WEAKLY CONTRACTIVE OPERATORS IN JLELI-SAMET GENERALIZED METRIC SPACES

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This article brings together two types of operators that have been studied in the literature: Geraghty-type mappings and Kannan-type contractions. The use of a mixture of these classes leads to fixed point results in the framework of the metric spaces generalized in the sense of Jleli and Samet. A partial order relation is used to prove our results, covering some important particular cases already studied. In addition, some examples and comments emphasize the importance and applicability of the new theorems.

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1. Introduction

The core of this area of study comes from the celebrated Banach fixed point principle, that has fundamental applications in mathematics, such as topology or partial differential equations. As the research on this domain grew up, developments have been made with respect to the original contractive condition. The Kannan condition or Chatterjea contraction completed and provided more profound meaning to the classic Banach inequality. The occurence of the well known article of Geraghty [6], in which the author presents an innovative weak contraction, led to a revolution in the theory of fixed points. Various types of improvements of the original work have been made until today. The work of Seong-Hoo Cho *et al.* [14] modified the original Geraghty-type contraction. In the paper of Parvaneh *et al.* [12], there are presented the so called rational Geraghty type contractions. An interesting study of fixed point results for Geraghty-type contractions in different spaces can be found in the work [17] due to Wiriyapongsanon *et al.*. A research for common fixed points regarding mappings with different contractive properties is made in Faraji *et al.* [5].

Theorems found for different contractive operators depend closely on the nature of the spaces involved in the study. In the majority of cases, results are proved in the settings of classical metric spaces. As time passed, the need for more general contexts of work appeared. A natural way of extending the notion of metric space is to modify one of the three axioms in defining them. A wide number of classes of spaces developed in the literature and fixed points results have been stated, see, for example [1], [2], [3], [4], [8], [9] [11], [13], [15], [16].

Our aim is to combine all elements to give a new solution to the problem of Geraghty-type contraction in Jleli-Samet context. Using adequate classes of functions, we are going to define Geraghty-Kannan contractions as a combination of all these notions in a new contractive inequality. There will be considered generalized spaces endowed with a partial order, fact which makes our proofs valid for different structures on these spaces. A consistent number of examples and corollaries will make this article proper for further studies.

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2. **Preliminaries**

From the beginning, we familiarize the reader with the notion of generalized metric space, as it has been defined in the original work of Jlei and Samet [7]. The main change that has been made resides in the third condition of such a space. The triangle inequality has been replaced by a more general relation, as follows, obtaining a class which contains a significant number of generalized metrics.

Definition 2.1. ([7]) Choosing a nonempty set X, define the application $D\colon X\times X\to[0,\infty]$ (playing the role of a "distance"). For every element $x_*\in X$, consider the set of all sequences $\{x_n\}_n\subset X$ that converge to x_* in the "metric" D

$$\mathcal{C}(D,X,x_*)=\big\{\{x_n\}_n\subset X:\lim_{n\to\infty}D(x_n,x_*)=0\big\}.$$

We are going to say (according to the original paper [7]) that D is a JS-metric on X, if the following axioms are accomplished:

($D1$) For every $x,\ y\in X$, the next implication holds true

$$D(x,y)=0\Longrightarrow x=y;$$

($D2$) For every $x,\ y\in X,\ D(x,y)=D(y,x)$;

($D3$) There is a constant $C>0$ such that for every $x,\ y\in X$, and for every sequence such that $\{x_n\}_n\in\mathcal{C}(D,X,x)$, the inequality below is ensured:

$$D(x,y)\le C\limsup_{n\to\infty}D(x_n,y).$$

We underline that this class contains other metrics besides the classic ones (metric, b-metric, dislocated metric and so on).

In our paper, the pair (X,D) will designate a Jleli-Samet metric space (or simply JS-space).

It is said that the sequence $\{x_n\}$ is convergent to $x\in X$ if

$$\lim_{n\to\infty}D(x_n,x)=0.$$

A direct computation will reveal that every convergent sequence in (X,D) has a unique limit.

$\{x_n\}_n$ is a Cauchy sequence in (X,D) if

$$\lim_{m,n\to\infty}D(x_m,x_n)=0.$$

Keeping some good properties from the usual metrics, it can be easily proved that all convergent sequences are in fact D-Cauchy ones. Since every classic metric is a Jleli-Samet metric, it is quite obvious that the converse remains false. A JS-space X in which all D-Cauchy sequences are D-convergent to some element in X it will be called D-complete.

Let us define some useful sets that will play an important role in the proofs of ours results in the following section. Let $\delta_{n_0}(D,T,\tilde{x})$ be defined as

$$\delta_{n_0}(D,T,\tilde{x})=\sup(\{D(T^n\tilde{x},T^m\tilde{x}):n,m\in\mathbb{N},n,m\ge n_0\}),$$

where $n_0\in\mathbb{N}$.

Consider also

$$\delta(D,T,\tilde{x})=\sup(\{D(T^n\tilde{x},T^m\tilde{x}):n,m\in\mathbb{N}\}).$$

The orbit of an element $\tilde{x}$ by a mapping $T\colon X\to X$ is defined as follows:

$$\mathcal{O}_T(\tilde{x})=\{T^n\tilde{x}:n\in\mathbb{N}\}.$$

In order to make our results more general, we work with a binary relation on the space (X,D), defined as a subset of the Cartesian product $X\times X$. We are going to denote it by $\mathbf{P}$. As a consequence of our notation, the fact that $(a,b)\in\mathbf{P}$ will be denoted by $a\mathbf{P}b$.

Let us remind the fact that a binary relation on X is called a partial order on X if it is reflexive, transitive and antisymmetric.

Introduce the following set

$$E_{\leq} = \{(x, y) \in X \times X : x \leq y\}$$

where $\leq$ is a partial order on X, which will be used in the equel. A binary relation $\mathbf{P}$ is said to be a preorder if it is only reflexive and transitive.

A useful notion in the study of ordered spaces is the concept of regularity.

Definition 2.2. ([7]) The JS-space (X, D) will be called D-regular if the following fact is accomplished: for every sequence $\{s_n\}_n \subset X$ for which $(s_n, s_{n+1}) \in E_{\leq}$ for every n that is large enough, if $\{s_n\}_n$ is D-convergent to $s_* \in X$, then there exists a subsequence $\{s_{n_k}\}_k$ of $\{s_n\}_n$ such that $(s_{n_k}, s_*) \in E_{\leq}$, for every k large enough.

Definition 2.3. ([10]) For a given preorder $\mathbf{P}$, a sequence $\{s_n\}_n \subset X$ is $\mathbf{P}$-nondecreasing if $s_n \mathbf{P} s_{n+1}$ for all $n \in \mathbb{N}$.

A kind of monotony can be defined for any preorder $\mathbf{P}$, as in the next lines.

Definition 2.4. ([10]) The Jleli-Samat generalized metric space (X, D) is called $\mathbf{P}$-nondecreasing-regular, where $\mathbf{P}$ is a preorder, if for every sequence $\{s_n\}_n \in \mathcal{C}(D, X, s_*)$ which is $\mathbf{P}$-nondecreasing, it follows that $s_n \mathbf{P} s_*$, for all $n \in \mathbb{N}$.

In order to be able to study operators defined on ordered spaces, we need the next definition.

Definition 2.5. Let (X, D) be a Jleli-Samet space having a binary relation $\mathbf{P}$, that is in fact a preorder, and $T\colon X \to X$. The operator T is called P-nondecreasing if $x\mathbf{P}y$ implies $Tx\mathbf{P}Ty$ for all x, $y \in X$.

The regularity of the Jleli-Samet spaces implies $\mathbf{P}$-nondecreasing-regularity, for any binary relation $\mathbf{P}$ (considered to be a partial order), but the converse does not happen.

Definition 2.6. ([10]) The Jleli-Samat metric space (X, D) is called $\mathbf{P}$-nondecreasing-complete if every $\{s_n\}_n$ that is D-Cauchy and $\mathbf{P}$-nondecreasing is in fact D-convergent in X.

Any Jleli-Samet metric space that is complete is also $\mathbf{P}$-nondecreasing complete, but the converse does not apply.

Definition 2.7. ([10]) A mapping $T\colon X \to X$ is said to be $\mathbf{P}$-nondecreasing-continuous at $q \in X$ if $\{Ts_n\}_n \in \mathcal{C}(D, X, Tq)$ for all $\mathbf{P}$-nondecreasing sequences $\{s_n\}_n \in \mathcal{C}(D, X, q)$. This function is called $\mathbf{P}$-nondecreasing-continuous if it is $\mathbf{P}$-nondecreasing-continuous at each point of X.

Let us define $\mathbf{S}$ the class of functions $\beta\colon [0, \infty) \to [0, 1)$ such that for any given sequence $\{t_n\}_n \subset [0, \infty)$, for which $\lim_{n\to\infty} \beta(t_n) = 1$, it follows that $\lim_{n\to\infty} t_n = 0$.

Define Θ the family of all continuous functions $\theta\colon [0, +\infty)^4 \to [0, +\infty)$, having the following properties

$$\begin{cases} \theta(0, t, s, u) = 0, & \text{for all}\, t, s, u \in [0, +\infty), \\ \theta(t, s, 0, u) = 0, & \text{for all}\, t, s, u \in [0, +\infty). \end{cases}$$

We give, without proof, the following lemma. It can be justified using the Bolzano-Weierstrass theorem.

Lemma 2.1. Let us take $\{p_n\}$ a sequence of positive real numbers knowing its limit $\lim_{n\to\infty} p_n = 0$ and let $\{c_{m,n}\}$ be a bounded sequence of positive numbers depending on two natural parameters. Then, the following relation is accomplished:

$$\lim_{k\to\infty} \sup_{m,n\geq k} \theta(c_{m,n}, c_{n,m}, p_n, p_m) = 0,$$

for all $\theta \in \Theta$.

3. **Main results**

We combine the notion of Geraghty function, theta function, and some ideas of contraction due to Kannan in order to develop our ideas. Firstly, one has to develop a technical procedure to prove that the Picard sequence is D-Cauchy. For this purpose, we have to prove a technical lemma.

Lemma 3.1. Let (X, D) to be a Jleli-Samet metric space and let $T\colon X \to X$ be a self-operator. Presume that the next conditions are fulfilled:

i) There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, \tilde{x}) < \infty$;

ii) There is a Geraghty-type function $\beta \in \mathbf{S}$ and $\theta \in \Theta$ such that:

$$\begin{aligned} D(Tx, Ty) \leq & \beta\left(\frac{D(x,Tx)+D(y,Ty)}{2}\right)\frac{D(x,Tx)+D(y,Ty)}{2} \\ & + \theta(D(y,Tx), D(x,Ty), D(x,Tx), D(y,Ty)) \end{aligned}$$

for all x, $y \in \mathcal{O}_T(\tilde{x})$;

iii) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$, for all $n \in \mathbb{N}$.

Then, the Picard sequence $\{T^n\tilde{x}\}_n$ is D-Cauchy.

Proof. To avoid trivial cases, one may assume that $D(T^n\tilde{x}, T^m\tilde{x}) > 0$ for all m, $n \in \mathbb{N}$. The first step in our proof is to show the asymptotically regularity of T in $\tilde{x}$. In order to simplify the notations, let us consider

$$D(T^n\tilde{x}, T^{n+1}\tilde{x}) = \alpha_n, \text{ for all } n > n_0 + 1.$$

If we look at the contractive inequality, we get:

$$\begin{aligned} D(T^n\tilde{x}, T^{n+1}\tilde{x}) & \leq & \beta\left(\frac{\alpha_{n-1}, \alpha_n}{2}\right)\frac{\alpha_{n-1}+\alpha_n}{2} \\ & & +\theta(D(T^n\tilde{x}, T^n\tilde{x}), D(T^{n-1}\tilde{x}, T^{n+1}\tilde{x}), \\ & & D(T^{n-1}\tilde{x}, T^n\tilde{x}), D(T^n\tilde{x}, T^{n+1}\tilde{x})) \\ & = & \beta\left(\frac{\alpha_{n-1}+\alpha_n}{2}\right)\frac{\alpha_{n-1}+\alpha_n}{2}. \end{aligned}$$

If we look again at the above inequality, it is clar that:

$$\alpha_n \leq \beta\left(\frac{\alpha_{n-1}, \alpha_n}{2}\right)\frac{\alpha_{n-1}+\alpha_n}{2} < \frac{\alpha_{n-1}+\alpha_n}{2},$$

so it is clear that $\alpha_n < \alpha_{n-1}$ for all $n > n_0$. Since $\{\alpha_n\}_n$ is a nondecreasing sequence of positive numbers, it is convergent to some $p \in \mathbb{R}_+$. Let us suppose that $p \neq 0$. Using the last inequality from above, we get, after passing through superior limit:

$$1 = \limsup_{n\to\infty} \frac{2\alpha_n}{\alpha_{n-1}+\alpha_n} \leq \limsup_{n\to\infty} \beta\left(\frac{\alpha_{n-1}+\alpha_n}{2}\right),$$

meaning that there is a subsequence $\{n_t\}_t$ for which

$$\lim_{t\to\infty} \frac{\alpha_{n_t-1}+\alpha_{n_t}}{2} = 0.$$

The last relation contradicts our assumption. In conclusion, we have:

$$\lim_{n\to\infty} D(T^n\tilde{x}, T^{n+1}\tilde{x}) = 0.$$

We can take $m, n \geq k+1$, $m' = m-1$, and $n' = n-1$, with $m' \geq n' \geq k$. Let us denote by

$$\eta_{n',m'} = D(T^{n'}\tilde{x}, T^{m'}\tilde{x}).$$

Combining all together in the contractive inequality (3.1), we obtain:

$$\begin{aligned} D(T^m\tilde{x}, T^n\tilde{x}) &= D(T^{n'+1}\tilde{x}, T^{m'+1}\tilde{x}) \\ &\leq \beta\left(\frac{\eta_{n',n'+1}+\eta_{m',m'+1}}{2}\right)\frac{\eta_{n',n'+1}+\eta_{m',m'+1}}{2} \\ &\quad +\theta(D(T^{m'}\tilde{x}, T^{n'+1}\tilde{x}), D(T^{n'}\tilde{x}, T^{m'+1}\tilde{x}), \\ &\quad D(T^{n'}\tilde{x}, T^{n'+1}\tilde{x}), D(T^{m'}\tilde{x}, T^{m'+1}\tilde{x})). \end{aligned}$$

Observe that each of the numbers from the above inequality $D(T^{m'}\tilde{x}, T^{n'+1}\tilde{x})$, $D(T^{n'}\tilde{x}, T^{m'+1}\tilde{x})$, $D(T^{n'}\tilde{x}, T^{n'+1}\tilde{x})$, and $D(T^{m'}\tilde{x}, T^{m'+1}\tilde{x})$ are in Ω_k. We must use the sequence $\{\delta_k\}_k$ in order to study the D-Cauchy convergence for the Picard sequence. It is obvious that $\{\delta_k\}_k$ is a nonincreasing sequence of positive numbers. Hence, it is convergent to some $l \in [0, \delta_{n_0}]$. Taking supremum over k, the contractive inequality shows that:

$$\begin{aligned} \delta_{k+1} \leq & \sup_{n',m'\geq k} \beta\left(\frac{\eta_{n',n'+1}+\eta_{m',m'+1}}{2}\right) \sup_{n',m'\geq k} \frac{\eta_{n',n'+1}+\eta_{m',m'+1}}{2} \\ & + \sup_{n',m'\geq k} \theta(D(T^{m'}\tilde{x}, T^{n'+1}\tilde{x}), D(T^{n'}\tilde{x}, T^{m'+1}\tilde{x}), \\ & D(T^{n'}\tilde{x}, T^{n'+1}\tilde{x}), D(T^{m'}\tilde{x}, T^{m'+1}\tilde{x})). \end{aligned}$$

Using Lemma 2.1 and the fact that T is asymptotically convergent to $\tilde{x}$, we can say that:

$$\begin{aligned} \lim_{k\to\infty} \sup_{n',m'\geq k} & \theta(D(T^{m'}\tilde{x}, T^{n'+1}\tilde{x}), D(T^{n'}\tilde{x}, T^{m'+1}\tilde{x}), D(T^{n'}\tilde{x}, T^{n'+1}\tilde{x}), \\ & D(T^{m'}\tilde{x}, T^{m'+1}\tilde{x})) = 0, \end{aligned}$$

and

$$\lim_{k\to\infty} \sup_{n',m'\geq k} \frac{\eta_{n',n'+1}+\eta_{m',m'+1}}{2} = 0.$$

It is easy to see that $\lim_{k\to\infty} \delta_{k+1} = 0$, and it follows that the sequence $\{T^n\tilde{x}\}$ is D-Cauchy. □

We are ready now to prove our main results from this section. It can be seen that Kannan contractive condition is strong enough to skip the third condition imposed in previous theorems.

Theorem 3.1. *Let (X, D) be a Jleli-Samet space for which there is a preorder $\mathbf{P}$ such that the space is $\mathbf{P}$-nondecreasing-complete. Let $T\colon X \to X$ be a mapping and suppose that the following conditions are fulfilled:*

i) *There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, \tilde{x}) < \infty$;*

ii) *There is a Geraghty-type function $\beta \in \mathbf{S}$ and $\theta \in \Theta$ such that:*

$$\begin{aligned} D(Tx, Ty) \leq & \beta\left(\frac{D(x,Tx)+D(y,Ty)}{2}\right)\frac{D(x,Tx)+D(y,Ty)}{2} \\ & + \theta(D(y,Tx), D(x,Ty), D(x,Tx), D(y,Ty)) \end{aligned}$$

for all x, $y \in \mathcal{O}_T(\tilde{x})$;

iii) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$, for all $n \in \mathbb{N}$;

iv) T *is* **P**-*nondecreasing-continuous, and* **A**-*nondecreasing.*

Then, the Picard sequence $\{T^n\tilde{x}\}_n$ *is* D*-convergent to a fixed point* ω *of* T*, and* $D(\omega,\omega) = 0$*. If there is another fixed point of* T*,* ω'*, then if* $D(\omega,\omega') < \infty$ *and* $D(\omega',\omega') = 0$*, then* $\omega = \omega'$*.*

Proof. First of all, the last condition tells us that the sequence $\{T^n\tilde{x}\}_n$ is **P**-nondecreasing and Lemma 3.1 ensures us that the same sequence is D-convergence to some point $\omega \in X$ for the reason that the space is supposed to be **P**-nondecreasing-complete. By the **P**-nondecreasing-continuity of T, it is obvious that:

$$\{Tx_n\}_n \xrightarrow{D} T\omega.$$

The existence part of the theorem is now proved since $T\omega = \omega$, and $\omega \in X$ is a fixed point of T. Let us show the uniqueness that we have following the assumptions. Suppose that there is another fixed point of T, denoted by ω', having $D(\omega,\omega') < \infty$ and $D(\omega',\omega') = 0$. In the contractive inequality we have:

$$\begin{aligned} D(\omega,\omega') &= D(T\omega,T\omega') \\ &\leq \beta(\frac{D(\omega,T\omega)+D(\omega',T\omega')}{2})\frac{D(\omega,T\omega)+D(\omega',T\omega')}{2} \\ &\quad +\theta(D(\omega',T\omega),D(\omega,T\omega'),D(\omega,T\omega),D(\omega',T\omega')). \end{aligned}$$

Now, if we take into consideration the axiom $(D3)$ from the definition of Jleli-Samet metric, we can see that:

$$D(\omega,\omega) \leq C \limsup_{n\to\infty} D(T^n\tilde{x},\omega) = 0,$$

because $\{x_n\}_n \in \mathcal{C}(D,T,\tilde{x})$, so $D(\omega,\omega) = 0$. Similarly, $D(\omega,T\omega) = 0$. Now, the contractive inequality shows that:

$$D(\omega,\omega') \leq 0.$$

In conclusion, we have that $D(\omega,\omega') = 0$ and $\omega = \omega'$. □

One more result can be proved, using similar settings.

Theorem 3.2. *Let* (X,D) *be a complete Jleli-Samet space and* $T\colon X \to X$ *a self mapping. Presume that the following conditions are fulfilled:*

i) *There is* $\tilde{x} \in X$*, and* $n_0 \in \mathbb{N}$ *for which we have* $\delta_{n_0}(D,T,\tilde{x}) < \infty$*;*

ii) *There is a Geraghty-type function* $\beta \in \mathbf{S}$ *and* $\theta \in \Theta$ *such that:*

$$\begin{aligned} D(Tx,Ty) \leq& \beta\left(\frac{D(x,Tx)+D(y,Ty)}{2}\right)\frac{D(x,Tx)+D(y,Ty)}{2} \\ &+\theta(D(y,Tx),D(x,Ty),D(x,Tx),D(y,Ty)) \end{aligned}$$

for all x*,* $y \in \mathcal{O}'_T(\tilde{x}) = \mathcal{O}_T(\tilde{x}) \cup \{\omega \in X : \lim_{n\to\infty} D(T^n\tilde{x},\omega) = 0\}$*;*

iii) $D(T^n\tilde{x},T^n\tilde{x}) = 0$*, for all* $n \in \mathbb{N}$*.*

Then the Picard sequence $\{T^n\tilde{x}\}_n$ *is* D*-convergent to a fixed point of* T*,* ω*. If* $D(\omega,\omega) < \infty$*, and* $\omega' \in X$ *is another fixed point of* T *with* $D(\omega,\omega') < \infty$*,* $D(\omega',\omega') = 0$*, then* $\omega = \omega'$*.*

Proof. We can choose the trivial preorder on X, i.e $x\mathbf{P}y$, for all x, $y \in X$. In this context, T becomes **P**-nondecreasing and the space (X,D) is **P**-nondecreasing-complete.

Since the Picard sequence $\{T^n\tilde{x}\}$ is **P**-nondecreasing, and D-Cauchy, there exists $\omega \in X$ such that we have $\{x_n\} \xrightarrow{D} \omega$. Observe that $\omega \in \mathcal{O}'_T(x_*)$ and, if we take into consideration the contractive inequality, we obtain:

$$\begin{aligned} D(T^{n+1}\tilde{x}, T\omega) &= D(TT^n\tilde{x}, T\omega) \\ &\leq \beta\left(\frac{D(T^n\tilde{x}, T^{n+1}\tilde{x} + D(\omega, T\omega)}{2}\right)\left(\frac{D(T^n\tilde{x}, T^{n+1}\tilde{x})}{2}\right. \\ &\quad \left. + \frac{D(\omega, T\omega)}{2}\right) + \theta(D(\omega, T^{n+1}\tilde{x}), D(T^n\tilde{x}, T\omega)), \\ &\quad D(T^n\tilde{x}, T^{n+1}\tilde{x}), D(\omega, T\omega)). \end{aligned}$$

We know the fact that $\lim_{n\to\infty} D(T^n\tilde{x}, T^{n+1}\tilde{x}) = 0$, and $\lim_{n\to\infty} D(T^n\tilde{x}, \omega) = D(\omega, \omega) = 0$, and we can apply Lemma 3.1 also. Passing through the limit over n and using the properties of boundedness for Geraghty functions, the contractive inequality becomes $D(\omega, T\omega) \leq \frac{D(\omega, T\omega)}{2}$.

It follows that $D(\omega, T\omega) = 0$, and our first statement of the theorem is proved.

We can consider now $\omega' \in X$, another fixed point for the operator T, having $D(\omega, \omega') < \infty$ and $D(\omega', \omega') = 0$, as above. Then, we can say that:

$$\begin{aligned} D(\omega, \omega') \leq &\beta\left(\frac{D(\omega, T\omega) + D(\omega', T\omega')}{2}\right)\frac{D(\omega, T\omega) + D(\omega', T\omega')}{2} \\ &+ \theta(D(\omega', T\omega), D(\omega, T\omega'), D(\omega, T\omega), D(\omega', T\omega')). \end{aligned}$$

Since we know, from the last proof, that $D(\omega, T\omega) = D(\omega, \omega) = 0$, and $D(\omega', T\omega') = D(\omega', \omega') = 0$, we can say that: $D(\omega, \omega') = 0$, so $\omega = \omega'$. □

We can particularize our functions to obtain some already well known results in the literature. For instance, if we take $\theta = 0$, we have the following theorem.

Corollary 3.1. Let (X, D) be a complete Jleli-Samet space and $T\colon X \to X$ a self mapping. Presume that the following conditions are fulfilled:

i) There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, \tilde{x}) < \infty$;

ii) There is a Geraghty-type function $\beta \in \mathbf{S}$ such that:

$$D(Tx, Ty) \leq \beta\left(\frac{D(x, Tx) + D(y, Ty)}{2}\right)\frac{D(x, Tx) + D(y, Ty)}{2}$$

for all x, $y \in \mathcal{O}'_T(\tilde{x}) = \mathcal{O}_T(\tilde{x}) \cup \{\omega \in X : \lim_{n\to\infty} D(T^n\tilde{x}, \omega) = 0\}$;

iii) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$, for all $n \in \mathbb{N}$.

Then $\{T^n\tilde{x}\}_n$ is D-convergent to a fixed point of T, ω. If $D(\omega, \omega) < \infty$, and $\omega' \in X$ is another fixed point of T with $D(\omega, \omega') < \infty$, $D(\omega', \omega') = 0$, then $\omega = \omega'$.

We can take now $\theta(t, u, v, w) = \tau \inf\{t, u, v, w\}$, τ a positive constant, t, u, v, $w \in [0, \infty)$, in Theorem 3.2, and the next corollary is valid.

Corollary 3.2. Let (X, D) be a complete Jleli-Samet space and $T\colon X \to X$ a self mapping. Presume that the following conditions are fulfilled:

i) There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, \tilde{x}) < \infty$;

ii) There is a Geraghty-type function $\beta \in \mathbf{S}$ and $\tau > 0$ such that:

$$\begin{aligned} D(Tx, Ty) \leq &\beta\left(\frac{D(x, Tx) + D(y, Ty)}{2}\right)\frac{D(x, Tx) + D(y, Ty)}{2} \\ &+ \tau \inf\{(D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty))\} \end{aligned}$$

for all x, $y \in \mathcal{O}'_T(\tilde{x}) = \mathcal{O}_T(\tilde{x}) \cup \{\omega \in X : \lim_{n\to\infty} D(T^n\tilde{x}, \omega) = 0\}$;

iii) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$, for all $n \in \mathbb{N}$.

Then $\{T^n\tilde{x}\}_n$ is D-convergent to a fixed point of T, ω. If $D(\omega, \omega) < \infty$, and $\omega' \in X$ is another fixed point of T with $D(\omega, \omega') < \infty$, $D(\omega', \omega') = 0$, then $\omega = \omega'$.

Another specific case is when $\theta(t,u,v,w) = \tau tuvw$, where $\tau > 0$ is a constant, t, u, v, $w > 0$, and the next consequence of Theorem 3.2 is given below.

Corollary 3.3. Let (X, D) be a complete Jleli-Samet space and $T: X \to X$ a self mapping. Presume that the following conditions are fulfilled:

i) There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, \tilde{x}) < \infty$;

ii) There is a Geraghty-type function $\beta \in \mathbf{S}$ and $\tau > 0$ such that:

$$D(Tx,Ty) \leq \beta\left(\frac{D(x,Tx)+D(y,Ty)}{2}\right)\frac{D(x,Tx)+D(y,Ty)}{2} + \tau D(y,Tx)D(x,Ty)D(x,Tx)D(y,Ty)$$

for all x, $y \in \mathcal{O}'_T(\tilde{x}) = \mathcal{O}_T(\tilde{x}) \cup \{\omega \in X : \lim_{n\to\infty} D(T^n\tilde{x}, \omega) = 0\}$;

iii) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$, for all $n \in \mathbb{N}$.

Then $\{T^n\tilde{x}\}_n$ is D-convergent to a fixed point of T, ω. If $D(\omega,\omega) < \infty$, and $\omega' \in X$ is another fixed point of T with $D(\omega,\omega') < \infty$, $D(\omega',\omega') = 0$, then $\omega = \omega'$.

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