THE EXISTENCE OF GLOBAL ATTRACTOR FOR A SIXTH-ORDER PHASE-FIELD EQUATION IN H^k SPACE

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In this paper, by using the regularity estimates for the semigroups, iteration technique and the classical existence theorem of global attractors, we studied the existence of global attractor for a sixth-order phase-field equation in the fractional power spaces $H^k(\Omega)$, where $0 \le k < \infty$.

Keywords: phase-field equation, iteration technique, global attractor, regularity estimates.

1. Introduction

In [11], the authors proposed a higher order nonlinear Willmore regularization in the Ginzburg-Landau free energy, which takes into account strongly anisotropic crystal and epitaxial growth during the growth and coarsening of thin films. The modified Ginzburg-Landau free energy is in the following form:

$$\psi_{MGL} = \int_\Omega [\gamma(\nu)\frac{1}{\epsilon}(\frac{\epsilon^2}{2}|\nabla\rho|^2 + F(\rho)) + \frac{\beta}{2}\frac{1}{\epsilon^3}\omega^2]dx, \tag{1}$$

where ρ is the order parameter, Ω is the domain occupied by the material(we assume that it is a bounded and regular domain of R^n, n=1, 2 or 3), $\gamma(\nu)$ is a function describing the anisotropy effects, $\nu = \nabla\rho/|\nabla\rho|$ (in what follows, ν also denotes the unit outer normal to the boundary Γ of Ω), ϵ is a small parameter about the measure of the interface transition layer thickness, $\omega = f(\rho) - \Delta\rho$, $f = F'$ is nonlinear Willmore regularization.

Consider mass conservation, i.e. $\partial\rho/\partial t = -divh$, where h is the mass flux which is related to the chemical potential μ by the constitutive relation $h = -M\nabla\mu$, and that the chemical potential is a variational derivative of ψ_{MGL} with respect to ρ, we end up with the following sixth-order anisotripic phase-field equation

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = \dfrac{1}{\epsilon}\nabla \cdot (M\nabla\mu), \\ \mu = \dfrac{1}{\epsilon}(\gamma f(\rho) - \epsilon^2\nabla \cdot m) + \dfrac{\beta}{\epsilon^2}(f'(\rho)\omega - \epsilon^2\Delta\omega), \\ m = \gamma(\nu)\nabla\rho + |\nabla\rho|P\nabla_\nu\gamma(\nu), \\ \omega = \dfrac{1}{\epsilon}(f(\rho) - \epsilon^2\Delta\rho), \end{cases} \tag{2}$$

where M is the mobility, m describes the anisotropic gradient, $P = I-\nu\otimes\nu$ is the projection matrix, I is the identity matrix, ∇_ν represents the gradient with respect to the components of the normal vector, respectively.

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Latterly, Miranville [6] supposed $\gamma(\nu) = 1$, $\beta = 1$, $\epsilon = 1$ and the equation is isotropic, provided the following modification of the Ginzburg-Landau free energy

$$\psi_{MGL} = \int_\Omega (\frac{1}{2}|\nabla\rho|^2 + F(\rho) + \frac{1}{2}\omega^2)dx, \tag{3}$$

and the following sixth-order phase-field equation

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = \nabla \cdot (M\nabla\mu), \\ \mu = f(\rho) - \Delta\rho + \omega f^{'}(\rho) - \Delta\omega, \\ \omega = f(\rho) - \Delta\rho, \end{cases} \tag{4}$$

The author also supposed that the nonlinear function $f(\rho)$ is of class C^2 and

$$\begin{cases} f(0) = 0, \quad f'(s) \geq -c_0, \quad c_0 \geq 0, \ s \in \mathbb{R}, \\ f(s)s \geq c_1 F(s) - c_2 \geq -c_2', \quad c_1 > 0, \ c_2, c_2' \geq 0, \ s \in \mathbb{R}, \\ F(s) = \int_0^s f(\tau)d\tau, \\ sf(s)f'(s) - f^2(s) \geq c_3 f^2(s) - c_4, \quad c_3 > 0, \ c_4 \geq 0, \ s \in \mathbb{R}, \\ |f'(s)| \leq \epsilon|f(s)| + c_5, \quad \epsilon > 0, \ c_5 \geq 0, \ s \in \mathbb{R}, \\ sf''(s) \geq 0, \quad s \in \mathbb{R}, \end{cases} \tag{5}$$

studied the asymptotic behavior, in terms of finite-dimensional attractors, for the initial-boundary value problems of equation (4) together with the mobility $M \equiv 1$.

Remark 1.1. *Eq.(4) can be rewritten, equivalently, as*

$$\frac{\partial\rho}{\partial t} = \Delta^3\rho - \Delta^2\rho - \Delta^2 f(\rho) - \Delta(f'(\rho)\Delta\rho) + \Delta(f'(\rho)f(\rho)) + \Delta f(\rho), \ t > 0, x \in \Omega, \tag{6}$$

In this paper, by using the regularity estimates for the semigroups, iteration technique and the classical existence theorem of global attractors, we consider the long time behavior of solutions for the initial-boundary value problem for Eq.(6). On the basis of physical considerations, as usual Eq.(6) is supplemented with the following boundary value conditions

$$\frac{\partial\rho}{\partial\nu}|_{\partial\Omega} = \frac{\partial\Delta\rho}{\partial\nu}|_{\partial\Omega} = \frac{\partial\Delta^2\rho}{\partial\nu}|_{\partial\Omega} = 0, \tag{7}$$

and the initial condition

$$\rho(x,0) = \rho_0(x), \ x \in \Omega. \tag{8}$$

Remark 1.2. *The dynamic properties of diffusion equation and diffusion system such as the global asymptotical behaviors of solutions and global attractors are important for the study of diffusion model, which ensure the stability of diffusion phenomena and provide the mathematical foundation for the study of diffusion dynamics. There are many studies on the existence of global attractors for the diffusion equations on bounded domains and unbounded domains, see for example [1, 2, 3, 4, 7, 10].*

This article is organized as follows. In section 2, we give some preparations for our consideration and the main theorem of this article. In section 3, we prove the existence of global attractors for problem (6)-(8) in the Sobolev space $H^k(\Omega)$ with any $k \geq 0$.

2. Preliminary

First of all, we give the following lemma on global existence and uniqueness of solution to problem (6)-(8).

Lemma 2.1. *Assume $\rho_0 \in H^3(\Omega)$ and the nonlinear function $f(s)$ satisfies (5). Then, problem (6)-(8) has a unique (weak) solution ρ such that*

$$\rho \in L^\infty(0,T;H_{\frac{1}{2}}) \cap L^2(0,T;H^6(\Omega)), \ \forall T > 0.$$

The proof of existence is based on the classical Galerkin method and the a priori estimates (see [6]). Thanks to the above existence lemma, we know that there exists a continuous operator semigroup $\{S(t)\}_{t \geq 0}$ in H^3 satisfying

$$S(t)\rho_0 = \rho(t, \rho_0), \ t \geq 0.$$

Futhermore, by the classical existence theorem of global attractors (see [10]) and on a priori estimates, we give the following lemma on the existence of the global attractor of problem (6)-(8) in $H^3(\Omega)$, which can be found in Miranville [6].

Lemma 2.2. [6] *Assume $\rho_0 \in H^3(\Omega)$ and the nonlinear function $f(s)$ satisfies (5). Then, the solution of problem (6)-(8) has a global attractor in H^3.*

Now, in order to consider the global attractors for Eq.(6) in the H^k space, we introduce the define as follows

$$\begin{cases} H = \left\{ \rho \in L^2(\Omega), \quad \dfrac{\partial \rho}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}, \\[2mm] H_{\frac{1}{2}} = \left\{ \rho \in H^3(\Omega) \bigcap H, \quad \dfrac{\partial \rho}{\partial \nu}\Big|_{\partial\Omega} = \dfrac{\partial \Delta \rho}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}, \\[2mm] H_1 = \left\{ \rho \in H^6(\Omega) \bigcap H, \quad \dfrac{\partial \rho}{\partial \nu}\Big|_{\partial\Omega} = \dfrac{\partial \Delta \rho}{\partial \nu}\Big|_{\partial\Omega} = \dfrac{\partial \Delta^2 \rho}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}. \end{cases} \quad (9)$$

In this article, we let $G(\rho) = -\Delta^2 \rho - \Delta^2 f(\rho) - \Delta(f'(\rho)\Delta\rho) + \Delta(f'(\rho)f(\rho)) + \Delta f(\rho)$ be a nonlinear function and assume that the linear operator $L = \Delta^3 : H_1 \to H$ in (9) is a sectorial operator, which generates an analytic semigroup e^{tL}, and L induces the fractional power operators and fractional order spaces as follows

$$\mathcal{L}^\alpha = (-L)^\alpha : \ H_\alpha \to H, \ \alpha \in \mathbb{R}, \quad (10)$$

where $H_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$. By the semigroup theory of linear operators, $H_\beta \subset H_\alpha$ is a compact inclusion for any $\beta > \alpha$.

For sectorial operators, we have the following lemma.

Lemma 2.3. [8, 9] *Assume that L is a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $Re\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha(\mathcal{L} = -L)$ we have*

(i)$T(t) : H \to H_\alpha$ is bounded for all $\alpha \in \mathbb{R}$ and $t > 0$;

(ii)$T(t)\mathcal{L}^\alpha x = \mathcal{L}T(t)x, \ \forall x \in H_\alpha$;

(iii)For each $t > 0$, $\mathcal{L}^\alpha T(t) : H \to H$ is bounded, and

$$\|\mathcal{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t},$$

where $\delta > 0$ and $C_\alpha > 0$ is a constant depending only on α;

(iv)The H_α-norm can be defined by $\|x\|_{H_\alpha} = \|\mathcal{L}^\alpha x\|_H$.

Finally, we give the main theorem of this article, which provides the existence of global attractors of Eq.(6) in any kth space H^k.

Theorem 2.1. *Assume $\rho_0 \in H^k(\Omega)$ ($k \in \mathbb{R}^+$) and the nonlinear function $f(s)$ satisfies (5), then the solution ρ of problem (6)-(8) possesses a global attractor $\mathscr{A}$ in the space $H^k(\Omega)$ which attracts all the bounded set of H^k in the H^k-norm.*

Remark 2.1. *Recently, Miranville[6] studied the global dynamics of the initial-boundary value problem of sixth order phase-field equation. The author supposed that the initial data $\rho_0 \in H^3(\Omega)$, established the existence and uniqueness of global weak solutions and proved the existence of bounded absorbing set. Hence, the H^3-global attractor had been obtained straightforward. In this manuscript, based on Miranville's results, we are just going to study the existence of global attractor in fractional space $H^k(\Omega)$, where $k \in [0, \infty)$. The main tools we used is the properties of sectorial operator and the iteration technique while Miranville's main tools are energy inequality and a priori estimates. The result of Theorem 2.1 can be viewed as an improvement of Miranvill' work[6].*

3. Proof of Theorem 2.1

On the basis of Ma and Wang [5], it's well known that the solution $\rho(t, \rho_0)$ of problem (6)-(8) can be expressed as

$$\rho(t, \rho_0) = e^{tL}\rho_0 + \int_0^t e^{(t-\tau)L} G(\rho) d\tau. \tag{11}$$

where $L = \Delta^3$ and $G(\rho) = \Delta g(\rho) = -\Delta^2 \rho - \Delta^2 f(\rho) - \Delta(f'(\rho)\Delta\rho) + \Delta(f'(\rho)f(\rho)) + \Delta f(\rho)$. Then, (11) means

$$\rho(t, \rho_0) = e^{tL}\rho_0 + \int_0^t e^{(t-\tau)L} \Delta g(\rho) d\tau = e^{tL}\rho_0 + \int_0^t (-L)^{\frac{1}{3}} e^{(t-\tau)L} g(\rho) d\tau. \tag{12}$$

In order to prove Theorem 2.1, we first prove the following lemma.

Lemma 3.1. *For any bounded set $U \in H_\alpha$, there exists a constant $C > 0$ such that*

$$\|\rho(t, \rho_0)\|_{H\alpha} \le C, \ \forall t \ge 0, \ \rho_0 \in U \subset H_\alpha, \ \alpha > 0. \tag{13}$$

Proof. For $\alpha = \frac{1}{2}$, this follows from Lemma 2.2, i.e. for any bounded set $U \subset H_{\frac{1}{2}}$, there exists a constant C, $C > 0$ such that

$$\|\rho(t, \rho_0)\|_{H_{\frac{1}{2}}} \le C, \ \forall t \ge 0, \ \rho_0 \in U \subset H_{\frac{1}{2}}, \tag{14}$$

Then we only need to prove (13) for any $\alpha \ge \frac{1}{2}$.

Step 1. We prove that for any bounded set $U \subset H_\alpha$ ($\frac{1}{2} \le \alpha < \frac{2}{3}$), there exists a constant $C > 0$ such that

$$\|\rho(t, \rho_0)\|_{H\alpha} \le C, \ \forall t \ge 0, \ \rho_0 \in U, \ \frac{1}{2} \le \alpha < \frac{2}{3}. \tag{15}$$

We claim that $g: \ H_{\frac{1}{2}} \to \ H$ is bounded, by Sobolev embedding theorem, we have

$$H_{\frac{1}{2}} \hookrightarrow H^2(\Omega), \ H_{\frac{1}{2}} \hookrightarrow W^{1,4}(\Omega), \ H_{\frac{1}{2}} \hookrightarrow L^\infty(\Omega).$$

Then, we obtain

$$
\begin{aligned}
\|g(\rho)\|_H^2 &= \int_\Omega |-\Delta\rho - \Delta f(\rho) - f'(\rho)\Delta\rho + f'(\rho)f(\rho) + f(\rho)|^2 dx \\
&\le C \int_\Omega (|\Delta\rho|^2 + |\Delta f(\rho)|^2 + |f'(\rho)\Delta\rho|^2 + |f'(\rho)f(\rho)|^2 + |f(\rho)|^2) dx \\
&\le C(\|\Delta\rho\|^2 + \|f''(\rho)|\nabla\rho|^2\|^2 + \|f'(\rho)\Delta\rho\|^2 \\
&\quad + \|f'(\rho)\Delta\rho\|^2 + \|f'(\rho)f(\rho)\|^2 + \|f(\rho)\|^2) \\
&\le C(\|\rho\|_{H^2}^2 + \|f''(\rho)\|_{L^\infty}^2\|\rho\|_{W^{1,4}}^4 + \|f'(\rho)\|_{L^\infty}^2\|\rho\|_{H^2}^2 \\
&\quad + \|f'(\rho)f(\rho)\|_{L^\infty}^2 + \|f(\rho)\|_{L^\infty}^2) \\
&\le C(\|\rho\|_{H^2}^2 + \|\rho\|_{W^{1,4}}^4 + C) \le C(\|\rho\|_{H_{\frac{1}{2}}}^2 + \|\rho\|_{H_{\frac{1}{2}}}^4 + C)
\end{aligned}
\tag{16}
$$

which means that $g : H_{\frac{1}{2}} \to H$ is bounded. By (12), (14) and (16) we find that

$$
\begin{aligned}
\|\rho(t,\rho_0)\|_{H\alpha} =& \|e^{tL}\rho_0 + \int_0^t (-L)^{\frac{1}{3}} e^{(t-\tau)L} g(\rho) d\tau\|_{H\alpha} \\
\leq& C\|\rho_0\|_{H\alpha} + \int_0^t \|(-L)^{\frac{1}{3}+\alpha} e^{(t-\tau)L} g(\rho)\|_H d\tau \\
\leq& C\|\rho_0\|_{H\alpha} + \int_0^t \|(-L)^{\frac{1}{3}+\alpha} e^{(t-\tau)L}\| \cdot \|g(\rho)\|_H d\tau \\
\leq& C\|\rho_0\|_{H\alpha} + C \int_0^t (t-\tau)^{-\beta} e^{-\delta(t-\tau)} d\tau \\
\leq& C\|\rho_0\|_{H\alpha} + C \int_0^t \tau^{-\beta} e^{-\delta\tau} d\tau \\
\leq& C, \quad \forall t \geq 0, \ \rho_0 \in U \subset H_\alpha,
\end{aligned}
\tag{17}
$$

where $\beta = \frac{1}{3} + \alpha, (0 < \beta < 1)$. Hence, (15) is valid.

Step 2. We prove that for any bounded set $U \subset H_\alpha$ ($\frac{2}{3} \leq \alpha < \frac{5}{6}$), there exists a constant $C > 0$ such that

$$
\|\rho(t,\rho_0)\|_{H\alpha} \leq C, \ \forall t \geq 0, \ \rho_0 \in U, \ \frac{2}{3} \leq \alpha < \frac{5}{6}.
\tag{18}
$$

We claim that $g : H_\alpha \to H_{\frac{1}{6}}$ is bounded, by Sobolev embedding theorem,we have

$$
H_\alpha \hookrightarrow H^3(\Omega), \ H_\alpha \hookrightarrow W^{1,6}(\Omega), \ H_\alpha \hookrightarrow W^{1,4}(\Omega),
$$

$$
H_\alpha \hookrightarrow W^{2,4}(\Omega), \ H_\alpha \hookrightarrow H(\Omega), \ H_\alpha \hookrightarrow L^\infty(\Omega)
$$

where $\frac{1}{2} \leq \alpha < \frac{2}{3}$.

Then,we obtain

$$
\begin{aligned}
\|g(\rho)\|_{H_{\frac{1}{6}}}^2 =& \int_\Omega |-\nabla\Delta\rho - \nabla\Delta f(\rho) - \nabla(f'(\rho)\Delta\rho) + \nabla(f'(\rho)f(\rho)) + \nabla f(\rho)|^2 dx \\
\leq& C \int_\Omega (|\nabla\Delta\rho|^2 + |\nabla\Delta f(\rho)|^2 + |\nabla(f'(\rho)\Delta\rho)|^2 + |\nabla(f'(\rho)f(\rho))|^2 \\
& + |\nabla f(\rho)|^2) dx \\
\leq& C(\|\rho\|_{H^3}^2 + \|f'''(\rho)|\nabla\rho|^3\|^2 + \|f''(\rho)\nabla\rho\Delta\rho\|^2 + \|f'(\rho)\nabla\Delta\rho\|^2 \\
& + \|f''(\rho)\nabla\rho\Delta\rho\|^2 + \|f'(\rho)\nabla\Delta\rho\|^2 + \|f''(\rho)f(\rho)\nabla\rho\|^2 \\
& + \||f'(\rho)|^2\nabla\rho\|^2 + \||f'(\rho)|^2\nabla\rho\|^2) \\
\leq& C(\|\rho\|_{H^3}^2 + \|f'''(\rho)\|_{L^\infty}^2 \|\rho\|_{W^{1,6}}^6 + \|f''(\rho)\|_{L^\infty}^4 \|\rho\|_{W^{1,4}}^4 \\
& + \|\rho\|_{W^{2,4}}^4 + \|f'(\rho)\|_{L^\infty}^2 \|\rho\|_{H^3}^2 + \|f''(\rho)f(\rho)\|_{L^\infty}^2 \|\rho\|_H^2 \\
& + \|f'(\rho)\|_{L^\infty}^4 \|\rho\|_H^2 + \|f'(\rho)\|_{L^\infty}^2 \|\rho\|_H^2) \\
\leq& C(\|\rho\|_{H_\alpha}^2 + \|\rho\|_{H_\alpha}^6 + \|\rho\|_{H_\alpha}^4)
\end{aligned}
\tag{19}
$$

which means that $g : H_\alpha \to H_{\frac{1}{6}}$ is bounded. On the basis of step 1 and (19), we deduce that

$$
\begin{aligned}
\|\rho(t,\rho_0)\|_{H_\alpha} =& \|e^{tL}\rho_0 + \int_0^t (-L)^{\frac{1}{3}} e^{(t-\tau)L} g(\rho) d\tau\|_{H_\alpha} \\
\le& C\|\rho_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{6}+\alpha} e^{(t-\tau)L} g(\rho)\|_{H_{\frac{1}{6}}} d\tau \\
\le& C\|\rho_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{6}+\alpha} e^{(t-\tau)L}\| \cdot \|g(\rho)\|_{H_{\frac{1}{6}}} d\tau \\
\le& C\|\rho_0\|_{H_\alpha} + C \int_0^t (t-\tau)^{-\beta} e^{-\delta(t-\tau)} d\tau \\
\le& C\|\rho_0\|_{H_\alpha} + C \int_0^t \tau^{-\beta} e^{-\delta\tau} d\tau \le C, \quad \forall t \ge 0, \ \rho_0 \in U \subset H_\alpha,
\end{aligned}
\tag{20}
$$

where $\beta = \frac{1}{6} + \alpha, (0 < \beta < 1)$. Hence, (18) is valid.

Step 3. We prove that for any bounded set $U \subset H_\alpha$ ($\frac{5}{6} \le \alpha < 1$), there exists a constant $C > 0$ such that

$$
\|\rho(t,\rho_0)\|_{H_\alpha} \le C, \ \forall t \ge 0, \ \rho_0 \in U, \ \frac{5}{6} \le \alpha < 1.
\tag{21}
$$

We claim that $g : H_\alpha \to H_{\frac{1}{3}}$ is bounded, by Sobolev embedding theorem, we have

$$
H_\alpha \hookrightarrow H^4(\Omega), \ H_\alpha \hookrightarrow W^{1,8}(\Omega), \ H_\alpha \hookrightarrow W^{2,4}(\Omega), \ H_\alpha \hookrightarrow W^{1,4}(\Omega), \ H_\alpha \hookrightarrow W^{3,4}(\Omega),
$$
$$
H_\alpha \hookrightarrow H^2(\Omega), \ H_\alpha \hookrightarrow L^\infty(\Omega)
$$

where $\frac{2}{3} \le \alpha < \frac{5}{6}$.

Then, we obtain

$$
\begin{aligned}
&\|g(\rho)\|^2_{H_{\frac{1}{3}}} \\
=& \int_\Omega |-\Delta^2\rho - \Delta^2 f(\rho) - \Delta(f'(\rho)\Delta\rho) + \Delta(f'(\rho)f(\rho)) + \Delta f(\rho)|^2 dx \\
\le& C \int_\Omega (|\Delta^2\rho|^2 + |\Delta^2 f(\rho)|^2 + |\Delta(f'(\rho)\Delta\rho)|^2 + |\Delta(f'(\rho)f(\rho))|^2 + |\Delta f(\rho)|^2) dx \\
\le& C(\|\rho\|^2_{H^4} + \|f^{(4)}(\rho)|\nabla\rho|^4\|^2 + \|f'''(\rho)|\nabla\rho|^2\Delta\rho\|^2 + \|f''(\rho)|\Delta\rho|^2\|^2 \\
&+ \|f''(\rho)\nabla\rho\nabla\Delta\rho\|^2 + \|f'(\rho)\Delta^2\rho\|^2 + \|f'''(\rho)|\nabla\rho|^2\Delta\rho\|^2 + \|f''(\rho)|\Delta\rho|^2\|^2 \\
&+ \|f''(\rho)\nabla\rho\nabla\Delta\rho\|^2 + \|f'(\rho)\Delta^2\rho\|^2 + \|f'''(\rho)f(\rho)|\nabla\rho|^2\|^2 + \|f''(\rho)f(\rho)\Delta\rho\|^2 \\
&+ \|f''(\rho)f'(\rho)|\nabla\rho|^2\|^2 + \||f'(\rho)|^2\Delta\rho\|^2 + \|f''(\rho)|\nabla\rho|^2\|^2 + \|f'(\rho)\Delta\rho\|^2) \\
\le& C(\|\rho\|^2_{H^4} + \|f^{(4)}(\rho)|\nabla\rho|^4\|^2 + \|f'''(\rho)|\nabla\rho|^2\|^4 + \|\Delta\rho\|^4 + \|f''(\rho)|\Delta\rho|^2\|^2 \\
&+ \|f''(\rho)\nabla\rho\|^4 + \|\nabla\Delta\rho\|^4 + \|f'(\rho)\Delta^2\rho\|^2 + \|f'''(\rho)f(\rho)|\nabla\rho|^2\|^2 + \|f''(\rho)f(\rho)\Delta\rho\|^2 \\
&+ \|f''(\rho)f'(\rho)|\nabla\rho|^2\|^2 + \||f'(\rho)|^2\Delta\rho\|^2 + \|f''(\rho)|\nabla\rho|^2\|^2 + \|f'(\rho)\Delta\rho\|^2) \\
\le& C(\|\rho\|^2_{H^4} + \|f^{(4)}(\rho)\|^2_{L^\infty}\|\rho\|^8_{W^{1,8}} + \|f'''(\rho)\|^4_{L^\infty}\|\rho\|^8_{W^{1,8}} + \|\rho\|^4_{W^{2,4}} \\
&+ \|f''(\rho)\|^2_{L^\infty}\|\rho\|^4_{W^{2,4}} + \|f''(\rho)\|^4_{L^\infty}\|\rho\|^4_{W^{1,4}} + \|\rho\|^4_{W^{3,4}} + \|f'(\rho)\|^2_{L^\infty}\|\rho\|^2_{H^4} \\
&+ \|f'''(\rho)f(\rho)\|^2_{L^\infty}\|\rho\|^4_{W^{1,4}} + \|f''(\rho)f(\rho)\|^2_{L^\infty}\|\rho\|^2_{H^2} + \|f''(\rho)f'(\rho)\|^2_{L^\infty}\|\rho\|^4_{W^{1,4}} \\
&+ \|f'(\rho)\|^4_{L^\infty}\|\rho\|^2_{H^2} + \|f''(\rho)\|^2_{L^\infty}\|\rho\|^4_{W^{1,4}} + \|f'(\rho)\|^2_{L^\infty}\|\rho\|^2_{H^2}) \\
\le& C(\|\rho\|^2_{H_\alpha} + \|\rho\|^8_{H_\alpha} + \|\rho\|^4_{H_\alpha})
\end{aligned}
\tag{22}
$$

which means that $g: H_\alpha \to H_{\frac{1}{3}}$ is bounded. On the basis of step 2 and (22), we deduce that

$$\begin{aligned}
\|\rho(t,\rho_0)\|_{H\alpha} &= \|e^{tL}\rho_0 + \int_0^t (-L)^{\frac{1}{3}} e^{(t-\tau)L} g(\rho) d\tau\|_{H\alpha} \\
&\leq C\|\rho_0\|_{H\alpha} + \int_0^t \|(-L)^\alpha e^{(t-\tau)L} g(\rho)\|_{H_{\frac{1}{3}}} d\tau \\
&\leq C\|\rho_0\|_{H\alpha} + \int_0^t \|(-L)^\alpha e^{(t-\tau)L}\| \cdot \|g(\rho)\|_{H_{\frac{1}{3}}} d\tau \\
&\leq C\|\rho_0\|_{H\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau} d\tau \leq C, \quad \forall t \geq 0,\ \rho_0 \in U \subset H_\alpha,
\end{aligned} \tag{23}$$

Hence, (21) is valid.

In the same fashion as in the proof of (21), by iteration we can prove that for any bounded set $U \subset H_\alpha (\alpha > 0)$ there exists a constant $C > 0$ such that

$$\|\rho(t,\rho_0)\|_{H\alpha} \leq C,\ \forall t \geq 0,\ \rho_0 \in U \subset H_\alpha,\ \alpha \geq 0. $$

That is, for all $\alpha \geq 0$ the semigroup $S(t)$ generated by problem (6)-(8) is uniformly compact in H_α. The Lemma 3.1 is proved. $\qquad\square$

Lemma 3.2. *For any $\alpha \geq 0$, problem (6)-(8) has a bounded absorbing set in H_α. That is, for any bounded set $U \in H_\alpha$, there exists $T > 0$ and a constant $C > 0$ independent of ρ_0, such that*

$$\|\rho(t,\rho_0)\|_{H\alpha} \leq C,\ \forall t \geq T,\ \rho_0 \in U \subset H_\alpha. \tag{24}$$

Proof. For $\alpha = \frac{1}{2}$, this follows from Lemma 2.2. Then, we prove (24) for any $\alpha > \frac{1}{2}$. we proceed in the following steps.

Step 1. We prove that for any $\frac{1}{2} \leq \alpha < \frac{2}{3}$, the problem (6)-(8) has a bounded absorbing set in H_α.

By (12),we have

$$\rho(t,\rho_0) = e^{tL}\rho_0 + \int_T^t (-L)^{\frac{1}{3}} e^{(t-T)L} g(\rho) d\tau. \tag{25}$$

Assume B is the bounded absorbing set of the problem (6)-(8) and B satisfies $B \subset H_{\frac{1}{2}}$. In addition, we also assume the time $t_0 > 0$ such that $\rho(t,\rho_0) \in B,\ \forall t \geq t_0,\ \rho_0 \in U \subset H_\alpha, \alpha \geq \frac{1}{2}$. Note that $\|e^{tL}\| \leq Ce^{-\lambda_1^3 t}$, where $\lambda_1 > 0$ is the first eigenvalue of the equation

$$\begin{cases} -\Delta\rho = \lambda\rho, \\ \dfrac{\partial\rho}{\partial n} = 0. \end{cases} \tag{26}$$

Then for any given $T > 0$ and $\rho_0 \in U \subset H_\alpha (\alpha \geq \frac{1}{2})$, we can obtain

$$\lim_{t\to\infty} \|e^{(t-T)L}\rho(T,\rho_0)\|_{H_\alpha} = 0. \tag{27}$$

Adding (16) and (25) together, we have

$$\begin{aligned}
\|\rho(t,\rho_0)\|_{H\alpha} &\leq \|e^{(t-t_0)L}\rho(t_0,\rho_0)\|_{H_\alpha} + \int_{t_0}^t \|(-L)^{\frac{1}{3}+\alpha} e^{(t-T)L}\| \cdot \|g(\rho)\|_H d\tau \\
&\leq \|e^{(t-t_0)L}\rho(t_0,\rho_0)\|_{H_\alpha} + C\int_{t_0}^t \|(-L)^{\frac{1}{3}+\alpha} e^{(t-T)L}\| d\tau \\
&\leq \|e^{(t-t_0)L}\rho(t_0,\rho_0)\|_{H_\alpha} + C\int_0^{T-t_0} \tau^{-\frac{1}{3}-\alpha} e^{-\delta\tau} d\tau \\
&\leq \|e^{(t-t_0)L}\rho(t_0,\rho_0)\|_{H_\alpha} + C,
\end{aligned} \tag{28}$$

where $C > 0$ is a constant independent of ρ_0, Then by (27) and (28), we have that (24) hold for all $\frac{1}{2} \leq \alpha < \frac{2}{3}$.

Step 2. We prove that for any $\frac{2}{3} \leq \alpha < \frac{5}{6}$, the problem (6)-(8) has a bounded absorbing set in H_α.

Adding (19) and (25) together, we have

$$
\begin{aligned}
\|\rho(t, \rho_0)\|_{H\alpha} &\leq \|e^{(t-t_0)L}\rho(t_0, \rho_0)\|_{H_\alpha} + \int_{t_0}^t \|(-L)^{\frac{1}{6}+\alpha}e^{(t-T)L}\| \cdot \|g(\rho)\|_{H_{\frac{1}{6}}} d\tau \\
&\leq \|e^{(t-t_0)L}\rho(t_0, \rho_0)\|_{H_\alpha} + C\int_{t_0}^t \|(-L)^{\frac{1}{6}+\alpha}e^{(t-T)L}\|d\tau \\
&\leq \|e^{(t-t_0)L}\rho(t_0, \rho_0)\|_{H_\alpha} + C\int_0^{T-t_0} \tau^{-\frac{1}{6}-\alpha}e^{-\delta\tau}d\tau \\
&\leq \|e^{(t-t_0)L}\rho(t_0, \rho_0)\|_{H_\alpha} + C,
\end{aligned}
\tag{29}
$$

where $C > 0$ is a constant independent of ρ_0, Then by (27) and (29), we have that (24) hold for all $\frac{2}{3} \leq \alpha < \frac{5}{6}$.

Step 3. We can use the same method as the above step to prove that for any $\frac{5}{6} \leq \alpha < 1$, the problem (6)-(8) has a bounded absorbing set in H_α. By the iteration method, we can obtain that (24) holds for all $\alpha \geq \frac{1}{2}$. $\qquad\square$

Now, we give the proof of Theorem 2.1.

Proof. Combining Lemma 3.1 with Lemma 3.2, we completed the proof of Theorem 2.1. $\quad\square$

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