text{int}(L)$. Let $x' \in S_{DP}$, $y_1' \in F_1(x')$, $y_2' \in F_2(x')$, and there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with*

$$z_1' - z_2' \in (-L).$$

Suppose that $F_2 : X \to 2^Y$, $G_2 : X \to 2^Z$ are ρ_2-K-convex, ρ_2'-L-convex, with respect to e_2, e_2', respectively, on A. Assume that for any $x \in A$, $y_2 \in F_2(x)$, and $z_2 \in G_2(x)$, $\partial_s F_2(x; y_2) \neq \emptyset$ and $\partial_s G_2(x; z_2) \neq \emptyset$. If there exist $y^* \in K^+ \setminus \{\theta_{Y^*}\}$ and $z^* \in L^+$, satisfying

$$\rho_2 \langle y^*, e_2 \rangle + \rho_2' \langle z^*, e_2' \rangle \geq 0, \tag{4.5}$$

such that

$$y^* T_1 + z^* T_2 \in \partial(y^* F_1 + z^* G_1)(x'; \langle y^*, y_1' \rangle + \langle z^*, z_1' \rangle),$$
$$\forall T_1 \in \partial_s F_2(x; y_2), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x), \text{ and } z_2 \in G_2(x) \tag{4.6}$$

and

$$\langle z^*, z_1' - z_2' \rangle = 0, \tag{4.7}$$

then $(x', y_1' - y_2')$ is a weak minimizer of the problem (DP).

5. Set-Valued Semi-Infinite Programming Problems

We consider semi-infinite programming problems in the setting of set-valued maps. Let U be a countably infinite subset of $\mathbb{R}^p$, $\emptyset \neq A \subseteq \mathbb{R}^n$, and $F = (F_1, F_2, ..., F_m) : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$, $G : \mathbb{R}^n \times U \rightarrow 2^{\mathbb{R}}$ be two set-valued maps with

$$A \subseteq \operatorname{dom}(F) \text{ and } A \times U \subseteq \operatorname{dom}(G).$$

Let $B_1, B_2, ..., B_m$ be $n \times n$ positive semi-definite (symmetric) real matrices. Consider a set-valued semi-infinite programming problem (SP):

$$\begin{aligned} \underset{x \in A}{\text{minimize}} \quad & (F_1(x) + (x^T B_1 x)^{\frac{1}{2}}, F_2(x) + (x^T B_2 x)^{\frac{1}{2}}, ..., F_m(x) + (x^T B_m x)^{\frac{1}{2}}) \\ \text{subject to} \quad & G(x, u) \cap (-\mathbb{R}_+) \neq \emptyset, \forall u \in U. \end{aligned} \tag{SP}$$

The feasible set of the problem (SP) is defined by

$$S_{SP} = \{x \in A : G(x, u) \cap (-\mathbb{R}_+) \neq \emptyset, \forall u \in U\}.$$

Definition 5.1. *A point $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$, with $x' \in S_{SP}$ and $y' = (y_1', y_2', ..., y_m') \in F(x')$, is called a minimizer of the problem* (SP) *if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, with $x \in S_{SP}$ and $y = (y_1, y_2, ..., y_m) \in F(x)$,*

$$(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, ..., y_m + (x^T B_m x)^{\frac{1}{2}})$$
$$- (y_1' + (x'^T B_1 x')^{\frac{1}{2}}, y_2' + (x'^T B_2 x')^{\frac{1}{2}}, ..., y_m' + (x'^T B_m x')^{\frac{1}{2}}) \notin (-\mathbb{R}_+^m) \setminus \{\boldsymbol{0}_{\mathbb{R}^m}\}.$$

Definition 5.2. *A point $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$, with $x' \in S_{SP}$ and $y' = (y_1', y_2'..., y_m') \in F(x')$, is called a weak minimizer of the problem* (SP) *if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, with $x \in S_{SP}$ and $y = (y_1, y_2, ..., y_m) \in F(x)$,*

$$(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, ..., y_m + (x^T B_m x)^{\frac{1}{2}})$$
$$- (y_1' + (x'^T B_1 x')^{\frac{1}{2}}, y_2' + (x'^T B_2 x')^{\frac{1}{2}}, ..., y_m' + (x'^T B_m x')^{\frac{1}{2}}) \notin (-\operatorname{int}(\mathbb{R}_+^m)).$$

Let J be the index set, such that $U = \{u_j : j \in J\}$. Let $x' \in A$. Denote a set $J(x')$ by

$$J(x') = \{j \in J : 0 \in G(x', u_j)\}.$$

Throughout this chapter, we assume that $J(x') \neq \emptyset$. Let $\overline{x}_i \in \mathbb{R}^n$, $i = 1, 2, ..., m$. Define maps $.^T B_i \overline{x}_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2, ..., m$, by

$$(.^T B_i \overline{x}_i)(x) = x^T B_i \overline{x}_i, \forall x \in \mathbb{R}^n.$$

The gradient vector of $.^T B_i \overline{x}_i$, denoted by $\nabla(.^T B_i \overline{x}_i)$, is given by

$$\nabla(.^T B_i \overline{x}_i) = B_i \overline{x}_i.$$

Let $x' \in A$ and $j \in J(x')$. Define a set-valued map $G(., u_j) : \mathbb{R}^n \rightarrow 2^{\mathbb{R}}$ by

$$G(., u_j)(x) = G(x, u_j), \forall x \in \operatorname{dom}(G).$$

We establish the sufficient KKT conditions of the set-valued semi-infinite programming problem (SP) by using ρ-cone convexity assumptions.

Theorem 5.1. *(Sufficient optimality conditions) Let A be a nonempty convex subset of $\mathbb{R}^n$, $x' \in S_{SP}$, and $y' = (y_1', y_2', ..., y_m') \in F(x')$. Let $\overline{x}_i \in \mathbb{R}^n$, $i = 1, 2, ..., m$ and $z' = (z_j')_{j \in J}$, with $z_j' \in G(x', u_j) \cap (-\mathbb{R}_+)$. Let $\rho_i, \rho_i', \rho_j'' \in \mathbb{R}$, for $i = 1, 2, ..., m$ and $j \in J(x')$. Suppose that F_i, $.^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ_i'-$\mathbb{R}_+$-convex, and ρ_j''-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A. Assume that the contingent epiderivatives $D_\uparrow F_i(x', y_i')$ and $D_\uparrow G(., u_j)(x', z_j')$ exist. If there exist $y_i^* > 0$, $i = 1, 2, ..., m$, and $z_j^* \geq 0$, $j \in J(x')$, with $z_j^* \neq 0$, for finitely many j, and*

$$\sum_{i=1}^m y_i^*(\rho_i + \rho_i') + \sum_{j \in J(x')} z_j^* \rho_j'' \geq 0, \tag{5.8}$$

satisfying the following conditions

$$\Big(\sum_{i=1}^m y_i^*(D_\uparrow F_i(x', y_i') + (B_i \overline{x}_i)^T) + \sum_{j \in J(x')} z_j^* D_\uparrow G(., u_j)(x', z_j') \Big)(x - x')$$
$$\geq 0, \forall x \in A, \tag{5.9}$$

$$z_j^* z_j' = 0, \forall j \in J(x'), \tag{5.10}$$

$$\overline{x}_i^T B_i \overline{x}_i \leq 1, i = 1, 2, ..., m, \tag{5.11}$$

and

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m. \tag{5.12}$$

Then (x', y') is a weak minimizer of the problem (SP).

6. Set-Valued Minimax Programming Problems

Definition 6.1. *A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called upper semicontinuous if $F^+(V) = \{x \in \mathbb{R}^n : F(x) \subseteq V\}$ is open in $\mathbb{R}^n$ for any open set V in $\mathbb{R}^m$.*

Definition 6.2. *Let B be a nonempty subset of $\mathbb{R}^m$. Then B is called $\mathbb{R}_+^m$-semicompact if every open cover of complements of the form $\{(y_i + \mathbb{R}_+^m)^c : y_i \in B, i \in I\}$ has a finite subcover.*

Definition 6.3. *A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called $\mathbb{R}_+^m$-semicompact-valued if $F(x)$ is $\mathbb{R}_+^m$-semicompact, for all $x \in \mathrm{dom}(F)$.*

Corley [6] derived the existence results for maximization of upper semicontinuous and cone semicompact-valued set-valued maps.

Theorem 6.1. *[6] Let X, Y be real topological vector spaces, A be a nonempty compact subset of X, and K be an acute, (i.e., $\overline{K}$ is pointed) convex cone in Y. Let $F : X \to 2^Y$ be a K-semicompact-valued and upper semicontinuous set-valued map. Then there exists a maximal point of the problem* $\max \bigcup_{x \in A} F(x)$.

For simplicity, let us assume $X = \mathbb{R}^m$, $Y = \mathbb{R}$, and $K = \mathbb{R}_+$.
Let A be a nonempty subset of $\mathbb{R}^n$ and B be a nonempty compact subset of $\mathbb{R}^m$. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ and $G : \mathbb{R}^n \to 2^{\mathbb{R}^p}$ be two set-valued maps with

$$A \times B \subseteq \mathrm{dom}(\Phi) \text{ and } A \subseteq \mathrm{dom}(G).$$

Consider a set-valued minimax programming problem (MP):

$$\begin{aligned} \underset{x \in A}{\text{minimize}} \quad & \max \bigcup_{y \in B} \Phi(x, y) \\ \text{subject to} \quad & G(x) \cap (-\mathbb{R}_+^p) \neq \emptyset, \end{aligned} \tag{MP}$$

where the set-valued map $\Phi(x,.) : \mathbb{R}^m \to 2^{\mathbb{R}}$ is $\mathbb{R}_+$-semicompact-valued and upper semi-continuous on B, for all $x \in A$. Therefore, by Theorem 6.1, $\max \bigcup_{y \in B} \Phi(x,y)$ always exists, for all $x \in A$. As $\Phi(x,y) \subseteq \mathbb{R}$, for each $x \in A$ there exists only one maximal point of the problem $\max \bigcup_{y \in B} \Phi(x,y)$.

The feasible set of the problem (MP) is given by

$$S_{MP} = \{x \in A : G(x) \cap (-\mathbb{R}_+^p) \neq \emptyset\}.$$

For $x \in A$, define following sets by

$$I(x) = \{j : 0 \in G_j(x), 1 \leq j \leq p\},$$

$$J(x) = \{1, 2, ..., p\} \setminus I(x),$$

and

$$B(x) = \{b \in B : \max \bigcup_{y \in B} \Phi(x,y) \in \Phi(x,b)\}.$$

Under the assumptions, $B(x) \neq \emptyset$, for all $x \in A$.

Definition 6.4. *Let $x' \in S_{MP}$ and $z' = \max \bigcup_{y \in B} \Phi(x',y)$. Then (x', z') is called a minimizer of the problem* (MP) *if for all $x \in S_{MP}$ and $z = \max \bigcup_{y \in B} \Phi(x,y)$,*

$$z' \leq z.$$

The sufficient KKT conditions of the set-valued minimax programming problem (MP) are established by using ρ-cone convexity assumptions.

Theorem 6.2. *(Sufficient optimality conditions) Let A be a nonempty convex subset of $\mathbb{R}^n$, $x' \in S_{MP}$, and $z' = \max \bigcup_{y \in B} \Phi(x',y)$. Assume that there exist a positive integer k, $z_i^* \geq 0$, $y_i \in B(x')$, $(1 \leq i \leq k)$ with $\sum_{i=1}^{k} z_i^* \neq 0$, and $w_j^* \geq 0$, $w_j' \in G_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$, such that*

$$\sum_{i=1}^{k} z_i^* D_\uparrow \Phi(.,y_i)(x',z')(x-x') + \sum_{j=1}^{p} w_j^* D_\uparrow G_j(x',w_j')(x-x') \geq 0, \tag{6.13}$$
$$\forall x \in A$$

and

$$w_j^* w_j' = 0, \forall j = 1, 2, ..., p. \tag{6.14}$$

Let $\rho_i, \rho_j' \in \mathbb{R}$, for $i = 1, 2, ..., k$ and $j = 1, 2, ..., p$. If $\Phi(.,y_i)$, $(1 \leq i \leq k)$ and G_j, $(1 \leq j \leq p)$ are ρ_i-$\mathbb{R}_+$-convex and ρ_j'-$\mathbb{R}_+$-convex, respectively, with respect to 1, on A, with

$$\sum_{i=1}^{k} z_i^* \rho_i + \sum_{j=1}^{p} w_j^* \rho_j' \geq 0, \tag{6.15}$$

then (x', z') is a minimizer of the problem (MP).

7. Set-Valued Optimization Problems via Contingent Epiderivative

Let X, Y, Z be real normed spaces and A be a nonempty subset of X. Let K and L be solid pointed convex cones of Y and Z, respectively. Suppose that $F : X \to 2^Y$ and $G : X \to 2^Z$ are two set-valued maps, with

$$A \subseteq \mathrm{dom}(F) \cap \mathrm{dom}(G).$$

Consider the set-valued optimization problem (P):

$$\begin{aligned}
&\underset{x \in A}{\text{minimize}} \quad F(x) \\
&\text{subject to} \quad G(x) \cap (-L) \neq \emptyset.
\end{aligned} \tag{P}$$

The feasible set of the problem (P) is given by

$$S_P = \{x \in A : G(x) \cap (-L) \neq \emptyset\}.$$

We introduce the notions of ρ-(η, θ)-cone preinvexity and ρ-(η, θ)-cone invexity of set-valued maps.

Definition 7.1. *Let A be an invex subset of a real normed space X with respect to $\eta : A \times A \to X$, $e \in \mathrm{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \mathrm{dom}(F)$. Then F is said to be ρ-(η, θ)-K-preinvex with respect to e on A if there exist a map $\theta : A \times A \to X$ and $\rho \in \mathbb{R}$, such that*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(x_2 + \lambda \eta(x_1, x_2)) + \rho \lambda (1 - \lambda)\|\theta(x_1, x_2)\|^2 e + K,$$
$$\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

Definition 7.2. *Let A be a nonempty subset of a real normed space X, $e \in \mathrm{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \mathrm{dom}(F)$. Let $x' \in A$ and $y' \in F(x')$. Assume that F is contingent epiderivable at (x', y'). Then F is said to be ρ-(η, θ)-K-invex with respect to e at (x', y') on A if there exist maps $\eta, \theta : A \times A \to X$ and $\rho \in \mathbb{R}$, with*

$$\eta(A, x') \subseteq \mathrm{dom}(D_{\uparrow}F(x', y')),$$

such that

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(\eta(x, x')) + \rho\|\theta(x, x')\|^2 e + K, \forall x \in A.$$

We give the following example of ρ-(η, θ)-cone invex set-valued map for some suitable ρ, η and θ, which is not an η-invex set-valued map for any η.

Example 7.1. *Consider a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ defined by*

$$F(\lambda) = \begin{cases} \{(x, \sqrt{x}) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, \sqrt{-x}) : x \in [-4, 0]\}, & \text{if } \lambda < 0. \end{cases}$$

Take $K = \mathbb{R}_+^2$. Here F is not η-invex at $(0, (0, 0))$ on $\mathbb{R}$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\rho = -1$, and $e = \mathbf{1}_{\mathbb{R}^2}$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that, $\theta(\lambda, 0) = \begin{cases} 0, & \text{if } \lambda \geq 0, \\ 2, & \text{if } \lambda < 0. \end{cases}$
We prove that F is ρ-(η, θ)-$\mathbb{R}_+^2$-invex with respect to $\mathbf{1}_{\mathbb{R}^2}$ at $(0, (0, 0))$ on $\mathbb{R}$. $\qquad \square$

We establish the KKT sufficient optimality conditions of the problem (P) by using contingent epiderivative and ρ-(η, θ)-cone preinvexity assumptions.

Theorem 7.1. *(Sufficient optimality conditions) Let A be an invex subset of a real normed space X with respect to a map $\eta : A \times A \to X$, $x' \in S_P$, $y' \in F(x')$, and $z' \in G(x') \cap (-L)$. Let $e \in \mathrm{int}(K)$ and $e' \in \mathrm{int}(L)$. Let $\rho_1, \rho_2 \in \mathbb{R}$ and $\eta, \theta : A \times A \to X$ be two maps. Assume*

that F, G are ρ_1-(η, θ)-K-preinvex, ρ_2-(η, θ)-L-preinvex with respect to e, e', respectively, on A. Let F and G be contingent epiderivable at (x', y') and (x', z'), respectively, with

$$\eta(A, x') \subseteq \mathrm{dom}(D_\uparrow F(x', y')) \cap \mathrm{dom}(D_\uparrow G(x', z')).$$

If there exists $(y^*, z^*) \in K^+ \times L^+$, with $y^* \neq \theta_{Y^*}$ and

$$\rho_1 \langle y^*, e \rangle + \rho_2 \langle z^*, e' \rangle \geq 0, \tag{7.16}$$

such that

$$\langle y^*, D_\uparrow F(x', y')(\eta(x, x')) \rangle + \langle z^*, D_\uparrow G(x', z')(\eta(x, x')) \rangle \geq 0, \forall x \in A \tag{7.17}$$

and

$$\langle z^*, z' \rangle = 0. \tag{7.18}$$

Then (x', y') is a weak minimizer of the problem (P).

We illustrate the Theorem 7.1 by the following example.

Example 7.2. *We consider a primal problem* (P)*, where $A = \mathbb{R}$, $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ is defined like Example 7.1, and $G : \mathbb{R} \to 2^{\mathbb{R}^2}$ is defined as*

$$G(\lambda) = \begin{cases} \{(x, x^2) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, x) : x > 8\}, & \text{if } \lambda < 0. \end{cases}$$

Take $K = \mathbb{R}^2_+$ and $L = \mathbb{R}^2_+$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\rho_1 = -1$, and $\rho_2 = 2$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that,

$$\theta(\lambda, 0) = \begin{cases} 0, & \text{if } \lambda \geq 0, \\ 2, & \text{if } \lambda < 0. \end{cases}$$

We prove that G is ρ_2-(η, θ)-$\mathbb{R}^2_+$-invex with respect to $\mathbf{1}_{\mathbb{R}^2}$ at $(1, (0, 0))$ on $\mathbb{R}$. From Example 7.1, F is ρ_1-(η, θ)-$\mathbb{R}^2_+$-invex with respect to $\mathbf{1}_{\mathbb{R}^2}$ at $(1, (0, 0))$ on $\mathbb{R}$. It is clear that for $y^* = z^* = (1, 1)$, Eqs. (7.17) and (7.18) are satisfied. Therefore, $(\lambda', (x', y')) = (1, (0, 0))$ is a weak minimizer of the problem (P). $\square$

8. Set-Valued Optimization Problems via Second-Order Contingent Epiderivative

We introduce second-order ρ-(η, θ)-cone invexity of set-valued maps via second-order contingent epiderivative.

Definition 8.1. *Let A be a nonempty subset of a real normed space X, $e \in int(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \mathrm{dom}(F)$. Let $x', u \in A$, $y' \in F(x')$, and $v \in F(u) + K$. Assume that F is second-order contingent epiderivable at (x', y') in the direction $(u - x', v - y')$. Then F is said to be second-order ρ-(η, θ)-K-invex with respect to e at (x', y') in the direction $(u - x', v - y')$ on A if there exist maps $\eta, \theta : A \times A \to X$, and $\rho \in \mathbb{R}$, with*

$$\eta(A, x') \subseteq \mathrm{dom}(D^2_\uparrow F(x', y', u - x', v - y')),$$

such that

$$F(x) - y' \subseteq D^2_\uparrow F(x', y', u - x', v - y')(\eta(x, x')) + \rho \|\theta(x, x')\|^2 e + K, \forall x \in A.$$

We construct the following set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$, which is second-order ρ-(η, θ)-$\mathbb{R}^2_+$-invex for some ρ, η and θ, but is not second-order η-invex, for any η.

Example 8.1. *Let a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ be defined by*

$$F(\lambda) = \begin{cases} \{(x, \sqrt{x}) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, \sqrt{-x}) : x \in [-4, 0]\}, & \text{if } \lambda < 0. \end{cases}$$

Let $K = \mathbb{R}_+^2$. Here F is not second-order η-invex at $(0, (0,0))$ in the direction $(-1, (0,0))$ on $\mathbb{R}$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\rho = -1$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that, $\theta(\lambda, 0) = \begin{cases} 0, & \text{if } \lambda \geq 0, \\ 2, & \text{if } \lambda < 0. \end{cases}$

We prove that F is second-order ρ-(η, θ)-$\mathbb{R}_+^2$-invex with respect to $\mathbf{1}_{\mathbb{R}^2}$ at $(0, (0,0))$ in the direction $(-1, (0,0))$ on $\mathbb{R}$. □

We establish the second-order KKT sufficient optimality conditions of the problem (P) via second-order contingent epiderivative and second-order ρ-(η, θ)-cone invexity assumptions.

Theorem 8.1. *(Sufficient optimality conditions) Let x' be a feasible point of the problem (P), $y' \in F(x')$, and $z' \in G(x') \cap (-L)$. Let $u \in A$, $v \in F(u) + K$, $w \in G(u) + L$, $e \in \text{int}(K)$, and $e' \in \text{int}(L)$. Let $\rho_1, \rho_2 \in \mathbb{R}$ and $\eta, \theta : A \times A \to X$ be two maps. Assume that F is second-order ρ_1-(η, θ)-K-invex with respect to e at (x', y') in the direction $(u - x', v - y')$ and G is second-order ρ_2-(η, θ)-L-invex with respect to e' at (x', z') in the direction $(u - x', w - z')$, on A. Suppose that there exists $(y^*, z^*) \in K^+ \times L^+$, with $y^* \neq \theta_{Y*}$, satisfying*

$$\rho_1 \langle y^*, e \rangle + \rho_2 \langle z^*, e' \rangle \geq 0, \tag{8.19}$$

such that

$$\begin{aligned} &\langle y^*, D_\uparrow^2 F(x', y', u - x', v - y') \eta(x, x') \rangle \\ &+ \langle z^*, D_\uparrow^2 G(x', z', u - x', w - z') \eta(x, x') \rangle \geq 0, \forall x \in A \end{aligned} \tag{8.20}$$

and

$$\langle z^*, z' \rangle = 0. \tag{8.21}$$

Then (x', y') is a weak minimizer of the problem (P).

We illustrate the Theorem 8.1 by the following example.

Example 8.2. *We consider a primal problem (P), where $X = \mathbb{R}$, the set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ is given in Example 8.1, and $G : \mathbb{R} \to 2^{\mathbb{R}^2}$ is defined as*

$$G(\lambda) = \begin{cases} \{(x, x^2) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, x) : x \geq 8\}, & \text{if } \lambda < 0. \end{cases}$$

Let $K = \mathbb{R}_+^2$ and $L = \mathbb{R}_+^2$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\rho_1 = -1$, and $\rho_2 = 2$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that, $\theta(\lambda, 0) = \begin{cases} 0, & \text{if } \lambda \geq 0, \\ 2, & \text{if } \lambda < 0. \end{cases}$

It is clear that for $y^* = z^* = (1, 1)$, Eqs. (8.20) and (8.21) are satisfied. Therefore, $(\lambda', (x', y')) = (0, (0,0))$ is a weak minimizer of the problem (P). □

9. Conclusions

In this paper, we introduce set-valued fractional programming problems, set-valued D. C. optimization problems, set-valued semi-infinite programming problems, and set-valued minimax programming problems. We establish the optimality conditions of various types of set-valued optimization problems via ρ-cone convexity assumptions.

REFERENCES

[1] *T. Antczak and A. Pitea*, Parametric approach to multitime multiobjective fractional variational problems under (F, ρ)-convexity, Optim. Control Appl. Methods, **37**(2016), No. 5, 831–847.

[2] *J. P. Aubin and H. Frankowska*, Set-Valued Analysis, Birhäuser, Boston, 1990.

[3] *J. P. Aubin*, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, in: Mathematical Analysis and Applications, Part A, L. Nachbin (ed.), Academic Press, New York, pp. 160–229, 1981.

[4] *J. Borwein*, Multivalued convexity and optimization: a unified approach to inequality and equality constraints, Math. Program, **13**(1977), No. 1, 183–199.

[5] *A. Cambini, L. Martein and M. Vlach*, Second order tangent sets and optimality conditions, Math. Japonica, **49**(1999), No. 3, 451–461.

[6] *H. W. Corley*, Existence and Lagrangian duality for maximizations of set-valued functions, J. Optim. Theory Appl., **54**(1987), No. 3, 489–501.

[7] *K. Das and C. Nahak*, Sufficient optimality conditions and duality theorems for set-valued optimization problem under generalized cone convexity, Rend. Circ. Mat. Palermo, **63**(2014), No. 3, 329–345.

[8] *K. Das and C. Nahak*, Sufficiency and duality of set-valued optimization problems via higher-order contingent derivative, J. Adv. Math. Stud., **8**(2015), No. 1, 137–151.

[9] *K. Das and C. Nahak*, Optimality conditions for approximate quasi efficiency in set-valued equilibrium problems, SeMA J., **73**(2016), No. 2, 183–199.

[10] *K. Das and C. Nahak*, Set-valued fractional programming problems under generalized cone convexity, Opsearch, **53**(2016), No. 1, 157–177.

[11] *K. Das and C. Nahak*, Approximate quasi efficiency of set-valued optimization problems via weak subdifferential, SeMA J., **74**(2017), No. 4, 523–542.

[12] *K. Das and C. Nahak*, Optimization problems with difference of set-valued maps under generalized cone convexity, J. Appl. Math. Inform., **35**(2017), No. 1–2, 147–163.

[13] *K. Das and C. Nahak*, Set-valued minimax programming problems under generalized cone convexity, Rend. Circ. Mat. Palermo, **66**(2017), No. 3, 361–374.

[14] *K. Das and C. Nahak*, Optimality conditions for set-valued minimax fractional programming problems, SeMA J., **77**(2020), No. 2, 161–179.

[15] *K. Das and C. Nahak*, Optimality conditions for set-valued minimax programming problems via second-order contingent epiderivative, J. Sci. Res., **64**(2020), No. 2, 313–321.

[16] *K. Das and C. Nahak*, Sufficiency and duality in set-valued optimization problems under (p, r)-ρ-(η, θ)-invexity, Acta Univ. Apulensis, **62**(2020), 93–110.

[17] *K. Das and C. Nahak*, Sufficiency and duality of set-valued semi-infinite programming problems under generalized cone convexity, Acta Univ. M. Belii Ser. Math., **2020**(2020), 95–111.

[18] *K. Das and C. Nahak*, Set-valued optimization problems via second-order contingent epiderivative, Yugosl. J. Oper. Res., **31**(2021), No. 1, 75–94.

[19] *X. L. Guo, S. J. Li and K. L. Teo*, Subdifferential and optimality conditions for the difference of set-valued mappings, Positivity, **16**(2012), No. 2, 321–337.

[20] *J. Jahn, A. Khan and P. Zeilinger*, Second-order optimality conditions in set optimization, J. Optim. Theory Appl., **125**(2005), No. 2, 331–347.

[21] *J. Jahn and R. Rauh*, Contingent epiderivatives and set-valued optimization, Math. Method Oper. Res., **46**(1997), No. 2, 193–211.

[22] *S. K. Mishra and M. Jaiswal*, Optimality conditions and duality for nondifferentiable multiobjective semi-infinite programming, Vietnam J. Math., **40**(2012), No. 2&3, 331–343.

[23] *A. Pitea and M. Postolache*, Duality theorems for a new class of multitime multiobjective variational problems, J. Glob. Optim., **53**(2012), No. 1, 47–58.

[24] *A. Pitea and M. Postolache*, Minimization of vectors of curvilinear functionals on the second order jet bundle. Necessary conditions, Optim. Lett., **6**(2012), No. 3, 459–470.

[25] *A. Pitea and M. Postolache*, Minimization of vectors of curvilinear functionals on the second order jet bundle. Sufficient efficiency conditions, Optim. Lett., **6**(2012), No. 8, 1657–1669.

[26] *B. Sheng and S. Liu*, Kuhn-Tucker condition and Wolfe duality of preinvex set-valued optimization, Appl. Math. Mech.-Engl., **27**(2006), 1655–1664.

[27] *S. K. Zhu, S. J. Li and K. L. Teo*, Second-order Karush-Kuhn-Tucker optimality conditions for set-valued optimization, J. Glob. Optim., **58**(2014), No. 4, 673–692.