GENERAL IMPLICIT SUBGRADIENT EXTRAGRADIENT METHODS FOR MONOTONE BILEVEL EQUILIBRIUM PROBLEMS

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In this paper, we introduce the general implicit subgradient extragradient method for solving the monotone bilevel equilibrium problem (MBEP) with a general system of variational inclusions (GSVI) and a common fixed-point problem of finitely many nonexpansive mappings and a strictly pseudocontractive mapping (CFPP) constraints. The strong convergence result for the proposed algorithm is established under the monotonicity assumption of the cost bifunctions with Lipschitz-type continuous conditions recently presented by Mastroeni in the auxiliary problem principle, and also applied for finding a common solution of variational inequality, variational inclusion and fixed-point problems.

Keywords: general implicit subgradient extragradient method, monotone bilevel equilibrium problem, convex minimization problem, strictly pseudocontractive mapping.

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1. **Introduction**

Let $(\mathcal{H}, \|\cdot\|)$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. Given a nonempty, closed and convex set $C \subset \mathcal{H}$. We denote by $\mathrm{Fix}(\Gamma)$ the fixed-point set of a self-mapping Γ on C. The mapping $\Gamma : C \to C$ is said to be strictly pseudocontractive if $\exists \xi \in [0, 1)$ s.t. $\|\Gamma u - \Gamma v\|^2 \le \|u - v\|^2 + \xi\|(I - \Gamma)u - (I - \Gamma)v\|^2\ \forall u, v \in C$. Let A be a self-mapping on $\mathcal{H}$. Consider the classical variational inequality problem (VIP) of finding $u^* \in C$ such that $\langle Au^*, v - u^*\rangle \ge 0\ \forall v \in C$. We denote by $\mathrm{VI}(C, A)$ the solution set of the VIP. The extragradient method invented first by Korpelevich [19] in 1976 has become one of the most effective methods for solving the VIP. It was shown in [19] that, if $\mathrm{VI}(C, A) \neq \emptyset$, this method converges weakly to a solution of the VIP. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it via various techniques; see e.g., [2, 6–8, 11–13, 15, 20, 31, 35, 38]. In particular, Censor et al. [11] modified Korpelevich's extragradient method and first introduced the subgradient extragradient method, in which the second projection onto C is replaced by a projection onto a half-space:

$$\begin{cases} v^k = P_C(u^k - \tau A u^k), \\ C_k = \{v \in \mathcal{H} : \langle u^k - \tau A u^k - v^k, v - v^k\rangle \le 0\}, \\ u^{k+1} = P_{C_k}(u^k - \tau A v^k), \quad \forall k \ge 0. \end{cases}$$

Suppose that $A_1, A_2 : \mathcal{H} \to \mathcal{H}$ are single-valued mappings and let $B_1, B_2 : C \to 2^{\mathcal{H}}$ are multi-valued mappings with $B_j u \neq \emptyset\ \forall u \in C, j = 1, 2$. The general system of variational

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inclusions (GSVI) is to find $(u^*, v^*) \in C \times C$ satisfying

$$\begin{cases} 0 \in \lambda_1(A_1 v^* + B_1 u^*) + u^* - v^*, \\ 0 \in \lambda_2(A_2 u^* + B_2 v^*) + v^* - u^*. \end{cases} \tag{1}$$

In particular, if $A_1 = A_2 = A,\ B_1 = B_2 = B$ and $u^* = v^*$, then problem (1) reduces to the variational inclusion (VI) ([9]). It is known that problem (1) has been transformed into a fixed point problem in the following way.

Lemma 1.1 ([10]). *Let $B_1, B_2 : C \to 2^{\mathcal{H}}$ be two maximal monotone operators. Then for given $u^*, v^* \in C$, (u^*, v^*) is a solution of problem (1.1) if and only if $u^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed-point set of the mapping $G := J_{\lambda_1}^{B_1}(I - \lambda_1 A_1) J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)$, and $v^* = J_{\lambda_2}^{B_2}(I - \lambda_2 A_2) u^*$.*

Furthermore, suppose that the mappings $A_1, A_2 : C \to \mathcal{H}$ are inverse-strongly monotone and the mapping $\Gamma : C \to C$ is asymptotically nonexpansive with a sequence $\{\theta_k\}$. Very recently, via a modified extragradient approach, Cai et al. [5] suggested a viscosity implicit rule for finding an element in the common solution set Ω of variational inequalities for $A_i, i = 1, 2$ and the fixed-point problem of Γ, i.e., for any given $u_1 \in C$, $\{u^k\}$ is the sequence generated by

$$\begin{cases} p^k = s_k u^k + (1 - s_k) q^k, \\ v^k = P_C(p^k - \lambda_2 A_2 p^k), \\ q^k = P_C(v^k - \lambda_1 A_1 v^k), \\ u^{k+1} = P_C[\beta_k f(u^k) + (I - \beta_k \rho F) \Gamma^k q^k], \end{cases} \tag{2}$$

where $\{\beta_k\}, \{s_k\} \subset (0, 1]$ are such that

(i) $\lim_{k\to\infty} \beta_k = 0,\ \sum_{k=1}^{\infty} \beta_k = \infty$ and $\sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty$;

(ii) $\lim_{k\to\infty} \frac{\theta_k}{\beta_k} = 0$;

(iii) $0 < \varepsilon \le s_k \le 1$ and $\sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty$;

(iv) $\sum_{k=1}^{\infty} \|\Gamma^{k+1} q^k - \Gamma^k q^k\| < \infty$.

They proved the strong convergence of $\{u^k\}$ to an element $u^* \in \Omega$, which solves the VIP: $\langle (\rho F - f) u^*, v - u^* \rangle \ge 0\ \forall v \in \Omega$.

Very recently, Ceng et al. [8] suggested a modified inertial subgradient extragradient method for finding a common solution of the VIP with pseudomonotone and Lipschitz continuous mapping $A : \mathcal{H} \to \mathcal{H}$ and the common fixed-point problem (CFPP) of finitely many nonexpansive mappings $\{\Gamma_i\}_{i=1}^N$ on $\mathcal{H}$. Under some suitable conditions, they proved strong convergence of the constructed sequence to a common solution of the VIP and CFPP.

Suppose that $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ is a bifunction such that $\Phi(x, x) = 0,\ \forall x \in C$. The equilibrium problem (shortly, $\mathrm{EP}(C, \Phi)$) is to find $\hat{u} \in C$ such that

$$\Phi(\hat{u}, v) \ge 0, \quad \forall v \in C. \tag{3}$$

The solution set of $\mathrm{EP}(C, \Phi)$ is denoted by $\mathrm{Sol}(C, \Phi)$. It is worth mentioning that the EP (3) is a unified model of several problems, namely, variational inequality problems ([14, 40, 45, 47–50, 53, 54, 56, 57]), optimization problems ([24]), saddle point problems, complementarity problems, fixed point problems ([25, 27–30, 33, 34, 39, 52]), Nash equilibrium problems ([22, 26]), split problems ([16, 17, 23, 32, 41, 43, 44, 46, 51, 55]). Many algorithms have been suggested and studied for solving the EP (3) and its extended versions; see [2, 6, 7, 9, 13, 26, 36, 42] and references therein. Very recently, Anh and An [2] introduced the monotone bilevel equilibrium problem (MBEP) with the fixed-point problem (FPP) constraint, i.e., a strongly monotone equilibrium problem $\mathrm{EP}(\Omega, \Psi)$ over the common solution set Ω of another monotone equilibrium problem $\mathrm{EP}(C, \Phi)$ and the fixed-point problem of a $\mathcal{K}$-demicontractive mapping Γ:

$$\text{Find } u^* \in \Omega \text{ such that } \Psi(u^*, v) \geq 0, \quad \forall v \in \Omega, \tag{4}$$

where $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ such that $\Psi(u,u) = 0$, $\forall u \in C$ and $\Omega = \mathrm{Sol}(C, \Phi) \cap \mathrm{Fix}(\Gamma)$.

Choose the parameter sequences $\{\lambda_k\}$ and $\{\beta_k\}$ such that

$$\begin{cases} \{\lambda_k\} \subset (a,b) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}),\ \lim_{k\to\infty} \lambda_k = \lambda, \\ \beta_k \downarrow 0,\ 2\beta_k\eta - \beta_k^2 S^2 < 1,\ \sum_{k=0}^{\infty} \beta_k = +\infty, \\ 0 < \tau < \min\{\eta, S\},\ 0 < \beta_k < \min\{\frac{1}{\tau}, \frac{2\eta-2\tau}{S^2-\tau^2}, \frac{2\eta}{S^2}\}, \end{cases} \tag{5}$$

where S is a constant associated with Ψ. The following modified subgradient extragradient method is proposed in [2] for finding a unique element of $\mathrm{Sol}(\Omega, \Psi)$.

Algorithm 1.1. *Choose an initial point $u^0 \in C$ and $\{\alpha_k\} \subset [\alpha, \bar{\alpha}] \subset (0, 1 - \mathcal{K})$. The parameter sequences $\{\lambda_k\}$ and $\{\beta_k\}$ satisfy the conditions* (5). *Compute u^{k+1} $(k \geq 0)$ as follows:*
Step 1. Compute $v^k = \mathrm{argmin}\{\lambda_k \Phi(u^k, v) + \frac{1}{2}\|v - u^k\|^2 : v \in C\}$ and $p^k = \mathrm{argmin}\{\lambda_k \Phi(v^k, p) + \frac{1}{2}\|p - u^k\|^2 : p \in C_k\}$, where $C_k = \{y \in \mathcal{H} : \langle u^k - \lambda_k w^k - v^k, y - v^k\rangle \leq 0\}$ and $w^k \in \partial_2 \Phi(u^k, v^k)$.
Step 2. Compute $q^k = (1 - \alpha_k)p^k + \alpha_k \Gamma p^k$ and $u^{k+1} = \mathrm{argmin}\{\beta_k \Psi(q^k, q) + \frac{1}{2}\|q - q^k\|^2 : q \in C\}$. Set $k := k + 1$ and return to Step 1.

In this paper, we introduce the general implicit subgradient extragradient method for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. The strong convergence result for the proposed algorithm is established under the monotonicity assumption of the cost bifunctions with Lipschitz-type continuous conditions recently presented by Mastroeni in the auxiliary problem principle. Our results improve and extend the corresponding results announced by some others, e.g., Cai et al. [5], Anh and An [2], and Ceng et al. [8].

2. **Preliminaries**

Assume that C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Given a sequence $\{y^k\} \subset \mathcal{H}$, we denote by $y^k \to y$ (resp., $y^k \rightharpoonup y$) the strong (resp., weak) convergence of $\{y^k\}$ to y. A bifunction $\Psi : C \times C \to \mathcal{R}$ is said to be

(i) η-strongly monotone, if $\Psi(y,z) + \Psi(z,y) \leq -\eta\|y - z\|^2, \forall y,z \in C$;

(ii) monotone, if $\Psi(y,z) + \Psi(z,y) \leq 0, \forall y,z \in C$;

(iii) Lipschitz-type continuous with constants $c_1, c_2 > 0$, if $\Psi(y,z) + \Psi(z,w) \geq \Psi(y,w) - c_1\|y - z\|^2 - c_2\|z - w\|^2, \forall y,z,w \in C$.

Also, recall that a mapping $F : C \to \mathcal{H}$ is said to be

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t. $\|Fy - Fz\| \leq L\|y - z\|,\ \forall y,z \in C$;

(ii) monotone if $\langle Fy - Fz, y - z\rangle \geq 0,\ \forall y,z \in C$;

(iii) pseudomonotone if $\langle Fz, y - z\rangle \geq 0 \Rightarrow \langle Fy, y - z\rangle \geq 0,\ \forall y,z \in C$;

(iv) η-strongly monotone if $\exists \eta > 0$ s.t. $\langle Fy - Fz, y - z\rangle \geq \eta\|y - z\|^2,\ \forall y,z \in C$;

(v) α-inverse-strongly monotone if $\exists \alpha > 0$ s.t. $\langle Fy - Fz, y - z\rangle \geq \alpha\|Fy - Fz\|^2,\ \forall y,z \in C$.

For each point $z \in \mathcal{H}$, we know that there exists a unique nearest point in C, denoted by $P_C z$, such that $\|z - P_C z\| \leq \|z - y\|,\ \forall y \in C$. The mapping P_C is said to be the metric projection of $\mathcal{H}$ onto C.

Lemma 2.1 ([18])**.** *The following hold:*

(i) $\langle y - z, P_C y - P_C z\rangle \geq \|P_C y - P_C z\|^2,\ \forall y,z \in \mathcal{H}$;

(ii) $\langle z - P_C z, y - P_C z\rangle \le 0,\ \forall z \in \mathcal{H}, y \in C$;
(iii) $\|z - y\|^2 \ge \|z - P_C z\|^2 + \|y - P_C z\|^2,\ \forall z \in \mathcal{H}, y \in C$;
(iv) $\|z - y\|^2 = \|z\|^2 - \|y\|^2 - 2\langle z - y, y\rangle,\ \forall y, z \in \mathcal{H}$;
(v) $\|sy + (1-s)z\|^2 = s\|y\|^2 + (1-s)\|z\|^2 - s(1-s)\|y - z\|^2,\ \forall y, z \in \mathcal{H}, s \in [0,1]$.

Recall that the mapping $\varGamma : C \to C$ is a ξ-strict pseudocontraction for some $\xi \in [0,1)$ if and only if the inequality holds $\langle \varGamma y - \varGamma z, y - z\rangle \le \|y - z\|^2 - \frac{1-\xi}{2}\|(I-\varGamma)y - (I-\varGamma)z\|^2\ \forall y, z \in C$. From [1] we know that if $\varGamma$ is a ξ-strictly pseudocontractive mapping, then $\varGamma$ satisfies Lipschitz condition $\|\varGamma y - \varGamma z\| \le \frac{1+\xi}{1-\xi}\|y - z\|\ \forall y, z \in C$.

Lemma 2.2. *Let $\varGamma : C \to C$ be a ξ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma + \delta)\xi \le \gamma$. Then $\|\gamma(y - z) + \delta(\varGamma y - \varGamma z)\| \le (\gamma + \delta)\|y - z\|,\ \forall y, z \in C$.*

Let $B : C \to 2^{\mathcal{H}}$ be a set-valued operator with $Bx \ne \emptyset\ \forall x \in C$. B is said to be monotone if for each $x, y \in C$, one has $\langle u - v, x - y\rangle \ge 0\ \forall u \in Bx, v \in By$. Also, B is said to be maximal monotone if $(I + \lambda B)C = \mathcal{H}$ for all $\lambda > 0$. For a monotone operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$.

Proposition 2.1 ([20])**.** *Let $B : C \to 2^{\mathcal{H}}$ be a maximal monotone operator. Then the following statements hold:*
(i) the resolvent identity: $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^B x)\ \forall \lambda, \mu > 0,\ x \in \mathcal{H}$;
(ii) if J_λ^B is a resolvent of B for $\lambda > 0$, then J_λ^B is a firmly nonexpansive mapping with $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$.

Let $A : \mathcal{H} \to \mathcal{H}$ be an α-inverse-strongly monotone mapping and $B : C \to 2^{\mathcal{H}}$ be a maximal monotone operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \forall \lambda > 0$.

Proposition 2.2 ([20])**.** *It is well known that (i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0,\ \forall \lambda > 0$ and (ii) $\|y - T_\lambda y\| \le 2\|y - T_r y\|$ for $0 < \lambda \le r$ and $y \in C$.*

Lemma 2.3. *Let the mapping $A : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone. Then, for a given $\lambda \ge 0$, $\|(I - \lambda A)u - (I - \lambda A)v\|^2 \le \|u - v\|^2 - \lambda(2\alpha - \lambda)\|Au - Av\|^2$. In particular, if $0 \le \lambda \le 2\alpha$, then $I - \lambda A$ is nonexpansive.*

Utilizing Proposition 2.1 (ii) and Lemma 2.3, we immediately obtain the following result.

Lemma 2.4. *Let $B_1, B_2 : C \to 2^{\mathcal{H}}$ be two maximal monotone operators. Let the mappings $A_1, A_2 : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : \mathcal{H} \to C$ be defined as $G := J_{\lambda_1}^{B_1}(I - \lambda_1 A_1)J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)$. If $0 \le \lambda_1 \le 2\alpha$ and $0 \le \lambda_2 \le 2\beta$, then $G : \mathcal{H} \to C$ is nonexpansive.*

Lemma 2.5 ([11])**.** *Let $A : C \to \mathcal{H}$ be pseudomonotone and continuous. Given a point $x \in C$. Then $\langle Ax, y - x\rangle \ge 0,\ \forall y \in C \Leftrightarrow \langle Ay, y - x\rangle \ge 0,\ \forall y \in C$.*

Lemma 2.6 ([1])**.** *Let $\varGamma : C \to C$ be a ξ-strict pseudocontraction. Then $I - \varGamma$ is demiclosed at zero, i.e., if $\{z_n\}$ is a sequence in C such that $z_n \rightharpoonup z \in C$ and $(I - \varGamma)z_n \to 0$, then $(I - \varGamma)z = 0$, where I is the identity mapping of $\mathcal{H}$.*

Lemma 2.7 ([21])**.** *Let $\{\mathcal{T}_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\mathcal{T}_{k_j}\}$ of $\{\mathcal{T}_k\}$ which satisfies $\mathcal{T}_{k_j} < \mathcal{T}_{k_j+1}$ for each integer $j \ge 1$. Define the sequence $\{\tau(k)\}_{k \ge k_0}$ of integers as follows:*

$$\tau(k) = \max\{j \le k : \mathcal{T}_j < \mathcal{T}_{j+1}\},$$

where integer $k_0 \ge 1$ such that $\{j \le k_0 : \mathcal{T}_j < \mathcal{T}_{j+1}\} \ne \emptyset$. Then, the following hold:
(i) $\tau(k_0) \le \tau(k_0 + 1) \le \cdots$ and $\tau(k) \to \infty$;
(ii) $\mathcal{T}_{\tau(k)} \le \mathcal{T}_{\tau(k)+1}$ and $\mathcal{T}_k \le \mathcal{T}_{\tau(k)+1},\ \forall k \ge k_0$.

On the other hand, the normal cone $N_C(u)$ of C at $u \in C$ is defined as $N_C(u) = \{w \in \mathcal{H} : \langle w, v-u\rangle \le 0,\ \forall v \in C\}$. The subdifferential of a convex function $g : C \to \mathcal{R} \cup \{+\infty\}$ at $u \in C$ is defined by $\partial g(u) = \{w \in \mathcal{H} : g(v) - g(u) \ge \langle w, v-u\rangle,\ \forall v \in C\}$.

In this paper, we are devoted to finding a solution $x^* \in \mathrm{Sol}(\Omega, \Psi)$ of the problem $\mathrm{EP}(\Omega, \Psi)$, where $\Omega = \bigcap_{i=0}^N \mathrm{Fix}(\Gamma_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$ with $\Gamma_0 := \Gamma$. We assume always that the following hold: (i) Γ_i is a nonexpansive self-mapping on $\mathcal{H}$ for $i = 1, ..., N$ and $\Gamma : \mathcal{H} \to \mathcal{H}$ is a ξ-strictly pseudocontractive mapping with $\xi \in [0, 1)$; (ii) $B_1, B_2 : C \to 2^{\mathcal{H}}$ are two maximal monotone operators, and $A_1, A_2 : \mathcal{H} \to \mathcal{H}$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively; (iii) $G : \mathcal{H} \to C$ is defined as $G := J_{\lambda_1}^{B_1}(I - \lambda_1 A_1) J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)$ where $0 < \lambda_1 < 2\alpha$ and $0 < \lambda_2 < 2\beta$. Choose the sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}$, $\{s_k\}$ such that

(H1) $\beta_k + \gamma_k + \delta_k = 1\ \forall k \ge 1$, $0 < \liminf_{k\to\infty} \delta_k$ and $(\gamma_k + \delta_k)\xi \le \gamma_k$;

(H2) $\limsup_{k\to\infty} \beta_k < 1$ and $0 < \liminf_{k\to\infty} \varepsilon_k \le \limsup_{k\to\infty} \varepsilon_k < 1$;

(H3) $\sum_{k=1}^{\infty} s_k = \infty$, $\lim_{k\to\infty} s_k = 0$, and $2s_k\nu - s_k^2 S^2 < 1$;

(H4) $\{\alpha_k\} \subset (a, b) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ and $\lim_{k\to\infty} \alpha_k = \tilde{\alpha}$;

(H5) $0 < \lambda < \min\{\nu, S\}$ and $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{S^2 - \lambda^2}, \frac{2\nu}{S^2}\}$.

In terms of Xu and Kim [37], we write $\Gamma_k := \Gamma_{k \bmod N}$ for integer $k \ge 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, i.e., if $k = jN + q$ for some integers $j \ge 0$ and $0 \le q < N$, then $\Gamma_k = \Gamma_N$ if $q = 0$ and $\Gamma_k = \Gamma_q$ if $0 < q < N$.

Algorithm 2.1. *Given $x^1 \in \mathcal{H}$ and $\zeta \in (0, 1)$ arbitrarily. The sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}$, $\{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}, \{s_k\}$ satisfy the conditions (H1)-(H5). Calculate x^{k+1} as follows:*

Step 1. *Compute*

$$\begin{cases} \bar{u}^k = \varepsilon_k x^k + (1 - \varepsilon_k)(\zeta \Gamma_k \bar{u}^k + (1 - \zeta) G \bar{u}^k), \\ \bar{v}^k = J_{\lambda_2}^{B_2}(\bar{u}^k - \lambda_2 A_2 \bar{u}^k). \end{cases}$$

Step 2. *Compute*

$$\begin{cases} \bar{q}^k = J_{\lambda_1}^{B_1}(\bar{v}^k - \lambda_1 A_1 \bar{v}^k), \\ y^k = \mathrm{argmin}\{\alpha_k \Phi(\bar{q}^k, y) + \frac{1}{2}\|y - \bar{q}^k\|^2 : y \in C\}. \end{cases}$$

Step 3. *Choose $\bar{w}^k \in \partial_2 \Phi(\bar{q}^k, y^k)$, and compute*

$$\begin{cases} C_k = \{v \in \mathcal{H} : \langle \bar{q}^k - \alpha_k \bar{w}^k - y^k, v - y^k\rangle \le 0\}, \\ z^k = \mathrm{argmin}\{\alpha_k \Phi(y^k, z) + \frac{1}{2}\|z - \bar{q}^k\|^2 : z \in C_k\}. \end{cases}$$

Step 4. *Compute*

$$\begin{cases} \tilde{p}^k = \beta_k z^k + \gamma_k G \tilde{p}^k + \delta_k \Gamma G \tilde{p}^k, \\ \bar{p}^k = G \tilde{p}^k, \\ x^{k+1} = \mathrm{argmin}\{s_k \Psi(\bar{p}^k, t) + \frac{1}{2}\|t - \bar{p}^k\|^2 : t \in C\}. \end{cases}$$

Set $k := k + 1$ and return to Step 1.

We need the following technical propositions.

Proposition 2.3 ([4]). *Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathcal{R} \cup \{+\infty\}$ be subdifferentiable. Then, $\bar{x}$ is a solution to the following convex minimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(\bar{x}) + N_C(\bar{x})$, where ∂g denotes the subdifferential of g.*

Proposition 2.4 ([3]). *Let X and Y be two sets, $\mathcal{G}$ be a set-valued map from Y to X, and W be a real valued function defined on $X \times Y$. The marginal function M is defined by*

$$M(y) = \{x^* \in \mathcal{G}(y) : W(x^*, y) = \sup\{W(x, y) : x \in \mathcal{G}(y)\}\}.$$

If W and $\mathcal{G}$ are continuous, then M is upper semicontinuous.

Next, we assume that two bifunctions $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ and $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ satisfy the following conditions where some notation is adopted from [2]:

$\mathbf{Ass}_\Phi$:

(Φ_1) $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(\Gamma_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi) \neq \emptyset$ with $\Gamma_0 := \Gamma$.

(Φ_2) Φ is monotone and Lipschitz-type continuous with constants $c_1, c_2 > 0$, and Φ is weakly continuous, i.e., $\{x^k \rightharpoonup \hat{x}$ and $y^k \rightharpoonup \hat{y}\} \Rightarrow \{\Phi(x^k, y^k) \to \Phi(\hat{x}, \hat{y})\}$.

$\mathbf{Ass}_\Psi$:

(Ψ_1) Ψ is ν-strongly monotone and weakly continuous.

(Ψ_2) There exist the mappings $\bar{\Psi}_i : C \times C \to \mathcal{H}$ and $\hat{\psi}_i : C \to \mathcal{H}$ for each $i \in \{1, ..., m\}$ such that $\bar{\Psi}_i(x, y) + \bar{\Psi}_i(y, x) = 0$, $\|\bar{\Psi}_i(x, y)\| \leq \bar{L}_i \|x - y\|$ and $\|\hat{\psi}_i(x) - \hat{\psi}_i(y)\| \leq \hat{L}_i \|x - y\|$ for all $x, y \in C$, and $\Psi(x, y) + \Psi(y, z) \geq \Psi(x, z) + \sum_{i=1}^{m} \langle \bar{\Psi}_i(x, y), \hat{\psi}_i(y - z) \rangle$, $\forall x, y, z \in C$.

(Ψ_3) For any sequence $\{y^k\} \subset C$ such that $y^k \to d$, we have $\limsup_{k\to\infty} \frac{|\Psi(d, y^k)|}{\|y^k - d\|} < +\infty$.

It is easy to see that if the bifunction Ψ satisfies the condition $\mathbf{Ass}_\Psi(\Psi_2)$, then Ψ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2} \sum_{i=1}^{m} \bar{L}_i \hat{L}_i$.

3. Main results

In this section, utilizing the general implicit subgradient extragradient method, we present convergence analysis of the iterative algorithm for solving the MBEP with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem $\mathrm{EP}(\Omega, \Psi)$ over the common solution set Ω of another monotone equilibrium problem $\mathrm{EP}(C, \Phi)$, the general system of variational inclusions (GSVI) and the CFPP of finitely many nonexpansive mappings $\{\Gamma_i\}_{i=1}^{N}$ and a strictly pseudocontractive mapping Γ, where $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(\Gamma_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$ with $\Gamma_0 := \Gamma$.

Theorem 3.1. *Assume that $\{x^k\}$ is the sequence constructed by Algorithm 2.1. Let the bifunctions Ψ, Φ satisfy the assumptions $\mathbf{Ass}_\Phi$-$\mathbf{Ass}_\Psi$. Then, under the conditions (H1)-(H5), the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω, Ψ).*

Proof. Choose an element $\bar{p} \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(\Gamma_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$ arbitrarily, where $G = J_{\lambda_1}^{B_1}(I - \lambda_1 A_1) J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)$ with $0 < \lambda_1 < 2\alpha$ and $0 < \lambda_2 < 2\beta$. We divide the proof into several steps as follows:

Step 1. We show that the following inequality holds

$$\|z^k - \bar{p}\|^2 \leq \|\bar{q}^k - \bar{p}\|^2 - (1 - 2\alpha_k c_1)\|y^k - \bar{q}^k\|^2 - (1 - 2\alpha_k c_2)\|z^k - y^k\|^2, \quad \forall k \geq 1.$$

Indeed, by Proposition 2.3, we know that for $y^k = \mathrm{argmin}\{\alpha_k \Phi(\bar{q}^k, y) + \frac{1}{2}\|y - \bar{q}^k\|^2 : y \in C\}$, there exists $\bar{w}^k \in \partial_2 \Phi(\bar{q}^k, y^k)$ such that $\alpha_k \bar{w}^k + y^k - \bar{q}^k \in -N_C(y^k)$, which hence yields

$$\langle \alpha_k \bar{w}^k + y^k - \bar{q}^k, x - y^k \rangle \geq 0, \quad \forall x \in C. \tag{6}$$

From the definition of $\bar{w}^k \in \partial_2 \Phi(\bar{q}^k, y^k)$, it follows that

$$\alpha_k[\Phi(\bar{q}^k, x) - \Phi(\bar{q}^k, y^k)] \geq \langle \alpha_k \bar{w}^k, x - y^k \rangle, \quad \forall x \in \mathcal{H}. \tag{7}$$

Adding (7) and (6), we get

$$\alpha_k[\Phi(\bar{q}^k, x) - \Phi(\bar{q}^k, y^k)] + \langle y^k - \bar{q}^k, x - y^k \rangle \geq 0, \quad \forall x \in C. \tag{8}$$

It follows from $z^k \in C_k$ and the definition of C_k that $\langle \bar{q}^k - \alpha_k \bar{w}^k - y^k, v - y^k \rangle \leq 0$, and hence

$$\alpha_k \langle \bar{w}^k, z^k - y^k \rangle \geq \langle \bar{q}^k - y^k, z^k - y^k \rangle. \tag{9}$$

Putting $x = z^k$ in (7), we get $\alpha_k[\Phi(\bar{q}^k, z^k) - \Phi(\bar{q}^k, y^k)] \geq \alpha_k\langle \bar{w}^k, z^k - y^k\rangle$. Adding (9) and the last inequality, we have

$$\alpha_k[\Phi(\bar{q}^k, z^k) - \Phi(\bar{q}^k, y^k)] \geq \langle \bar{q}^k - y^k, z^k - y^k\rangle. \tag{10}$$

By Proposition 2.3, we know that for $z^k = \operatorname{argmin}\{\alpha_k\Phi(y^k, y) + \frac{1}{2}\|y - \bar{q}^k\|^2 : y \in C_k\}$, there exist $\bar{h}^k \in \partial_2\Phi(y^k, z^k)$ and $\bar{t}^k \in N_{C_k}(z^k)$ such that $\alpha_k\bar{h}^k + z^k - \bar{q}^k + \bar{t}^k = 0$. So, we infer that $\alpha_k\langle \bar{h}^k, y - z^k\rangle \geq \langle \bar{q}^k - z^k, y - z^k\rangle\ \forall y \in C_k$, and $\Phi(y^k, y) - \Phi(y^k, z^k) \geq \langle \bar{h}^k, y - z^k\rangle, \forall y \in \mathcal{H}$. Putting $y = \bar{p} \in C \subset C_k$ in two last inequalities and later adding them, we get

$$\alpha_k[\Phi(y^k, \bar{p}) - \Phi(y^k, z^k)] \geq \langle \bar{q}^k - z^k, \bar{p} - z^k\rangle.$$

By the monotonicity of Φ, $\bar{p} \in \mathrm{Sol}(C, \Phi)$ and $y^k \in C$, we get $\Phi(y^k, \bar{p}) \leq -\Phi(\bar{p}, y^k) \leq 0$. Therefore, $-\alpha_k\Phi(y^k, z^k) \geq \langle \bar{q}^k - z^k, \bar{p} - z^k\rangle$. Combining this and the following Lipschitz-type continuity of Φ

$$\Phi(\bar{q}^k, y^k) + \Phi(y^k, z^k) \geq \Phi(\bar{q}^k, z^k) - c_1\|\bar{q}^k - y^k\|^2 - c_2\|y^k - z^k\|^2,$$

we obtain that

$$\begin{aligned} &\langle \bar{q}^k - z^k, z^k - \bar{p}\rangle \geq \alpha_k\Phi(y^k, z^k) \\ &\geq \alpha_k[\Phi(\bar{q}^k, z^k) - \Phi(\bar{q}^k, y^k)] - \alpha_k c_1\|\bar{q}^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2. \end{aligned}$$

This together with (10), implies that

$$\langle \bar{q}^k - z^k, z^k - \bar{p}\rangle \geq \langle \bar{q}^k - y^k, z^k - y^k\rangle - \alpha_k c_1\|\bar{q}^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2. \tag{11}$$

Therefore, applying the equality

$$\langle u, v\rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2) \quad \forall u, v \in \mathcal{H}, \tag{12}$$

for $\langle \bar{q}^k - z^k, z^k - \bar{p}\rangle$ and $\langle y^k - \bar{q}^k, z^k - y^k\rangle$ in (11), we obtain the desired result.

Step 2. We show that the following inequality holds

$$\|x^{k+1} - x\|^2 \leq \|\bar{p}^k - x\|^2 - \|x^{k+1} - \bar{p}^k\|^2 + 2s_k[\Psi(\bar{p}^k, x) - \Psi(\bar{p}^k, x^{k+1})], \quad \forall x \in C.$$

Indeed, since $x^{k+1} = \operatorname{argmin}\{s_k\Psi(\bar{p}^k, t) + \frac{1}{2}\|t - \bar{p}^k\|^2 : t \in C\}$, there exists $\bar{m}^k \in \partial_2\Psi(\bar{p}^k, x^{k+1})$ such that $0 \in s_k\bar{m}^k + x^{k+1} - \bar{p}^k + N_C(x^{k+1})$. By the definition of normal cone N_C and the subgradient $\bar{m}^k$, we get $\langle s_k\bar{m}^k + x^{k+1} - \bar{p}^k, x - x^{k+1}\rangle \geq 0, \forall x \in C$, and

$$s_k[\Psi(\bar{p}^k, x) - \Psi(\bar{p}^k, x^{k+1})] \geq \langle s_k\bar{m}^k, x - x^{k+1}\rangle, \quad \forall x \in C.$$

Adding two last inequalities, we get

$$2s_k[\Psi(\bar{p}^k, x) - \Psi(\bar{p}^k, x^{k+1})] + 2\langle x^{k+1} - \bar{p}^k, x - x^{k+1}\rangle \geq 0, \quad \forall x \in C. \tag{13}$$

Putting $u = x^{k+1} - \bar{p}^k$ and $v = x - x^{k+1}$ in (12), we get

$$2s_k[\Psi(x^{k+1}, x) - \Psi(\bar{p}^k, x^{k+1})] + \|\bar{p}^k - x\|^2 - \|x^{k+1} - \bar{p}^k\|^2 - \|x^{k+1} - x\|^2 \geq 0, \quad \forall x \in C.$$

This attains the desired result.

Step 3. We show that if x^* is a solution of the MBEP with the GSVI and CFPP constraints, then $\|x^{k+1} - \bar{p}^k_*\| \leq \eta_k\|\bar{p}^k - x^*\| \leq (1 - \lambda s_k)\|\bar{p}^k - x^*\|$, where $\bar{p}^k_* = \operatorname{argmin}\{s_k\Psi(x^*, v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\}$, $\eta_k = \sqrt{1 - 2s_k\nu + s_k^2S^2}$, $0 < \lambda < \min\{\nu, S\}$, $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{S^2 - \lambda^2}\}$, and $S = \sum_{i=1}^m \bar{L}_i\hat{L}_i$. Indeed, put $\bar{p}^k_* = \operatorname{argmin}\{s_k\Psi(x^*, v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\}$. By the similar arguments to those of (13), we also get

$$s_k[\Psi(x^*, x) - \Psi(x^*, \bar{p}^k_*)] + \langle \bar{p}^k_* - x^*, x - \bar{p}^k_*\rangle \geq 0, \quad \forall x \in C. \tag{14}$$

Setting $x = \bar{p}^k_* \in C$ in (13) and $x = x^{k+1} \in C$ in (14), respectively, we obtain that

$$\begin{aligned} &s_k[\Psi(\bar{p}^k, \bar{p}^k_*) - \Psi(\bar{p}^k, x^{k+1})] + \langle x^{k+1} - \bar{p}^k, \bar{p}^k_* - x^{k+1}\rangle \geq 0, \\ &s_k[\Psi(x^*, x^{k+1}) - \Psi(x^*, \bar{p}^k_*)] + \langle \bar{p}^k_* - x^*, x^{k+1} - \bar{p}^k_*\rangle \geq 0. \end{aligned}$$

Adding two last inequalities, we have

$$\begin{aligned}
0 &\le 2s_k[\Psi(\bar{p}^k,\bar{p}^k_*)-\Psi(\bar{p}^k,x^{k+1})+\Psi(x^*,x^{k+1})-\Psi(x^*,\bar{p}^k_*)]\\
&\quad+2\langle x^{k+1}-\bar{p}^k-\bar{p}^k_*+x^*,\bar{p}^k_*-x^{k+1}\rangle\\
&=2s_k[\Psi(\bar{p}^k,\bar{p}^k_*)-\Psi(\bar{p}^k,x^{k+1})+\Psi(x^*,x^{k+1})-\Psi(x^*,\bar{p}^k_*)]+\|\bar{p}^k-x^*\|^2\\
&\quad-\|x^{k+1}-\bar{p}^k-\bar{p}^k_*+x^*\|^2-\|x^{k+1}-\bar{p}^k_*\|^2,
\end{aligned} \tag{15}$$

where the last equality follows directly from (12).

Note that, under assumption $\mathbf{Ass}_\Psi(\Psi_2)$, it follows that

$$\Psi(\bar{p}^k,\bar{p}^k_*)-\Psi(x^*,\bar{p}^k_*)\le\Psi(\bar{p}^k,x^*)-\sum_{i=1}^m\langle\bar{\Psi}_i(\bar{p}^k,x^*),\hat{\psi}_i(x^*-\bar{p}^k_*)\rangle,$$

$$\Psi(x^*,x^{k+1})-\Psi(\bar{p}^k,x^{k+1})\le\Psi(x^*,\bar{p}^k)-\sum_{i=1}^m\langle\bar{\Psi}_i(x^*,\bar{p}^k),\hat{\psi}_i(\bar{p}^k-x^{k+1})\rangle.$$

Therefore, we have

$$\begin{aligned}
&\Psi(\bar{p}^k,\bar{p}^k_*)-\Psi(\bar{p}^k,x^{k+1})+\Psi(x^*,x^{k+1})-\Psi(x^*,\bar{p}^k_*)\\
&\le\Psi(\bar{p}^k,x^*)+\Psi(x^*,\bar{p}^k)-\sum_{i=1}^m\langle\bar{\Psi}_i(\bar{p}^k,x^*),\hat{\psi}_i(x^*-\bar{p}^k_*)\rangle-\sum_{i=1}^m\langle\bar{\Psi}_i(x^*,\bar{p}^k),\hat{\psi}_i(\bar{p}^k-x^{k+1})\rangle.
\end{aligned}$$

Then, using $\mathbf{Ass}_\Psi(\Psi_2)$, and the strong monotonicity of Ψ in $\mathbf{Ass}_\Psi(\Psi_1)$ that $\Psi(x,y)+\Psi(y,x)\le-\nu\|x-y\|^2\ \forall x,y\in C$, we get

$$\begin{aligned}
&\Psi(\bar{p}^k,\bar{p}^k_*)-\Psi(\bar{p}^k,x^{k+1})+\Psi(x^*,x^{k+1})-\Psi(x^*,\bar{p}^k_*)\\
&\le-\nu\|\bar{p}^k-x^*\|^2+\sum_{i=1}^m\langle\bar{\Psi}_i(\bar{p}^k,x^*),\hat{\psi}_i(\bar{p}^k-x^{k+1})-\hat{\psi}_i(x^*-\bar{p}^k_*)\rangle\\
&\le-\nu\|\bar{p}^k-x^*\|^2+\sum_{i=1}^m\bar{L}_i\hat{L}_i\|\bar{p}^k-x^*\|\|\bar{p}^k-x^{k+1}-x^*+\bar{p}^k_*\|\\
&=-\nu\|\bar{p}^k-x^*\|^2+S\|\bar{p}^k-x^*\|\|\bar{p}^k-x^{k+1}-x^*+\bar{p}^k_*\|.
\end{aligned} \tag{16}$$

Combining (15) and (16), we get

$$\begin{aligned}
0\ &\le(1-2s_k\nu)\|\bar{p}^k-x^*\|^2+2s_kS\|\bar{p}^k-x^*\|\|\bar{p}^k-x^{k+1}-x^*+\bar{p}^k_*\|\\
&\quad-\|x^{k+1}-\bar{p}^k-\bar{p}^k_*+x^*\|^2-\|x^{k+1}-\bar{p}^k_*\|^2\\
&=(1-2s_k\nu)\|\bar{p}^k-x^*\|^2-(\|x^{k+1}-\bar{p}^k-\bar{p}^k_*+x^*\|-s_kS\|\bar{p}^k-x^*\|)^2\\
&\quad+s_k^2S^2\|\bar{p}^k-x^*\|^2-\|x^{k+1}-\bar{p}^k_*\|^2\\
&\le(1-2s_k\nu+s_k^2S^2)\|\bar{p}^k-x^*\|^2-\|x^{k+1}-\bar{p}^k_*\|^2.
\end{aligned}$$

Note that $0\le\eta_k=\sqrt{1-2s_k\nu+s_k^2S^2}<1-\lambda s_k$. This ensures the desired result.

Step 4. We show that the sequence $\{x^k\}$ is bounded. Indeed, putting $X:=C,Y:=[0,1],\mathcal{G}(s):=C,\forall s\in Y,\ s:=s_k,W(x,s):=-s\Psi(x^*,x)-\frac{1}{2}\|x-x^*\|^2\ \forall(x,s)\in X\times Y$, we have that $M(s_k)=\mathrm{argmax}\{W(x,s_k):x\in C\}=\mathrm{argmin}\{s_k\Psi(x^*,x)+\frac{1}{2}\|x-x^*\|^2:x\in C\}=\{\bar{p}^k_*\}$. Note that M is continuous and $\lim_{k\to\infty}\bar{p}^k_*=x^*$. Since Ψ is continuous on C, we get $\lim_{k\to\infty}\Psi(x^*,\bar{p}^k_*)=\Psi(x^*,x^*)=0$. In terms of $\mathbf{Ass}_\Psi(\Psi_3)$, there exists a constant $\bar{M}(x^*)>0$ such that $|\Psi(x^*,\bar{p}^k_*)|\le\bar{M}(x^*)\|\bar{p}^k_*-x^*\|,\forall k\ge1$. Putting $x=x^*$ in (14) and using $\Psi(x^*,x^*)=0$, we get $-s_k\Psi(x^*,\bar{p}^k_*)+\langle\bar{p}^k_*-x^*,x^*-\bar{p}^k_*\rangle\ge0$, which hence yields

$$\|\bar{p}^k_*-x^*\|^2\le s_k[-\Psi(x^*,\bar{p}^k_*)]\le s_k\bar{M}(x^*)\|\bar{p}^k_*-x^*\|,\quad\forall k\ge1.$$

This immediately implies that $\|\bar{p}^k_*-x^*\|\le s_k\bar{M}(x^*),\forall k\ge1$. Also, according to Lemma 2.3 we know that $I-\lambda_1A_1$ and $I-\lambda_2A_2$ are nonexpansive mappings, where $\lambda_1\in(0,2\alpha)$ and $\lambda_2\in(0,2\beta)$. Note that the mapping $G:\mathcal{H}\to C$ is defined as $G:=J^{B_1}_{\lambda_1}(I-\lambda_1A_1)J^{B_2}_{\lambda_2}(I-\lambda_2A_2)$.

Hence, by Lemma 2.4, we know that G is nonexpansive. We write $y^* = J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)x^*$. Then, by Lemma 1.1, we get $x^* = J_{\lambda_1}^{B_1}(I - \lambda_1 A_1)y^* = Gx^*$. Thus we observe that

$$\begin{aligned}\|\bar{u}^k - x^*\| &\le \varepsilon_k\|x^k - x^*\| + (1-\varepsilon_k)[\zeta\|\Gamma_k\bar{u}^k - x^*\| + (1-\zeta)\|G\bar{u}^k - x^*\|] \\ &\le \varepsilon_k\|x^k - x^*\| + (1-\varepsilon_k)[\zeta\|\bar{u}^k - x^*\| + (1-\zeta)\|\bar{u}^k - x^*\|] \\ &= \varepsilon_k\|x^k - x^*\| + (1-\varepsilon_k)\|\bar{u}^k - x^*\|,\end{aligned}$$

which hence yields

$$\|\bar{u}^k - x^*\| \le \|x^k - x^*\|. \tag{17}$$

Since $\bar{v}^k = J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)\bar{u}^k$ and $\bar{q}^k = J_{\lambda_1}^{B_1}(I - \lambda_1 A_1)\bar{v}^k$, we have $\bar{q}^k = G\bar{u}^k$. Thus we get $\|\bar{q}^k - x^*\| = \|G\bar{u}^k - x^*\| \le \|\bar{u}^k - x^*\|$, which together with (17), yields

$$\|\bar{q}^k - x^*\| \le \|\bar{u}^k - x^*\| \le \|x^k - x^*\|.$$

This together with the result in Step 1, implies that

$$\|z^k - x^*\| \le \|\bar{q}^k - x^*\| \le \|\bar{u}^k - x^*\| \le \|x^k - x^*\|, \quad \forall k \ge 1. \tag{18}$$

Since Γ is ξ-strictly pseudocontractive such that $(\gamma_n + \delta_n)\xi \le \gamma_n$, by Lemma 2.2 we have

$$\begin{aligned}\|\tilde{p}^k - x^*\| &\le \beta_k\|z^k - x^*\| + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_k(G\tilde{p}^k - x^*) + \delta_k(\Gamma G\tilde{p}^k - x^*)]\| \\ &\le \beta_k\|z^k - x^*\| + (1-\beta_n)\|G\tilde{p}^k - x^*\| \\ &\le \beta_k\|z^k - x^*\| + (1-\beta_n)\|\tilde{p}^k - x^*\|,\end{aligned}$$

which together with (18) and $\bar{p}^k = G\tilde{p}^k$, yields

$$\|\bar{p}^k - x^*\| \le \|\tilde{p}^k - x^*\| \le \|z^k - x^*\| \le \|\bar{q}^k - x^*\| \le \|\bar{u}^k - x^*\| \le \|x^k - x^*\| \quad \forall k \ge 1. \tag{19}$$

So it follows that

$$\begin{aligned}&\|x^{k+1} - x^*\| \le \|x^{k+1} - \bar{p}_*^k\| + \|\bar{p}_*^k - x^*\| \le (1-\lambda s_k)\|\bar{p}^k - x^*\| + \|\bar{p}_*^k - x^*\| \\ &\le (1-\lambda s_k)\|x^k - x^*\| + s_k\bar{M}(x^*) \le \max\{\|x^k - x^*\|, \frac{\bar{M}(x^*)}{\lambda}\}.\end{aligned} \tag{20}$$

By induction, we get $\|x^k - x^*\| \le \max\{\|x^1 - x^*\|, \frac{\bar{M}(x^*)}{\lambda}\}\ \forall k \ge 1$. Thus, $\{x^k\}$ is bounded, and so are the sequences $\{\tilde{p}^k\}, \{\bar{p}^k\}, \{\bar{q}^k\}, \{y^k\}, \{z^k\}, \{\bar{u}^k\}, \{\bar{v}^k\}$.

Step 5. We show that if $x^{k_i} \rightharpoonup \hat{x}$, $\bar{q}^{k_i} - x^{k_i} \to 0$ and $\bar{q}^{k_i} - y^{k_i} \to 0$ for $\{k_i\} \subset \{k\}$, then $\hat{x} \in \mathrm{Sol}(C, \Phi)$. Indeed, noticing $\bar{q}^{k_i} - x^{k_i} \to 0$ and $\bar{q}^{k_i} - y^{k_i} \to 0$, we get

$$\|x^{k_i} - y^{k_i}\| \le \|x^{k_i} - \bar{q}^{k_i}\| + \|\bar{q}^{k_i} - y^{k_i}\| \to 0 \quad (i \to \infty). \tag{21}$$

So it follows from $x^{k_i} \rightharpoonup \hat{x}$ that $\bar{q}^{k_i} \rightharpoonup \hat{x}$ and $y^{k_i} \rightharpoonup \hat{x}$. Since $\{y^k\} \subset C$, $y^{k_i} \rightharpoonup \hat{x}$ and C is weakly closed, we know that $\hat{x} \in C$. By (8), we have

$$\alpha_{k_i}\Phi(\bar{q}^{k_i}, x) \ge \alpha_{k_i}\Phi(\bar{q}^{k_i}, y^{k_i}) + \langle y^{k_i} - \bar{q}^{k_i}, y^{k_i} - x\rangle, \quad \forall x \in C.$$

Taking the limit as $i \to \infty$ and using the assumptions that $\lim_{k\to\infty}\alpha_k = \tilde{\alpha} > 0$, $\Phi(\hat{x}, \hat{x}) = 0$, $\{y^{k_i}\}$ is bounded and Φ is weakly continuous, we obtain that $\tilde{\alpha}\Phi(\hat{x}, x) \ge 0$, $\forall x \in C$. This implies that $\hat{x} \in \mathrm{sol}(C, \Phi)$.

Step 6. We show that $x^k \to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraints. Indeed, set $\mathcal{T}_k = \|x^k - x^*\|^2$. Since Γ is ξ-strictly pseudocontractive such that $(\gamma_k + \delta_k)\xi \le \gamma_k$, using Lemma 2.2 and Lemma 2.1 (v) we obtain

$$\begin{aligned}\|\tilde{p}^k - x^*\|^2 &= \|\beta_k(z^k - x^*) + \gamma_k(G\tilde{p}^k - x^*) + \delta_k(\Gamma G\tilde{p}^k - x^*)\|^2 \\ &\le \beta_k\|z^k - x^*\|^2 + (1-\beta_k)\|G\tilde{p}^k - x^*\|^2 \\ &\quad -\beta_k(1-\beta_k)\|\tfrac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\|^2 \\ &\le \beta_k\|z^k - x^*\|^2 + (1-\beta_k)\|\tilde{p}^k - x^*\|^2 \\ &\quad -\beta_k(1-\beta_k)\|\tfrac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\|^2,\end{aligned}$$

which immediately leads to

$$\|\tilde{p}^k - x^*\|^2 \le \|z^k - x^*\|^2 - (1-\beta_k)\|\frac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\|^2. \tag{22}$$

By the results in Steps 1 and 2 we deduce from (19) and (22) that

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 \le \|\bar{p}^k - x^*\|^2 - \|x^{k+1} - \bar{p}^k\|^2 + 2s_k[\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1})] \\
&\le \|\bar{q}^k - x^*\|^2 - (1-2\alpha_k c_1)\|y^k - \bar{q}^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2 \\
&- (1-\beta_k)\|\frac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\|^2 - \|x^{k+1} - \bar{p}^k\|^2 \\
&+ 2s_k[\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1})] \\
&\le \|x^k - x^*\|^2 - (1-2\alpha_k c_1)\|y^k - \bar{q}^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2 \\
&- (1-\beta_k)\|\frac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\|^2 - \|x^{k+1} - \bar{p}^k\|^2 + s_k K,
\end{aligned} \tag{23}$$

where $\sup_{k\ge1}\{2|\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1})|\} \le K$ for some $K > 0$.

Finally, we show the convergence of $\{\mathcal{T}_k\}$ to zero by the following two cases: **Case 1.** Suppose that there exists an integer $k_0 \ge 1$ such that $\{\mathcal{T}_k\}$ is non-increasing. Then the limit $\lim_{k\to\infty} \mathcal{T}_k = \bar{\tau} < +\infty$ and $\mathcal{T}_k - \mathcal{T}_{k+1} \to 0 \quad (k \to \infty)$. From (23), we get

$$\begin{aligned}
&(1-2\alpha_k c_1)\|y^k - \bar{q}^k\|^2 + (1-2\alpha_k c_2)\|z^k - y^k\|^2 + (1-\beta_k) \\
&\times \|\frac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\|^2 + \|x^{k+1} - \bar{p}^k\|^2 \le \mathcal{T}_k - \mathcal{T}_{k+1} + s_k K,
\end{aligned} \tag{24}$$

Since $s_k \to 0$, $\mathcal{T}_k - \mathcal{T}_{k+1} \to 0$ and $\limsup_{k\to\infty} \beta_k < 1$, we obtain from $\{\alpha_k\} \subset (a,b) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ that

$$\lim_{k\to\infty} \|\frac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\| = 0, \tag{25}$$

and

$$\lim_{k\to\infty} \|y^k - \bar{q}^k\| = \lim_{k\to\infty} \|z^k - y^k\| = \lim_{k\to\infty} \|x^{k+1} - \bar{p}^k\| = 0. \tag{26}$$

We now show that $\|\bar{u}^k - \bar{q}^k\| \to 0$ as $k \to \infty$. Indeed, we set $y^* = J_{\lambda_2}^{B_2}(x^* - \lambda_2 A_2 x^*)$. Note that $\bar{v}^k = J_{\lambda_2}^{B_2}(\bar{u}^k - \lambda_2 A_2 \bar{u}^k)$ and $\bar{q}^k = J_{\lambda_1}^{B_1}(\bar{v}^k - \lambda_1 A_1 \bar{v}^k)$. Then $\bar{q}^k = G\bar{u}^k$. By Proposition 2.1 (ii) and Lemma 2.3 we have

$$\|\bar{v}^k - y^*\|^2 \le \|\bar{u}^k - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|A_2\bar{u}^k - A_2 x^*\|^2, \tag{27}$$

and

$$\|\bar{q}^k - x^*\|^2 \le \|\bar{v}^k - y^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|A_1\bar{v}^k - A_1 y^*\|^2. \tag{28}$$

Substituting (27) for (28), by (19) we get

$$\|\bar{q}^k - x^*\|^2 \le \|x^k - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|A_2\bar{u}^k - A_2 x^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|A_1\bar{v}^k - A_1 y^*\|^2. \tag{29}$$

Also, substituting (29) for (23), we get

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 \le \|\bar{q}^k - x^*\|^2 + s_k K \\
&\le \|x^k - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|A_2\bar{u}^k - A_2 x^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|A_1\bar{v}^k - A_1 y^*\|^2 + s_k K,
\end{aligned}$$

which immediately yields

$$\lambda_2(2\beta - \lambda_2)\|A_2\bar{u}^k - A_2 x^*\|^2 + \lambda_1(2\alpha - \lambda_1)\|A_1\bar{v}^k - A_1 y^*\|^2 \le \mathcal{T}_k - \mathcal{T}_{k+1} + s_k K.$$

Since $\lambda_1 \in (0, 2\alpha)$, $\lambda_2 \in (0, 2\beta)$, $s_k \to 0$ and $\mathcal{T}_k - \mathcal{T}_{k+1} \to 0$, we get

$$\lim_{k\to\infty} \|A_2\bar{u}^k - A_2 x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|A_1\bar{v}^k - A_1 y^*\| = 0. \tag{30}$$

On the other hand, from Lemma 2.1 (iv) and Proposition 2.1 (ii), we get

$$\begin{aligned}&\|\bar{q}^k - x^*\|^2 \le \langle \bar{v}^k - y^*, \bar{q}^k - x^*\rangle + \lambda_1\langle A_1y^* - A_1\bar{v}^k, \bar{q}^k - x^*\rangle\\ &\le \tfrac{1}{2}[\|\bar{v}^k - y^*\|^2 + \|\bar{q}^k - x^*\|^2 - \|\bar{v}^k - \bar{q}^k + x^* - y^*\|^2] + \lambda_1\|A_1y^* - A_1\bar{v}^k\|\|\bar{q}^k - x^*\|.\end{aligned}$$

This ensures that

$$\|\bar{q}^k - x^*\|^2 \le \|\bar{v}^k - y^*\|^2 - \|\bar{v}^k - \bar{q}^k + x^* - y^*\|^2 + 2\lambda_1\|A_1y^* - A_1\bar{v}^k\|\|\bar{q}^k - x^*\|. \quad (31)$$

Similarly, we get

$$\|\bar{v}^k - y^*\|^2 \le \|\bar{u}^k - x^*\|^2 - \|\bar{u}^k - \bar{v}^k + y^* - x^*\|^2 + 2\lambda_2\|A_2x^* - A_2\bar{u}^k\|\|\bar{v}^k - y^*\|. \quad (32)$$

Combining (31) and (32), by (19) we have

$$\begin{aligned}\|\bar{q}^k - x^*\|^2 \le{}& \|x^k - x^*\|^2 - \|\bar{u}^k - \bar{v}^k + y^* - x^*\|^2 - \|\bar{v}^k - \bar{q}^k + x^* - y^*\|^2\\ &+ 2\lambda_1\|A_1y^* - A_1\bar{v}^k\|\|\bar{q}^k - x^*\| + 2\lambda_2\|A_2x^* - A_2\bar{u}^k\|\|\bar{v}^k - y^*\|.\end{aligned} \quad (33)$$

Substituting (33) for (23), we get

$$\begin{aligned}&\|x^{k+1} - x^*\|^2 \le \|\bar{q}^k - x^*\|^2 + s_kK\\ &\le \|x^k - x^*\|^2 - \|\bar{u}^k - \bar{v}^k + y^* - x^*\|^2 - \|\bar{v}^k - \bar{q}^k + x^* - y^*\|^2\\ &\ +2\lambda_1\|A_1y^* - A_1\bar{v}^k\|\|\bar{q}^k - x^*\| + 2\lambda_2\|A_2x^* - A_2\bar{u}^k\|\|\bar{v}^k - y^*\| + s_kK.\end{aligned}$$

This immediately leads to

$$\begin{aligned}&\|\bar{u}^k - \bar{v}^k + y^* - x^*\|^2 + \|\bar{v}^k - \bar{q}^k + x^* - y^*\|^2\\ &\le \mathcal{T}_k - \mathcal{T}_{k+1} + 2\lambda_1\|A_1y^* - A_1\bar{v}^k\|\|\bar{q}^k - x^*\| + 2\lambda_2\|A_2x^* - A_2\bar{u}^k\|\|\bar{v}^k - y^*\| + s_kK.\end{aligned}$$

Since $s_k \to 0$ and $\mathcal{T}_k - \mathcal{T}_{k+1} \to 0$, we deduce from (30) that

$$\lim_{k\to\infty}\|\bar{u}^k - \bar{v}^k + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} +\|\bar{v}^k - \bar{q}^k + x^* - y^*\| = 0.$$

Thus,

$$\|\bar{u}^k - G\bar{u}^k\| = \|\bar{u}^k - \bar{q}^k\| \le \|\bar{u}^k - \bar{v}^k + y^* - x^*\| + \|\bar{v}^k - \bar{q}^k + x^* - y^*\| \to 0 \quad (k \to \infty). \ (34)$$

Utilizing the similar arguments to those of (34), we obtain

$$\lim_{k\to\infty}\|\tilde{p}^k - G\tilde{p}^k\| = \lim_{k\to\infty}\|\tilde{p}^k - \bar{p}^k\| = 0.$$

Noticing $\bar{u}^k = \varepsilon_k x^k + (1 - \varepsilon_k)(\zeta\Gamma_k\bar{u}^k + (1 - \zeta)G\bar{u}^k)$, we obtain from (19) and Lemma 2.1 (v) that

$$\begin{aligned}\|\bar{u}^k - x^*\|^2 &= \varepsilon_k\|x^k - x^*\|^2 + (1 - \varepsilon_k)[\zeta\|\Gamma_k\bar{u}^k - x^*\|^2 + (1 - \zeta)\|G\bar{u}^k - x^*\|^2\\ &\quad -\zeta(1 - \zeta)\|\Gamma_k\bar{u}^k - G\bar{u}^k\|^2] - \varepsilon_k(1 - \varepsilon_k)\|\zeta(x^k - \Gamma_k\bar{u}^k) + (1 - \zeta)(x^k - G\bar{u}^k)\|^2\\ &\le \varepsilon_k\|x^k - x^*\|^2 + (1 - \varepsilon_k)[\zeta\|\bar{u}^k - x^*\|^2 + (1 - \zeta)\|\bar{u}^k - x^*\|^2\\ &\quad -\zeta(1 - \zeta)\|\Gamma_k\bar{u}^k - G\bar{u}^k\|^2] - \varepsilon_k(1 - \varepsilon_k)\|\zeta(x^k - \Gamma_k\bar{u}^k) + (1 - \zeta)(x^k - G\bar{u}^k)\|^2\\ &= \varepsilon_k\|x^k - x^*\|^2 + (1 - \varepsilon_k)\|\bar{u}^k - x^*\|^2 - (1 - \varepsilon_k)\zeta(1 - \zeta)\|\Gamma_k\bar{u}^k - G\bar{u}^k\|^2\\ &\quad -\varepsilon_k(1 - \varepsilon_k)\|\zeta(x^k - \Gamma_k\bar{u}^k) + (1 - \zeta)(x^k - G\bar{u}^k)\|^2,\end{aligned}$$

which hence yields

$$\begin{aligned}\|\bar{u}^k - x^*\|^2 &\le \|x^k - x^*\|^2 - \tfrac{1-\varepsilon_k}{\varepsilon_k}\zeta(1 - \zeta)\|\Gamma_k\bar{u}^k - G\bar{u}^k\|^2\\ &\quad -(1 - \varepsilon_k)\|\zeta(x^k - \Gamma_k\bar{u}^k) + (1 - \zeta)(x^k - G\bar{u}^k)\|^2.\end{aligned}$$

This together with (23) and (19), implies that

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le \|\bar{q}^k - x^*\|^2 + s_kK \le \|\bar{u}^k - x^*\|^2 + s_kK\\ &\le \|x^k - x^*\|^2 - \tfrac{1-\varepsilon_k}{\varepsilon_k}\zeta(1 - \zeta)\|\Gamma_k\bar{u}^k - G\bar{u}^k\|^2\\ &\quad -(1 - \varepsilon_k)\|\zeta(x^k - \Gamma_k\bar{u}^k) + (1 - \zeta)(x^k - G\bar{u}^k)\|^2 + s_kK.\end{aligned}$$

So it follows that

$$\frac{1 - \varepsilon_k}{\varepsilon_k}\zeta(1-\zeta)\|\Gamma_k\bar{u}^k - G\bar{u}^k\|^2 + (1-\varepsilon_k)\|\zeta(x^k - \Gamma_k\bar{u}^k) + (1-\zeta)(x^k - G\bar{u}^k)\|^2 \le \mathcal{T}_k - \mathcal{T}_{k+1} + s_kK.$$

Since $s_k \to 0,\ \mathfrak{T}_k - \mathfrak{T}_{k+1} \to 0,\ \zeta \in (0,1)$ and $\limsup_{k\to\infty} \varepsilon_k < 1$, we get

$$\lim_{k\to\infty} \|\Gamma_k \bar{u}^k - G\bar{u}^k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|\zeta(x^k - \Gamma_k \bar{u}^k) + (1-\zeta)(x^k - G\bar{u}^k)\| = 0. \tag{35}$$

Noticing $\bar{u}^k - G\bar{u}^k = \varepsilon_k(x^k - G\bar{u}^k) + (1-\varepsilon_k)\zeta(\Gamma_k\bar{u}^k - G\bar{u}^k)$, we have

$$\begin{aligned} \varepsilon_k\|x^k - G\bar{u}^k\| &\le \|\bar{u}^k - G\bar{u}^k\| + (1-\varepsilon_k)\zeta\|\Gamma_k\bar{u}^k - G\bar{u}^k\| \\ &\le \|\bar{u}^k - G\bar{u}^k\| + \|\Gamma_k\bar{u}^k - G\bar{u}^k\|. \end{aligned}$$

From (34), (35) and $0 < \liminf_{k\to\infty} \varepsilon_k$, it follows that

$$\lim_{k\to\infty} \|x^k - G\bar{u}^k\| = 0, \tag{36}$$

which together with (35), implies that

$$\begin{aligned} \|\bar{u}^k - x^k\| &= (1-\varepsilon_k)\|\zeta(\Gamma_k\bar{u}^k - x^k) + (1-\zeta)(G\bar{u}^k - x^k)\| \\ &\le \|\zeta(\Gamma_k\bar{u}^k - G\bar{u}^k + G\bar{u}^k - x^k) + (1-\zeta)(G\bar{u}^k - x^k)\| \\ &\le \|\Gamma_k\bar{u}^k - G\bar{u}^k\| + \|G\bar{u}^k - x^k\| \to 0 \quad (k\to\infty). \end{aligned} \tag{37}$$

Combining (36) and (37), we have

$$\begin{aligned} \|x^k - Gx^k\| &\le \|x^k - G\bar{u}^k\| + \|G\bar{u}^k - Gx^k\| \\ &\le \|x^k - G\bar{u}^k\| + \|\bar{u}^k - x^k\| \to 0 \quad (k\to\infty). \end{aligned} \tag{38}$$

Moreover, from (34), (35) and (37), we infer that

$$\|\Gamma_k\bar{u}^k - \bar{u}^k\| \le \|\Gamma_k\bar{u}^k - G\bar{u}^k\| + \|G\bar{u}^k - \bar{u}^k\| \to 0 \quad (k\to\infty),$$

and hence

$$\begin{aligned} \|\Gamma_k x^k - x^k\| &\le \|\Gamma_k x^k - \Gamma_k\bar{u}^k\| + \|\Gamma_k\bar{u}^k - \bar{u}^k\| + \|\bar{u}^k - x^k\| \\ &\le 2\|x^k - \bar{u}^k\| + \|\Gamma_k\bar{u}^k - \bar{u}^k\| \to 0 \quad (k\to\infty). \end{aligned} \tag{39}$$

In addition, from (25) and (26) it follows that

$$\|z^k - \tilde{p}^k\| = (1-\beta_k)\|\frac{1}{1-\beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \Gamma G\tilde{p}^k)]\| \to 0 \quad (k\to\infty), \tag{40}$$

and

$$\|z^k - \bar{q}^k\| \le \|z^k - y^k\| + \|y^k - \bar{q}^k\| \to 0 \quad (k\to\infty), \tag{41}$$

and hence

$$\begin{aligned} &\|\bar{p}^k - \bar{q}^k\| \le \|\bar{p}^k - \tilde{p}^k\| + \|\tilde{p}^k - \bar{q}^k\| \\ &\le \|\bar{p}^k - \tilde{p}^k\| + \|\tilde{p}^k - z^k\| + \|z^k - \bar{q}^k\| \to 0 \quad (k\to\infty). \end{aligned} \tag{42}$$

Thus, using (26), (36) and (42), we have

$$\|x^{k+1} - x^k\| \le \|x^{k+1} - \bar{p}^k\| + \|\bar{p}^k - \bar{q}^k\| + \|\bar{q}^k - x^k\| \to 0 \quad (k\to\infty). \tag{43}$$

Note that

$$\|\bar{q}^k - G\bar{q}^k\| = \|G\bar{u}^k - G\bar{q}^k\| \le \|\bar{u}^k - \bar{q}^k\| \to 0 \quad (k\to\infty). \tag{44}$$

So it follows from (34), (41) and (42) that

$$\begin{aligned} \|z^k - G\tilde{p}^k\| &\le \|z^k - \bar{q}^k\| + \|\bar{q}^k - G\bar{q}^k\| + \|G\bar{q}^k - G\tilde{p}^k\| \\ &\le \|z^k - \bar{q}^k\| + \|\bar{q}^k - \bar{u}^k\| + \|\bar{q}^k - \tilde{p}^k\| \to 0 \quad (k\to\infty). \end{aligned} \tag{45}$$

Since $z^k - \tilde{p}^k = \gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \varGamma G\tilde{p}^k)$, we obtain from (40), (45) and $0 < \liminf_{k\to\infty} \delta_k$ that

$$\begin{aligned}\|z^k - \varGamma G\tilde{p}^k\| &= \frac{1}{\delta_k}\|z^k - \tilde{p}^k - \gamma_k(z^k - G\tilde{p}^k)\| \\ &\leq \frac{1}{\delta_k}(\|z^k - \tilde{p}^k\| + \gamma_k\|z^k - G\tilde{p}^k\|) \to 0 \quad (k \to \infty).\end{aligned} \tag{46}$$

Note that

$$\begin{aligned}\|G\bar{p}^k - \bar{p}^k\| &\leq \|G\bar{p}^k - G\bar{q}^k\| + \|G\bar{q}^k - \bar{q}^k\| + \|\bar{q}^k - \bar{p}^k\| \\ &\leq 2\|\bar{p}^k - \bar{q}^k\| + \|G\bar{q}^k - \bar{q}^k\|.\end{aligned}$$

So it follows from (40), (42), (44) and (46) that

$$\begin{aligned}\|\bar{p}^k - \varGamma G\bar{p}^k\| &\leq \|\bar{p}^k - z^k\| + \|z^k - \varGamma G\bar{p}^k\| \\ &\leq \|\bar{p}^k - \tilde{p}^k\| + \|\tilde{p}^k - z^k\| + \|z^k - \varGamma G\tilde{p}^k\| + \|\varGamma G\tilde{p}^k - \varGamma G\bar{p}^k\| \\ &\leq \tfrac{2}{1-\xi}\|\bar{p}^k - \tilde{p}^k\| + \|\tilde{p}^k - z^k\| + \|z^k - \varGamma G\tilde{p}^k\|,\end{aligned}$$

and hence

$$\begin{aligned}&\|\bar{p}^k - \varGamma\bar{p}^k\| \leq \|\bar{p}^k - \varGamma G\bar{p}^k\| + \|\varGamma G\bar{p}^k - \varGamma\bar{p}^k\| \\ &\leq \frac{2}{1-\xi}\|\bar{p}^k - \tilde{p}^k\| + \|\tilde{p}^k - z^k\| + \|z^k - \varGamma G\tilde{p}^k\| + \frac{1+\xi}{1-\xi}\|G\bar{p}^k - \bar{p}^k\| \\ &\leq \frac{2}{1-\xi}\|\bar{p}^k - \tilde{p}^k\| + \|\tilde{p}^k - z^k\| + \|z^k - \varGamma G\tilde{p}^k\| + \frac{1+\xi}{1-\xi}(2\|\bar{p}^k - \bar{q}^k\| + \|G\bar{q}^k - \bar{q}^k\|) \to 0.\end{aligned} \tag{47}$$

Meantime, it is easy to see from (26) and (43) that

$$\|x^k - \bar{p}^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - \bar{p}^k\| \to 0 \quad (k \to \infty). \tag{48}$$

Next we show that $\lim_{k\to\infty} \|x^k - x^*\| = 0$. In fact, since the sequences $\{\bar{p}^k\}$ and $\{x^k\}$ are bounded, we know that there exists a subsequence $\{\bar{p}^{k_i}\}$ of $\{\bar{p}^k\}$ converging weakly to $\hat{x} \in C$ and satisfying the equality

$$\liminf_{k\to\infty}[\Psi(x^*, \bar{p}^k) + \Psi(\bar{p}^k, x^{k+1})] = \lim_{i\to\infty}[\Psi(x^*, \bar{p}^{k_i}) + \Psi(\bar{p}^{k_i}, x^{k_i+1})]. \tag{49}$$

From (26) and (48) it follows that $x^{k_i} \rightharpoonup \hat{x}$ and $x^{k_i+1} \rightharpoonup \hat{x}$. Then, by the result in Step 5, we deduce that $\hat{x} \in \mathrm{Sol}(C, \Phi)$. We now show that $\lim_{k\to\infty} \|x^k - \varGamma_j x^k\| = 0$ for $j = 1, ..., N$. Note that for $j = 1, ..., N$,

$$\begin{aligned}\|x^k - \varGamma_{k+j}x^k\| &\leq \|x^k - x^{k+j}\| + \|x^{k+j} - \varGamma_{k+j}x^{k+j}\| + \|\varGamma_{k+j}x^{k+j} - \varGamma_{k+j}x^k\| \\ &\leq 2\|x^k - x^{k+j}\| + \|x^{k+j} - \varGamma_{k+j}x^{k+j}\|.\end{aligned}$$

Thus, from (39) and (43) we get $\lim_{k\to\infty} \|x^k - \varGamma_{k+j}x^k\| = 0$ for $j = 1, ..., N$. This immediately implies that

$$\lim_{k\to\infty} \|x^k - \varGamma_j x^k\| = 0 \quad \text{for } j = 1, ..., N. \tag{50}$$

Also, by (47) and (48) we have

$$\begin{aligned}\|x^k - \varGamma x^k\| &\leq \|x^k - \bar{p}^k\| + \|\bar{p}^k - \varGamma\bar{p}^k\| + \|\varGamma\bar{p}^k - \varGamma x^k\| \\ &\leq \|x^k - \bar{p}^k\| + \|\bar{p}^k - \varGamma\bar{p}^k\| + \frac{1+\xi}{1-\xi}\|\bar{p}^k - x^k\| \\ &= \frac{2}{1-\xi}\|x^k - \bar{p}^k\| + \|\bar{p}^k - \varGamma\bar{p}^k\| \to 0 \quad (k \to \infty).\end{aligned} \tag{51}$$

It is clear from (50) that $x^{k_i} - \varGamma_j x^{k_i} \to 0$ for $j = 1, ..., N$. Note that Lemma 2.6 guarantees the demiclosedness of $I - \varGamma_j$ at zero for $j = 1, ..., N$. So, we know that $\hat{x} \in \mathrm{Fix}(\varGamma_j)$. Since j is an arbitrary element in the finite set $\{1, ..., N\}$, we get $\hat{x} \in \cap_{j=1}^N \mathrm{Fix}(\varGamma_j)$. Also, note that Lemma 2.6 guarantees the demiclosedness of both $I - \varGamma$ and $I - G$ at zero. Since $\lim_{k\to\infty} \|x^k - \varGamma x^k\| = 0$ (due to (51)), we infer from $x^{k_i} \rightharpoonup \hat{x}$ that $\hat{x} \in \mathrm{Fix}(\varGamma)$, which hence yields $\hat{x} \in \bigcap_{j=0}^{N} \mathrm{Fix}(\varGamma_j)$. Meantime, from $x^{k_i} \rightharpoonup \hat{x}$ and $x^k - Gx^k \to 0$ (due to (38)) it follows that $\hat{x} \in \mathrm{Fix}(G)$. Consequently, $\hat{x} \in \bigcap_{j=0}^{N} \mathrm{Fix}(\varGamma_j) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi) = \varOmega$. In terms of (49), we have

$$\liminf_{k\to\infty}[\Psi(x^*, \bar{p}^k) + \Psi(\bar{p}^k, x^{k+1})] = \Psi(x^*, \hat{x}) \geq 0. \tag{52}$$

Since Ψ is ν-strongly monotone, we have

$$\limsup_{k\to\infty}[\Psi(x^*, \bar{p}^k) + \Psi(\bar{p}^k, x^*)] \leq \limsup_{k\to\infty}(-\nu\|\bar{p}^k - x^*\|^2) = -\nu\bar{\tau}. \tag{53}$$

Combining (52) and (53), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}[\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1})] \\
&= \limsup_{k\to\infty}[\Psi(\bar{p}^k, x^*) + \Psi(x^*, \bar{p}^k) - \Psi(x^*, \bar{p}^k) - \Psi(\bar{p}^k, x^{k+1})] \\
&\leq \limsup_{k\to\infty}[\Psi(\bar{p}^k, x^*) + \Psi(x^*, \bar{p}^k)] + \limsup_{k\to\infty}[-\Psi(x^*, \bar{p}^k) - \Psi(\bar{p}^k, x^{k+1})] \\
&= \limsup_{k\to\infty}[\Psi(\bar{p}^k, x^*) + \Psi(x^*, \bar{p}^k)] - \liminf_{k\to\infty}[\Psi(x^*, \bar{p}^k) + \Psi(\bar{p}^k, x^{k+1})] \\
&\leq -\nu\bar{\tau}.
\end{aligned} \tag{54}$$

We now claim that $\bar{\tau} = 0$. On the contrary, we assume $\bar{\tau} > 0$. Without loss of generality we may assume that $\exists k_0 \geq 1$ s.t.

$$\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1}) \leq -\frac{\nu\bar{\tau}}{2}, \quad \forall k \geq k_0, \tag{55}$$

which together with (23), implies that for all $k \geq k_0$,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - (1 - 2\alpha_k c_1)\|y^k - \bar{q}^k\|^2 - (1 - 2\alpha_k c_2)\|z^k - y^k\|^2 \\
&- (1 - \beta_k)\|\frac{1}{1 - \beta_k}[\gamma_k(z^k - G\tilde{p}^k) + \delta_k(z^k - \varGamma G\tilde{p}^k)]\|^2 - \|x^{k+1} - \bar{p}^k\|^2 \\
&+ 2s_k[\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1})] \\
&\leq \|x^k - x^*\|^2 + 2s_k[\Psi(\bar{p}^k, x^*) - \Psi(\bar{p}^k, x^{k+1})].
\end{aligned} \tag{56}$$

So it follows that for all $k \geq k_0$,

$$\mathcal{T}_k - \mathcal{T}_{k_0} \leq -\nu\bar{\tau}\sum_{j=k_0}^{k-1} s_j. \tag{57}$$

Since $\sum_{j=1}^{\infty} s_j = \infty$ and $\lim_{k\to\infty} \mathcal{T}_k = \bar{\tau}$, taking the limit in (57) as $k \to \infty$ we get

$$-\infty < \bar{\tau} - \mathcal{T}_{k_0} = \lim_{k\to\infty}(\mathcal{T}_k - \mathcal{T}_{k_0}) \leq \lim_{k\to\infty}[-\nu\bar{\tau}\sum_{j=k_0}^{k-1} s_j] = -\infty.$$

This reaches a contradiction. Therefore, $\lim_{k\to\infty} \mathcal{T}_k = 0$ and hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem $\mathrm{EP}(\varOmega, \Psi)$.

Case 2. Suppose that $\exists\{\mathcal{T}_{k_j}\} \subset \{\mathcal{T}_k\}$ s.t. $\mathcal{T}_{k_j} < \mathcal{T}_{k_j+1}$ $\forall j \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau : \mathcal{N} \to \mathcal{N}$ by $\tau(k) := \max\{j \leq k : \mathcal{T}_j < \mathcal{T}_{j+1}\}$. By Lemma 2.7, we get

$$\mathcal{T}_{\tau(k)} \leq \mathcal{T}_{\tau(k)+1} \quad \text{and} \quad \mathcal{T}_k \leq \mathcal{T}_{\tau(k)+1}. \tag{58}$$

Utilizing the same inferences as in (26) and (43), we can obtain that

$$\lim_{k\to\infty}\|x^{\tau(k)+1} - \bar{p}^{\tau(k)}\| = \lim_{k\to\infty}\|\bar{q}^{\tau(k)} - y^{\tau(k)}\| = \lim_{k\to\infty}\|y^{\tau(k)} - z^{\tau(k)}\| = 0, \tag{59}$$

and

$$\lim_{k\to\infty} \|x^{\tau(k)+1} - x^{\tau(k)}\| = 0. \tag{60}$$

Since $\{\bar{p}^k\}$ is bounded, there exists a subsequence of $\{\bar{p}^{\tau(k)}\}$ converging weakly to $\hat{x}$. Without loss of generality, we may assume that $\bar{p}^{\tau(k)} \rightharpoonup \hat{x}$. Then, utilizing the same inferences as in Case 1, we can obtain that $\hat{x} \in \Omega = \bigcap_{i=0}^N \mathrm{Fix}(\Gamma_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$. From $\bar{p}^{\tau(k)} \rightharpoonup \hat{x}$ and (59), we get $x^{\tau(k)+1} \rightharpoonup \hat{x}$. Using the condition $\{\alpha_k\} \subset (a, b) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$, we have $1 - 2\alpha_{\tau(k)}c_1 > 0$ and $1 - 2\alpha_{\tau(k)}c_2 > 0$. So it follows from (23) that

$$\begin{aligned}
&2s_{\tau(k)}[\Psi(\bar{p}^{\tau(k)}, x^{\tau(k)+1}) - \Psi(\bar{p}^{\tau(k)}, x^*)] \le \mathcal{T}_{\tau(k)} - \mathcal{T}_{\tau(k)+1} - \|x^{\tau(k)+1} - \bar{p}^{\tau(k)}\|^2 \\
&-(1 - 2\alpha_{\tau(k)}c_1)\|y^{\tau(k)} - \bar{q}^{\tau(k)}\|^2 - (1 - 2\alpha_{\tau(k)}c_2)\|z^{\tau(k)} - y^{\tau(k)}\|^2 \\
&-(1 - \beta_{\tau(k)})\|\tfrac{1}{1-\beta_{\tau(k)}}[\gamma_{\tau(k)}(z^{\tau(k)} - G\tilde{p}^{\tau(k)}) + \delta_{\tau(k)}(z^{\tau(k)} - \Gamma G\tilde{p}^{\tau(k)})]\|^2 \le 0,
\end{aligned}$$

which hence leads to

$$\Psi(\bar{p}^{\tau(k)}, x^{\tau(k)+1}) - \Psi(\bar{p}^{\tau(k)}, x^*) \le 0. \tag{61}$$

Since Ψ is ν-strongly monotone on C, we get

$$\nu\|\bar{p}^{\tau(k)} - x^*\|^2 \le -\Psi(\bar{p}^{\tau(k)}, x^*) - \Psi(x^*, \bar{p}^{\tau(k)}). \tag{62}$$

Combining (61) and (62), we deduce from $\mathbf{Ass}_\Psi(\Psi_1)$ and $\hat{x} \in \Omega$ that

$$\begin{aligned}
\nu\limsup_{k\to\infty}\|\bar{p}^{\tau(k)} - x^*\|^2 &\le \limsup_{k\to\infty}[-\Psi(\bar{p}^{\tau(k)}, x^{\tau(k)+1}) - \Psi(x^*, \bar{p}^{\tau(k)})] \\
&= -\Psi(\hat{x}, \hat{x}) - \Psi(x^*, \hat{x}) \le 0.
\end{aligned}$$

Hence, $\limsup_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 \le 0$. Thus, we get $\lim_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 = 0$. From (60), we get

$$\begin{aligned}
\|x^{\tau(k)+1} - x^*\|^2 - \|x^{\tau(k)} - x^*\|^2 &= 2\langle x^{\tau(k)+1} - x^{\tau(k)}, x^{\tau(k)} - x^*\rangle + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2 \\
&\le 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2.
\end{aligned}$$

Owing to $\mathcal{T}_k \le \mathcal{T}_{\tau(k)+1}$, we get

$$\begin{aligned}
&\|x^k - x^*\|^2 \le \|x^{\tau(k)+1} - x^*\|^2 \\
&\le \|x^{\tau(k)} - x^*\|^2 + 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2.
\end{aligned}$$

So it follows from (60) that $x^k \to x^*$ as $k \to \infty$. This completes the proof. □

4. **Concluding remarks**

In a real Hilbert space, let the GSVI and CFPP represent a general system of variational inclusions and a common fixed-point problem of finitely many nonexpansive mappings and a strictly pseudocontractive mapping, respectively. In this article, we have suggested a new iterative algorithm with the general implicit subgradient extragradient technique for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. The strong convergence result for the proposed algorithm to solve such a MBEP with the GSVI and CFPP constraints is established under some suitable assumptions. Furthermore, in the proposed method, the second minimization problem over a closed convex set is replaced by the subgradient projection onto some constructible half-space, and a new approach for solving the GSVI and CFPP via Mann implicit iterations is provided.

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