U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 3, 2014 ISSN 1223-7027

ANALYSIS OF NONLINEAR RANDOM VIBRATIONS BY STATISTICAL EQUIVALENT METHODS

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In this paper is given a method for improving the accuracy of the mean square response estimation of quadratic dissipative systems using a linear oscillatory system having the same exact solution with a statistically equivalent system with nonlinear dissipative characteristic. It is compared numerically the proposed method with the classical linearization method.

Keywords: quadratic dissipative system, statistical equivalent linearization, statistical equivalent non-linearization

1. Introduction

Stochastic equivalent linearization method is the most popular approach to the approximate analysis of nonlinear systems under random oscillator. The original version of Gaussian equivalent linearization was proposed by Caughey [1] and has been generalized by many authors (see e.g. [2]-[6]). It has been shown that this method is presently the simplest tool widely used for analysis of nonlinear stochastic problems. The use of equivalent linear systems for assessing the mean square response of the non-linear systems allows applying the transfer functions method which provides closed analytical solutions and is very efficient from computational point of view. However, a major limitation for its application is that its accuracy decreases as either the nonlinearity or input intensity increase. For this reason, there have been developed up o recent years a series of statistical equivalent linearization methods aimed to improve the approximation accuracy of statistical characteristics of nonlinear system output [7-13]. In this paper, the classical method of statistical equivalent linearization will be improved to increase the approximation accuracy of the mean square response of a SDOF system with quadratic dissipative characteristic excited by a Gaussian stationary white noise process. The method is applied in two steps. In the first step, the system with quadratic damping is replaced by a nonlinear equivalent system for which the exact mean square output can be obtained by solving the associated Fokker-Planck equation [14], [15]. In the second step, the parameters of the latter system

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are determined so that the exact solution of the equivalent nonlinear system to be equal with the exact solution of the linear system. It is shown that the proposed method, which combines the statistical linearization and non-linearization techniques, significantly improves the accuracy of classical linearization method.

2. Analytical model and input calibration.

In order to analyze comparatively the behavior of different oscillating systems excited by a white noise random process, it is necessary to develop a numerical simulation program to determine their mean square response. It is well known that the one sided power spectral density (PSD) of the theoretical white noise process has a constant value $2S_0$ within the infinite frequency range $[0,\infty)$, thus implying an infinite energy. However, due to the filtering properties of physical systems, the white noise process can be used as useful excitation model in many practical applications. According to Nyquist sampling theorem, the frequency bandwidth of the PSD, estimated from numerically simulated sample functions, is inherently limited. Therefore the intensity of simulated white noise excitation must be sized so that the system response belongs to the usual range of the considered practical application. In what follows, the way of solving this problem is illustrated in the case of a SDOF oscillator consisting of a sprung mass, spring and damper (Fig.1), modeling the automobile random vibrations excited by the road irregularities for a traveling speed V [16].

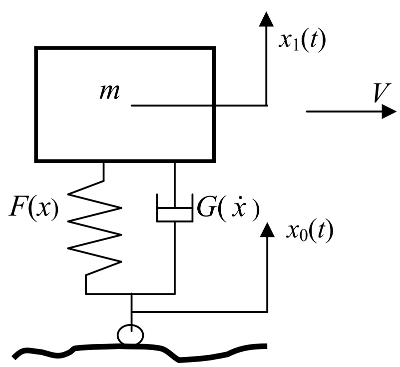


Fig. 1. Mechanical system with one degree of freedom, consisting of a mass, spring and damper

This model can be used to assess the automobile ride comfort, which is measured by the root mean square (r.m.s.) value of the sprung mass acceleration $\ddot{x}_1$ [16]. The equation of motion of the considered system is:

$$m\ddot{x}_1 + F(x) + G(\dot{x}) = 0\,,$$
$$x(0) = x^0, \dot{x}(0) = \dot{x}^0, \quad x^0, \dot{x}^0 \in R \tag{1}$$

where $x = x_1 - x_0$ is the relative displacement, $F(x)$ is the elastic force and $G(\dot{x})$ is the damping characteristic. Equation (1) can be rewritten as

$$\ddot{x}_1 + f(x) + g(\dot{x}) = 0\,, \tag{2}$$

where

$$f(x) = \frac{F(x)}{m},\; g(\dot{x}) = \frac{G(\dot{x})}{m}. \tag{3}$$

For the linear case

$$F(x) = kx \quad \text{and} \quad G(\dot{x}) = c\dot{x} \tag{4}$$

so that

$$f(x) = \frac{k}{m}x \quad \text{and} \quad g(\dot{x}) = \frac{c}{m}\dot{x}\ . \tag{5}$$

Denoting by $\omega_n = \sqrt{\dfrac{k}{m}}, \zeta = \dfrac{c}{2\sqrt{km}}$ the system undamped natural frequency and relative damping coefficient, the equation of motion becomes

$$\ddot{x} + 2\omega_n\zeta\dot{x} + \omega_n^2 x = z(t)\,, \quad t \in R_+ \tag{6}$$

where $z(t) = -\ddot{x}_0(t)$ is a Gaussian random white noise perturbation with

$$\mathrm{E}\big[z(t)\big] = 0\,, \quad \mathrm{E}\big[z(t)z(t+\tau)\big] = 2\pi S_0\delta(\tau) \tag{7}$$

It is known that the mean square solution of equation (6) is [15], [17]:

$$\sigma_x^2 = \frac{\pi S_0}{2\omega_n^3\zeta}, \sigma_{\dot{x}}^2 = \frac{\pi S_0}{2\omega_n\zeta}, \sigma_{\ddot{x}_1}^2 = \frac{\pi\omega_n S_0(1+4\zeta^2)}{2\zeta} \tag{8}$$

Knowing that the usual values for automobiles [5] are:

$$2\pi\cdot 0.9 < \omega_n < 2\pi\cdot 1.4\text{rad/s} \quad , \quad 0.15 < \zeta < 0.35 \quad \text{and} \quad 0.96\text{m/s}^2 < \sigma_{\ddot{x}_1} < 2\text{m/s}^2 \tag{9}$$

the parameters of the system and the intensity of the numerical simulated excitation will be chosen in order to cover the range of these values.

The constant value S_0 of the spectral density of the white noise excitation will be now determined, so that the solution obtained by numerical simulation to approximate with a good accuracy the exact mean response (8) for $\omega_n = 2\pi$ rad/s and all practical values of the relative damping coefficient, $0.1 \le \zeta \le 0.7$. The considered sampling interval was $\Delta t = 0.01s$, which corresponds to a 50Hz white noise bandwidth, according to the Nyquist criterion (if the sampling frequency is $f_e = 1/\Delta t = N/T$ [Hz], then the cut-off frequency is $f_c = f_e / 2$). This bandwidth can be viewed as infinite for the considered system which behaves like a band-pass filter with undamped natural frequency f_n=1 Hz. The numerical simulations were carried out using a program developed in Matlab-Simulink.

The program was run for different values of simulated white noise input power until the system r.m.s. acceleration value was obtained within the range of those measured for a car equipped by adjustable dampers, travelling on different types of roads [16]. The results of numerical simulations are presented in Table 1.

Table 1

Simulated values of the r.m.s. output of linear system

i	ζ_i	$\sigma_x(\zeta_i)$ [m]	$\sigma_{\dot{x}}(\zeta_i)$ [m/s]	$\sigma_{\ddot{x}}(\zeta_i)$ [m/s^2]
1	0.1	0.042	0.262	1.676
2	0.2	0.030	0.192	1.291
3	0.3	0.025	0.158	1.153
4	0.4	0.022	0.138	1.099
5	0.5	0.019	0.123	1.085
6	0.6	0.0175	0.112	1.093
7	0.7	0.016	0.104	1.112

Considering, however, that the parameter of practical interest is in this case the r.m.s. sprung mass acceleration, in order to calibrate the excitation of the linear system, S_0 will be determined so that the following expression reaches its minimum:

$$\mathcal{E}(S_0) = \sum_{i=1}^{n}\left[\sigma_{\ddot{x}_1}^2(\zeta_i) - \frac{\pi\omega_n S_0(1+4\zeta_i^2)}{2\zeta_i}\right]^2 = \min., \quad \frac{\partial\mathcal{E}(S_0)}{\partial S_0} = 0 \tag{10}$$

This gives

$$S_0 = \frac{2\sum_{i=1}^{n}\left[\sigma_{\ddot{x}_1}^2(\zeta_i)\dfrac{(1+4\zeta_i^2)}{\zeta_i}\right]}{\pi\omega_n\sum_{i=1}^{n}\dfrac{(1+4\zeta_i^2)^2}{\zeta_i^2}} \cong 0.029\,\mathrm{m^2s^{-3}} \tag{11}$$

The time history of calibrated input acceleration ($0 \le t \le 100\mathrm{s}$) and its amplitude spectrum are shown figures 2 and 3. Figure 4-6 present the r.m.s. output values obtained by numerical simulation, compared with the exact values (8), using the white noise intensity given by (11). These results show that the simulated solutions approximate well the exact solutions obtained for the same values of the relative damping coefficient ζ.

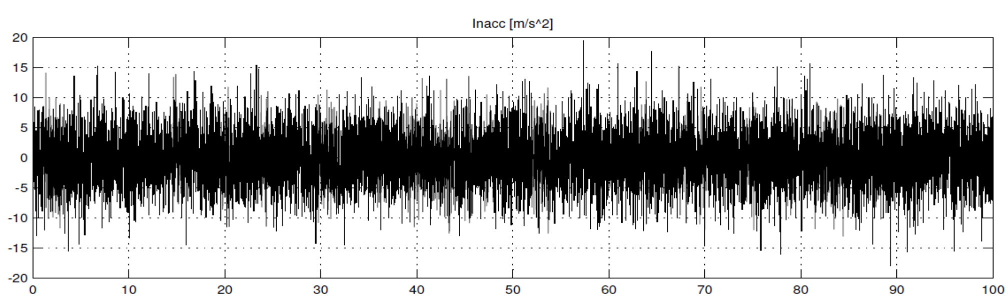


Fig.2. Simulated time history of input acceleration

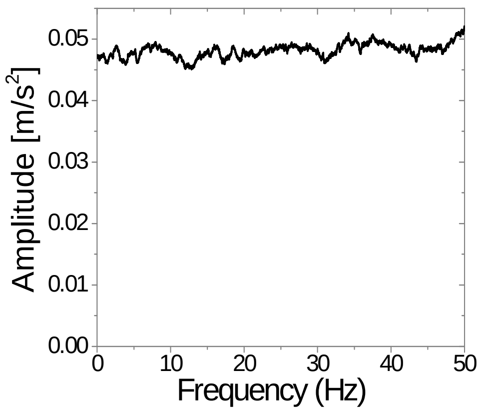


Fig.3. Amplitude spectrum of input acceleration

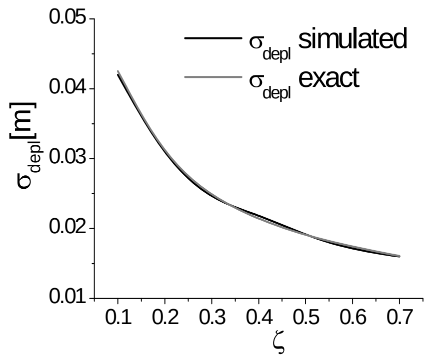


Fig.4. R.m.s. values of relative displacement

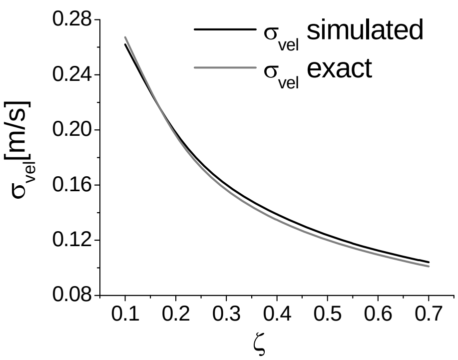


Fig.5. R.m.s. values of relative velocity

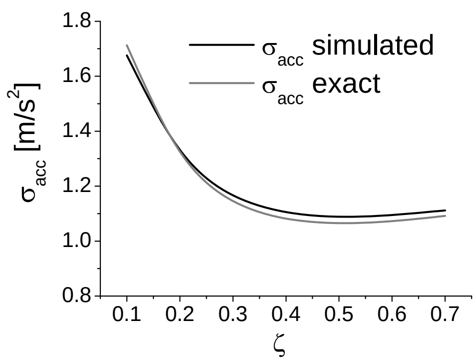


Fig.6. R.m.s. values of sprung mass acceleration

3. Classical equivalent linearization method

Consider now an oscillating system with linear elastic characteristic and quadratic dissipative characteristic.

$$\ddot{x} + q\dot{x}\left|\dot{x}\right| + \omega_n^2 x = z(t)$$
$$\mathrm{E}[z(t)] = 0\,, \quad \mathrm{E}\big[z(t)z(t+\tau)\big] = 2\pi S_0 \delta(\tau) \tag{12}$$

Using the statistical linearization method, the relative damping coefficient of the linear equivalent system is obtained by imposing the condition

$$\mathcal{E}(\zeta) = \mathrm{E}\left[(q|\dot{x}|\dot{x} - 2\omega_n\zeta\dot{x})^2\right] = \min. \,, \quad \frac{\partial\mathcal{E}}{\partial\zeta} = 0 \tag{13}$$

The expectations are calculated with respect to the Gaussian probability density function

$$p(\dot{x}) = \frac{1}{\sqrt{2\pi}\sigma_{\dot{x}}}\exp\left(-\frac{\dot{x}^2}{2\sigma_{\dot{x}}^2}\right), \ \sigma_{\dot{x}}^2 = \frac{\pi S_0}{2\omega_n\zeta} \tag{14}$$

yielding a nonlinear algebraic equation with the unknown quantity ζ. By solving this equation, one obtains

$$\zeta = \frac{1}{\omega_n}\left(q^2 S_0\right)^{1/3} \tag{15}$$

Relation (15) will be used to compare the results of the two analysis methods for the values of $\zeta_i, i = 1,2,...7$ used in table 1. Therefore, in case of classical linearization method, the coefficient q can be expressed as function of its equivalent relative damping value ζ as follows:

$$q_{(1)}(\zeta) = \sqrt{\frac{(\zeta\omega_n)^3}{S_0}} \tag{16}$$

4. Proposed equivalent linearization method

In order to improve the accuracy of equivalent linearization of system with quadratic damping (12) a two steps method is proposed. In the first step it is applied an equivalent non-linearization method consisting in replacing the system with quadratic damping by a system having a special type of nonlinear damping characteristic, for which the exact r.m.s. output can be calculated [15]. The equation of motion of this system is

$$\begin{gathered} \ddot{x} + c\dot{x}\sqrt{\dot{x}^2 + \omega_n^2 x^2} + \omega_n^2 x^2 = z(t) \\ \mathrm{E}[z(t)] = 0\,, \qquad \mathrm{E}\left[z(t)z(t+\tau)\right] = 2\pi S_0\delta(\tau) \end{gathered} \tag{17}$$

As one can see, the nonlinear dissipative characteristic $g(\dot{x}) = c\dot{x}\sqrt{\dot{x}^2 + \omega_n^2 x^2}$ depends explicitly on the system Hamiltonian $H = \frac{1}{2}(\dot{x}^2 + \omega_n^2 x^2)$. The joint probability density function $p(x,\dot{x})$ of the steady state solution of equation (17) can be determined by solving the associated Fokker-Planck equation, which in this case can be written as (see [14], [15]):

$$2c\sqrt{H}\,p(H) + \pi S_0\frac{d\,p(H)}{dH} = 0 \tag{18}$$

The solution of equation (19) can be written as

$$p(x,\dot{x}) = p(H) = \frac{3\omega_n}{2\pi\Gamma(3/2)}\left(\frac{3\pi S_0}{c}\right)^{-2/3} \exp\left[-\frac{c}{3\pi S_0}\left(\dot{x}^2+\omega_n^2 x^2\right)\right] \tag{19}$$

The r.m.s. output of system (17), determined by using the joint probability density function (19), is given by

$$\begin{aligned}
\sigma_x^2 &= \mathrm{E}\left[x^2\right] = \frac{(3\pi)^{2/3}\Gamma(4/3)}{2\Gamma(2/3)\omega_n^2}\left(\frac{S_0}{c}\right)^{2/3} = \frac{1.472}{\omega_n^2}\left(\frac{S_0}{c}\right)^{2/3} \\
\sigma_{\dot{x}}^2 &= \mathrm{E}\left[\dot{x}^2\right] = \frac{(3\pi)^{2/3}\Gamma(4/3)}{2\Gamma(2/3)}\left(\frac{S_0}{c}\right)^{2/3} = 1.472\left(\frac{S_0}{c}\right)^{2/3} \\
\sigma_{\ddot{x}_1}^2 &= \mathrm{E}\left[\ddot{x}_1^2\right] = 7.355 S_0^2\left(\frac{S_0}{c}\right)^{-2/3} + 1.472\omega_n^2\left(\frac{S_0}{c}\right)^{2/3}
\end{aligned} \tag{20}$$

The nonlinear system (17) can be directly replaced by an equivalent linear system (6) having same r.m.s. output. This condition, which is fulfilled if the r.m.s values of relative displacement from (6) and (20) are equal, yields

$$c = \frac{1}{\sqrt{S_0}}\left(1.472\frac{2\omega_n\zeta}{\pi}\right)^{3/2}, \tag{21}$$

Next, the following condition is imposed for nonlinear stochastic equivalence of systems (12) and (17):

$$\mathcal{E}(c) = \mathrm{E}\left[(q|\dot{x}|\dot{x} - c\dot{x}\sqrt{\dot{x}^2+\omega_n^2 x^2})^2\right] = \min, \quad \frac{\partial\mathcal{E}}{\partial \mathrm{c}} = 0 \tag{22}$$

where the expectations are calculated with respect to the joint probability density function (19). This condition leads to the following relation between the equivalent nonlinear damping coefficients c and q:

$$c = \frac{8}{3\pi}q \tag{23}$$

In the second step, a new relation between the coefficient of quadratic damping q and its equivalent relative damping coefficient value ζ is directly obtained from (21) and (23):

$$q_{(2)} = \frac{3\pi}{8}\frac{1}{\sqrt{S_0}}\left(1.472\frac{2\omega_n\zeta}{\pi}\right)^{3/2}. \tag{24}$$

5. Comparison of classical and proposed linearization methods

In what follows, the accuracy of classical and proposed linearization methods is assessed comparatively in terms of the r.m.s. output of system with quadratic damping and of its linear equivalent systems.

The results obtained by numerical integration of the nonlinear system (12) with damping coefficients $q_{(1)}(\zeta_i)$ and $q_{(2)}(\zeta_i)$ given by (16) and (24), respectively, are given in Tables 2 and 3.

Table 2

R.m.s. output of nonlinear system with quadratic damping (first method)

i	ζ_i	$q_{(1)}(\zeta_i)$ [m^{-1}]	$\sigma_x(q_{(1)})$ [m]	$\sigma_{\dot{x}}(q_{(1)})$ [m /s]	$\sigma_{\ddot{x}_1}(q_{(1)})$ [m/s^2]
1	0.1	2.92	0.043	0.271	1.74
2	0.2	8.27	0.0312	0.197	1.32
3	0.3	15.2	0.0255	0.162	1.17
4	0.4	23.4	0.0221	0.140	1.12
5	0.5	32.7	0.0197	0.126	1.11
6	0.6	43	0.0179	0.115	1.12
7	0.7	54.2	0.0166	0.106	1.14

Table 3

R.m.s. output of nonlinear system with quadratic damping (second method)

i	ζ_i	$q_{(2)}(\zeta_i)$ [m^{-1}]	$\sigma_x(q_{(2)})$ [m]	$\sigma_{\dot{x}}(q_{(2)})$ [m /s]	$\sigma_{\ddot{x}_1}(q_{(2)})$ [m/s^2]
1	0.1	3.13	0.0421	0.264	1.69
2	0.2	8.84	0.0304	0.192	1.299
3	0.3	16.24	0.0248	0.158	1.16
4	0.4	25	0.0215	0.137	1.112
5	0.5	35	0.0192	0.122	1.106
6	0.6	45.9	0.0175	0.111	1.121
7	0.7	57.8	0.0162	0.103	1.147

Figures 7-9 present comparatively the r.m.s. output values from Tables 2 and 3 with those from Table1

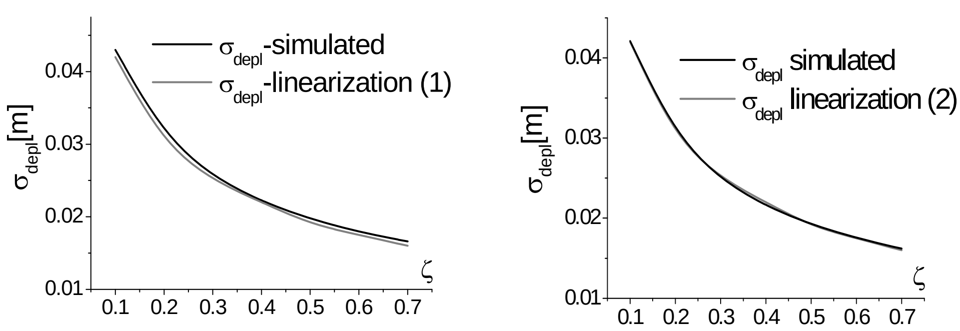


Fig.7. R.m.s. relative displacement

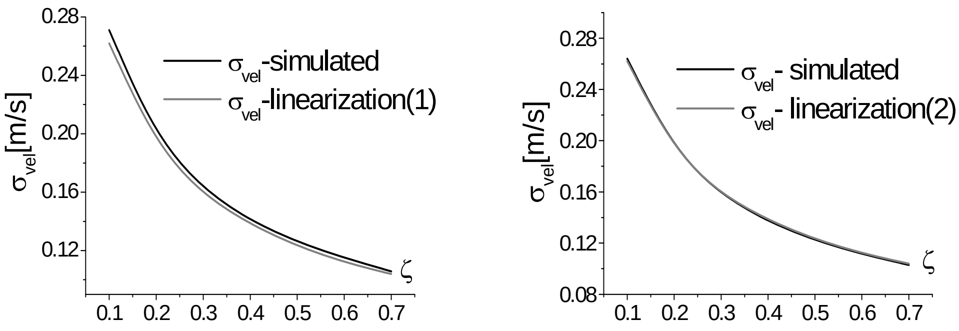


Fig.8. R.m.s. relative velocity

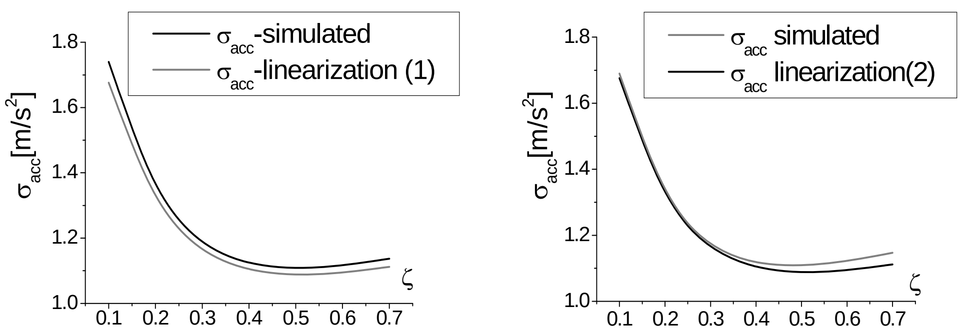


Fig.9. R.m.s. acceleration

Figure 10 shows the variation versus the equivalent linear damping ratio ζ of the relative errors

$$\varepsilon_{\mathrm{d}}^{(k)}(\zeta_i)=\frac{\left|\sigma_x(q_{(k)})-\sigma_x(\zeta_i)\right|}{\sigma_x(q_{(k)})},\quad \varepsilon_{\mathrm{v}}^{(k)}(\zeta_i)=\frac{\left|\sigma_{\dot{x}}(q_{(k)})-\sigma_{\dot{x}}(\zeta_i)\right|}{\sigma_{\dot{x}}(q_{(k)})} \qquad (25)$$

$$\varepsilon_{\mathrm{a}}^{(k)}(\zeta_i)=\frac{\left|\sigma_{\ddot{x}_1}(q_{(k)})-\sigma_{\ddot{x}_1}(\zeta_i)\right|}{\sigma_{\ddot{x}_1}(q_{(k)})},\quad k=1,2 .$$

obtained for the first and second linearization methods, where $\sigma_x(\zeta_i),\sigma_{\dot{x}}(\zeta_i)$ and $\sigma_{\ddot{x}_1}(\zeta_i)$ are calculated using relations (8). As one can see, the relative errors between the mean square response of the system with quadratic and its linear equivalent system is significantly reduced by application of second linearization method, within the range of most relevant damping ratio values ($0.15 \le \zeta \le 0.5$).

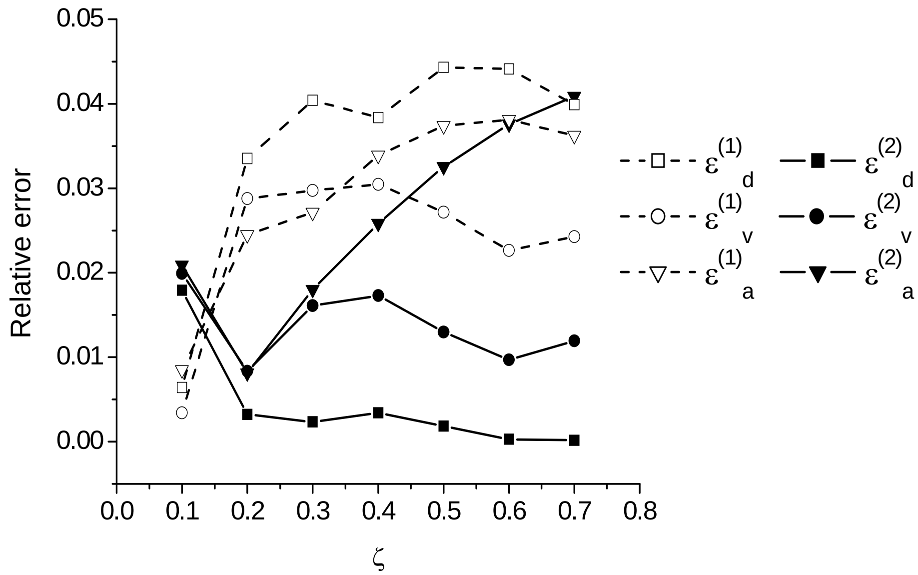
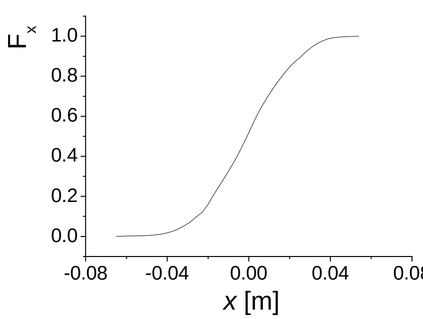
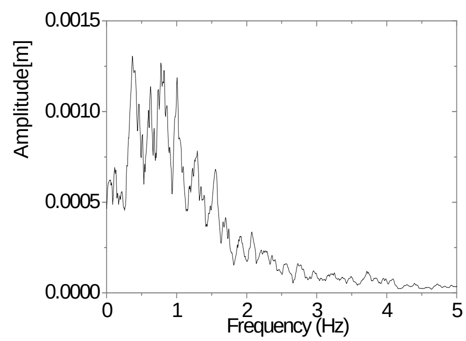


Fig.10. Relative errors of classical and proposed equivalent linearization methods

Figures 11-16 show comparatively the repartition functions and the FFT amplitude spectra of system output samples obtained by numerical simulations for $\zeta = 0.5,\ q_1 = 32.7[\mathrm{m}^{-1}],\ q_2 = 35[\mathrm{m}^{-1}]$ (see tables 2 and 3)

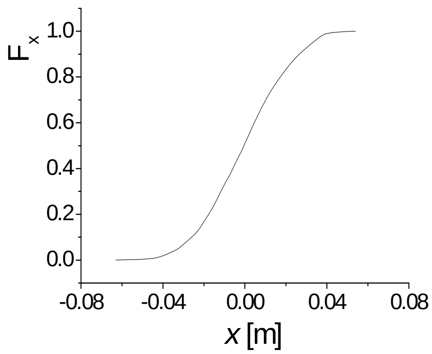


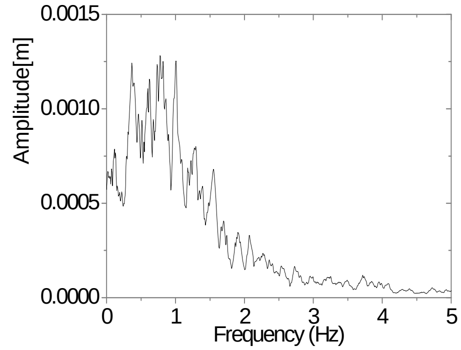


$\zeta_5 = 0.5$

Fig.11. Repartition function of relative displacement

Fig.12. FFT Amplitude spectrum of relative displacement

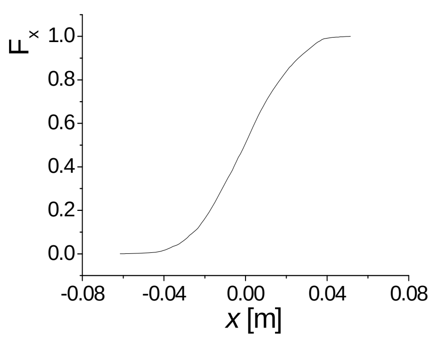


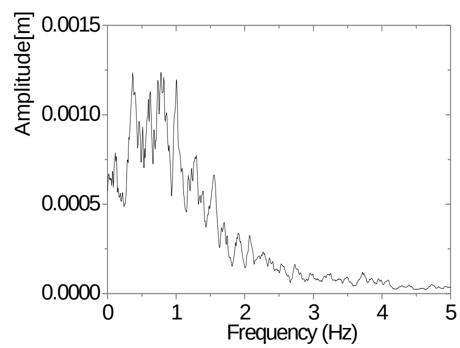


$q_{(1)} = 32.7[\mathrm{m}^{-1}]$

Fig.13. Repartition function of relative displacement

Fig.14. FFT Amplitude spectrum of relative displacement





$q_{(2)} = 35[\mathrm{m}^{-1}]$

Fig.15. Repartition function of relative displacement

Fig.16. FFT Amplitude spectrum of relative displacement

6. Conclusions

An equivalent linearization method with two steps was proposed to evaluate the r.m.s. response of oscillators with quadratic damping excited by white noise random process. The intensity of simulated white noise excitation and the model parameters were chosen as so the system output to be within the usual range of automobile random vibrations. The results can be used for assessing the optimum damping for ride comfort measured by the r.m.s. value of sprung mass acceleration. The classical linearization method was compared with a two step linearization procedure. The results obtained by numerical simulation have shown that the accuracy of proposed method is better than that of classical equivalent linearization method, for all damping coefficient values, relevant for the considered application.

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