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ON VARIOUS REPRESENTATIONS OF A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH SUBORDINATION

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By means of subordination, we introduce a new class $\mathcal{M}(n,\mu,[\phi])$ of analytic functions defined in the open unit disk. We present various representations for this class and point out some of their main consequences. Also, a sufficient class condition for a function f to be in the class $\mathcal{M}(1,\mu,[\phi])$ is studied by using the Briot-Bouquet differential subordination and its representation is considered by choosing an open door function. Some applications of this result (Theorem 3 below) are also considered.

Keywords: Starlike function; close-to-convex functions; subordination; Briot-Bouquet differential subordination
MSC2010: 30C45; 30C50.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ represents a linear space of all analytic functions defined in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For $a\in\mathbb{C}$, $k\in\ \mathbb{N}=\{1,2,\ldots\}$, let

$$\mathcal{H}[a,k]=\left\{f\in\mathcal{H}(\mathbb{U}):f(z)=a+a_kz^k+a_{k+1}z^{k+1}+\ldots\right\}.$$

We denote the subclass $\mathcal{H}[0,1]$ with $a_1=1$ by $\mathcal{A}$. Further, let $\mathcal{S}$ denote a class of functions $f\in\mathcal{A}$ which are univalent and let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ whose members are starlike in $\mathbb{U}$ satisfying the analytic condition that

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>0\ \ (\ z\in\mathbb{U}).$$

A class of close-to-convex functions $f\in\mathcal{A}$ satisfying for some $g\in\mathcal{S}^*$ and some $\alpha\in(-\pi/2,\pi/2)$ the condition that

$$\Re\left\{\frac{zf'(z)}{e^{i\alpha}g(z)}\right\}>0\ \ (z\in\mathbb{U}), \tag{1}$$

is denoted by $\mathcal{C}$, see [1].

For two functions p, $q\in\mathcal{H}(\mathbb{U})$, we say that p is subordinate to q in $\mathbb{U}$ and write $p(z)\prec q(z)$, $z\in\ \mathbb{U}$, if there exists a Schwarz function ω (analytic in $\mathbb{U}$ with $\omega(0)=0$, and $|\omega(z)|\le|z|$, $z\in\ \mathbb{U}$) such that $p(z)=q(\omega(z))$,

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$z \in \mathbb{U}$. Furthermore, if the function q is univalent in $\mathbb{U}$, then we have the following equivalence:

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}).$$

Let $\mathcal{P}$ be a class of functions $\phi(z)$ which are analytic with positive real part in $\mathbb{U}$ satisfying $\phi(0) = 1$.

In terms of the concept of subordination, we define in this paper a new class $\mathcal{M}(n, \mu, [\phi])$ by

$$\mathcal{M}(n, \mu, [\phi]) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} \prec \phi(z^n) \quad (\phi \in \mathcal{P}; z \in \mathbb{U}) \right\},$$

for some $n \in \mathbb{N}, \mu \geq 0$ and for some $g \in \mathcal{S}^*$ with $\frac{f(z)}{z} \neq 0$ in $\mathbb{U}$, where only principal values of the powers are considered.

On the other hand, if $f \in \mathcal{A}$ satisfies

$$\Re \left\{ \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} \right\} > 0 \ \ (z \in \mathbb{U})$$

for some $g \in \mathcal{S}^*$, and $\frac{f(z)}{z} \neq 0$ in $\mathbb{U}$ for some $\beta \in (0, \infty)$, then f is said to be a Bazilevič function of type β ([3], see also [4]) and is denoted by $f \in \mathcal{B}(\beta)$.

We denote the class $\mathcal{M}(n, 1, [\phi])$ by $\mathcal{C}(n, [\phi])$ and the class $\mathcal{M}(n, 0, [\phi])$ by $\mathcal{S}^*(n, [\phi])$. We note that if $\phi \in \mathcal{P}$ is such that $\phi(\mathbb{U})$ is symmetrical with respect to the real axis and starlike with respect to 1, then we have the following obvious relationships:

$$\mathcal{C}(1, [\phi]) = \mathcal{C}[\phi] \quad \text{and} \quad \mathcal{S}^*(1, [\phi]) = \mathcal{S}^*[\phi].$$

The classes $\mathcal{S}^*[\phi]$ and $\mathcal{C}[\phi]$ are, respectively, the Ma-Minda type classes of starlike and close-to-convex functions [2].

It may be noted here that the class $\mathcal{S}^*\left(n, \left[\frac{1+Az}{1+Bz}\right]\right) = \mathcal{S}^*(n, A, B)$ was earlier studied in [5] (see also [6]) for complex A and B $(B \neq 0, |A| \leq 1, |B| \leq 1)$. For real values of A and B, $\mathcal{S}^*\left[\frac{1+Az}{1+Bz}\right] = \mathcal{S}^*(A, B)$ $(-1 \leq B < A \leq 1)$ is the Janowski class of starlike functions studied by many authors (see, for example, [7], [8], [9] and also [10], [11]). The function $\frac{1+Az}{1+Bz}$ was studied by Kuroki *et al.* [12] for complex numbers A and B satisfying one of following conditions:

(i) $|A| \leq 1, |B| < 1, A \neq B, \Re\left(1 - A\overline{B}\right) \geq |A - B|$;
(ii) $|A| \leq 1, |B| = 1, A \neq B, 1 - A\overline{B} > 0$.

Further, for $-1 \leq A \leq 1$, $-1 \leq B \leq 1$ with $A \neq B$, the bilinear transformation $\frac{1+Az}{1+Bz}$ was considered in [13] for additional conditions that (in case $B \neq 0$) B and $B - A$ are of same sign.

Based on the superordinate function $\phi(z^n)$ involved in the definition of the class $\mathcal{M}(n, \mu, [\phi])$, we obtain various representations for this class and consider also some of its consequences. By using the Briot-Bouquet differential subordination, a class $\mathcal{L}(n, \alpha, \mu, [\phi])$ is further defined and a representative of this class is obtained with the use of an open door function.

2. The Class $\mathcal{M}(n, \mu, [\phi])$ and its Consequences

We first prove the following lemma when the superordinate function ϕ has complex coefficients.

Lemma 2.1. *Let ϕ be of the form:*

$$\phi(z) = 1 + \frac{Hz}{1+Bz} \quad (z \in \mathbb{U}), \tag{2}$$

where H and B $(B \neq 0, |B| \leq 1)$ are complex numbers. Then

$$\phi(z^n) \;= \frac{1}{n}\sum_{j=0}^{n-1} \phi_j(z), \quad n \in \mathbb{N}, \tag{3}$$

where

$$\phi_j(z) = 1 + \frac{H_j z}{1+B_j z} \tag{4}$$

$$\left(B_j = -\epsilon_j^{-1}\sqrt[n]{B}, \epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, \frac{B_j H}{B} = H_j, j = 0, 1, \ldots, n-1\right).$$

Proof. Let ϕ be of the form (2), then on writing $B = b^n$ in (2) with

$$\epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, \quad j = 0, 1, \ldots, n-1,$$

we get

$$\begin{aligned}
\phi(z^n) &= 1 + \frac{H}{b^n}\left(1 - \frac{1}{1+(bz)^n}\right) = 1 + \frac{H}{b^n}\left(1 - \frac{1}{(bz-\epsilon_0)(bz-\epsilon_1)\ldots(bz-\epsilon_{n-1})}\right) \\
&= 1 + \frac{H}{b^n}\left(1 - \sum_{j=0}^{n-1}\frac{1}{n\epsilon_j^{n-1}(bz-\epsilon_j)}\right) = 1 + \frac{H}{b^n}\left(1 - \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{1-b\epsilon_j^{-1}z}\right) \\
&= 1 + \frac{H}{nb^n}\sum_{j=0}^{n-1}\left(1 - \frac{1}{1-b\epsilon_j^{-1}z}\right) \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\left[1 + \frac{-b\epsilon_j^{-1}Hz}{b^n}\left(\frac{1}{1-b\epsilon_j^{-1}z}\right)\right] \qquad (5) \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\phi_j(z),
\end{aligned}$$

where, we put $-b\epsilon_j^{-1} = B_j$ and $B_j H/B = H_j$, since $b^n = B$. This proves the result. $\square$

In view of (5), for a given Schwarz function $w(z)$, we find Schwarz functions

$$w_j(z) = -\epsilon_j^{-1} w(z) \left(\epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, j = 0, 1, \ldots, n-1\right)$$

such that

$$\phi(w^n(z)) = \frac{1}{n}\sum_{j=0}^{n-1}\phi_j(w(z)) \;= \frac{1}{n}\sum_{j=0}^{n-1}\Phi(w_j(z)), \tag{6}$$

where

$$\Phi(z) = 1 + \frac{hz}{1+bz} \quad \left(B = b^n, h = \frac{bH}{B}; z \in \mathbb{U}\right). \tag{7}$$

Theorem 2.1. *Let $f \in \mathcal{M}(n, \mu, [\phi])$, then there exists an analytic function $p(z)$:*

$$p(z) \prec \phi(z^n) \quad (z \in \mathbb{U}) \tag{8}$$

such that

$$f(z) = \begin{cases} z\left[\mu \int\limits_0^1 p(xz)G^\mu(xz)x^{\mu-1}\mathrm{d}x\right]^{1/\mu}, & \text{if } \mu > 0, \\ z\exp\int\limits_0^z \frac{p(t)-1}{t}\mathrm{d}t, & \text{if } \mu = 0, \end{cases} \tag{9}$$

where $G(z) = \frac{g(z)}{z}$ *for some* $g \in \mathcal{S}^*$. *Furthermore, if* $f \in \mathcal{A}$ *with* $\frac{f(z)}{z} \neq 0$ *in* $\mathbb{U}$, *is represented by* (9) *for some analytic function* $p(z)$ *satisfying* (8) *and for some* $G(z) = \frac{g(z)}{z}$, *where* $g \in \mathcal{S}^*$, *then* $f \in \mathcal{M}(n, \mu, [\phi])$.

Proof. Let $f \in \mathcal{M}(n, \mu, [\phi])$, then

$$\frac{zf'(z)}{f^{1-\mu}(z)g^\mu(z)} \prec \phi(z^n) \ \ (z \in \mathbb{U})$$

for some $g \in \mathcal{S}^*$. Let

$$p(z) = \frac{zf'(z)}{f^{1-\mu}(z)g^\mu(z)}, \tag{10}$$

then $p(z)$ is an analytic function satisfying (8) and

$$(f(z)/z)^{\mu-1}f'(z) = p(z)(g(z)/z)^\mu. \tag{11}$$

Hence, on integrating (11), we get

$$f^\mu(z) - z^\mu = \mu \int\limits_0^z \frac{p(t)g^\mu(t) - t^\mu}{t}\mathrm{d}t, \text{ if } \mu > 0$$

and

$$\log\frac{f(z)}{z} = \int\limits_0^z \frac{p(t)-1}{t}\mathrm{d}t, \text{ if } \mu = 0,$$

which on simplification gives the representation (9). Furthermore, if $f(z)$ is represented by (9), then from it (for both the cases $\mu = 0, \mu > 0$), we get (10). Hence, from (8), we have

$$\frac{zf'(z)}{f^{1-\mu}(z)g^\mu(z)} \prec \phi(z^n) \ \ (z \in \mathbb{U}),$$

which evidently implies that $f \in \mathcal{M}(n, \mu, [\phi])$. □

If we choose $\mu = 1$ in (10) of Theorem 2.1, then we get the following result for the class $\mathcal{C}(n, [\phi])$.

Corollary 2.1. *Let* $\phi \in \mathcal{P}$ *and* $f \in \mathcal{C}(n, [\phi])$. *Then there exists an analytic function* $s(z)$ *satisfying the condition that*

$$s(z) \prec \phi(z^n) \ \ (z \in \mathbb{U}), \tag{12}$$

where $f(z)$ *is given by*

$$f(z) = \int\limits_0^z \frac{g(t)s(t)}{t}\mathrm{d}t \tag{13}$$

for some $g \in \mathcal{S}^*$. *Furthermore, if an analytic function* $s(z)$ *satisfies* (12), *then* $f(z)$ *represented by* (13) *(with* $\frac{f(z)}{z} \neq 0$ *in* $\mathbb{U}$*) belongs to the class* $\mathcal{C}(n, [\phi])$.

Theorem 2.2. *If $f \in \mathcal{M}(n, \mu, [\phi])$, where $\phi(z)$ is of the form (2), then there exist functions $f_j \in \mathcal{M}(1, \mu, [\Phi])$, $j = 0, 1, \ldots, n-1$ for which $\Phi(z)$ is given by (7) and*

$$f_j(z) = \begin{cases} z\left[\mu \int\limits_0^1 \Phi(w_j(xz)) G^\mu(xz) \mathrm{d}x\right]^{1/\mu}, & \text{if } \mu > 0, \\ z \exp \int\limits_0^z \frac{\Phi(w_j(t)) - 1}{t} \mathrm{d}t, & \text{if } \mu = 0, \end{cases} \tag{14}$$

where $G(z) = \frac{g(z)}{z}$ for some $g \in \mathcal{S}^*$ and

$$w_j(z) = -\epsilon_j^{-1} w(z) \left(\epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, j = 0, 1, \ldots, n-1\right)$$

for some Schwarz function $w(z)$. Moreover,

$$f(z) = \begin{cases} \left(\frac{1}{n} \sum_{j=0}^{n-1} f_j^\mu(z)\right)^{1/\mu}, & \text{if } \mu > 0, \\ \left(\prod\limits_{j=0}^{n-1} f_j(z)\right)^{1/n}, & \text{if } \mu = 0. \end{cases} \tag{15}$$

Proof. If $f \in \mathcal{M}(n, \mu, [\phi])$ such that $\phi(z)$ is of the form (2), then for a Schwarz function $w(z)$, we find the Schwarz functions $w_j(z)$ of the form:

$$w_j(z) = -\epsilon_j^{-1} w(z)\, \epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, j = 0, 1, \ldots, n-1.$$

Thus from (6), we get

$$p(z) = \frac{z f'(z)}{f^{1-\mu}(z) g^\mu(z)} = \frac{1}{n} \sum_{j=0}^{n-1} \Phi(w_j(z)),$$

where $\Phi(z)$ is given by (7). Hence, on replacing $p(z)$ by $\frac{1}{n} \sum_{j=0}^{n-1} \Phi(w_j(z))$ in the representation (9) (for both the cases), we get (15), where $f_j(z)$ is represented by (14) with $\frac{f_j(z)}{z} \neq 0$ $(j = 0, 1, \ldots, n-1)$ in $\mathbb{U}$ and belongs to the class $\mathcal{M}(1, \mu, [\Phi])$ for $\Phi(z)$ given by (7). This proves Theorem 2.2. □

Setting $\mu = 1$ in Theorem 2.2, we get the following result for the class $\mathcal{C}(n, [\phi])$.

Corollary 2.2. *If $f \in \mathcal{C}(n, [\phi])$, where $\phi(z)$ is of the form (2), then there exist functions $F_j \in \mathcal{C}[\Phi]$, $j = 0, 1, \ldots, n-1$, and*

$$f(z) = \frac{1}{n} \sum_{j=0}^{n-1} F_j(z), \tag{16}$$

where for a given Schwarz function $w(z)$, we find Schwarz functions

$$w_j(z) = -\epsilon_j^{-1} w(z) \left(\epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, j = 0, 1, \ldots, n-1\right)$$

such that

$$F_j(z) = \int\limits_0^z \frac{g(t) \Phi(w_j(t))}{t} \mathrm{d}t \tag{17}$$

and $\Phi(z)$ is given by (7).

For a complex number A $(\neq B, |A| \leq 1)$ and $H = A - B$ in (2), we get $\phi(z) = \frac{1+Az}{1+Bz}$ $(z \in \mathbb{U})$ and the class $\mathcal{M}(n, \mu, [\phi])$ then reduces to the class $\mathcal{M}(n, \mu, A, B)$. We thus get the following results (Corollaries 2.3 and 2.4) as worthwhile consequences from the results (3), (6) and (15) .

Corollary 2.3. *Let A and B $(B \neq 0)$ be complex numbers such that $|A| \leq 1, |B| \leq 1$. Then*

$$\frac{1 + Az^n}{1 + Bz^n} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1 + A_j z}{1 + B_j z}$$

$$\left(B_j = -\epsilon_j^{-1} \sqrt[n]{B}, \epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, A_j = \frac{B_j A}{B}, j = 0, 1, \ldots, n-1 \right)$$

and hence, for a given Schwarz function $w(z)$, we find Schwarz functions

$$w_j(z) = -\epsilon_j^{-1} w(z) \left(\epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, j = 0, 1, \ldots, n-1 \right)$$

such that

$$\frac{1 + Aw^n(z)}{1 + Bw^n(z)} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1 + aw_j(z)}{1 + bw_j(z)} \quad \left(b = \sqrt[n]{B}, a = \frac{bA}{B}; z \in \mathbb{U} \right).$$

Corollary 2.4. *If $f \in \mathcal{M}(n, \mu, A, B)$, then there exist functions $g_j \in \mathcal{M}(1, \mu, a, b)$, $j = 0, 1, \ldots, n-1$, and*

$$f(z) = \begin{cases} \left(\frac{1}{n} \sum_{j=0}^{n-1} g_j^{\mu}(z) \right)^{1/\mu}, & \text{if } \mu > 0, \\ \left(\prod_{j=0}^{n-1} g_j(z) \right)^{1/n}, & \text{if } \mu = 0, \end{cases} \tag{18}$$

where for a given Schwarz function $w(z)$, we find Schwarz functions

$$w_j(z) = -\epsilon_j^{-1} w(z) \left(\epsilon_j = e^{\frac{(2j+1)\pi i}{n}}, j = 0, 1, \ldots, n-1 \right)$$

such that

$$g_j(z) = \begin{cases} z \left[\mu \int_0^1 \frac{1+aw_j(xz)}{1+bw_j(xz)} G^{\mu}(xz) x^{\mu-1} \mathrm{d}x \right]^{1/\mu}, & \text{if } \mu > 0, \\ z \exp \int_0^z \left(\frac{1+aw_j(t)}{1+bw_j(t)} - 1 \right) \frac{\mathrm{d}t}{t}, & \text{if } \mu = 0, \end{cases} \tag{19}$$

where $G(z) = \frac{g(z)}{z}$ for some $g \in \mathcal{S}^*$ and $b = \sqrt[n]{B}, a = bA/B, z \in \mathbb{U}$.

3. The Classes $\mathcal{L}(n, \alpha, \mu, [\phi])$ and $\mathcal{M}(1, \mu, [\phi])$

In this section, we obtain a sufficiency condition for a function f to be in the class $\mathcal{M}(1, \mu, [\phi])$ by using its representation and by adopting the methods of the Briot-Bouquet differential subordination. For this purpose, we define a class $\mathcal{L}(n, \alpha, \mu, [\phi])$ of functions $f \in \mathcal{A}$ satisfying the condition that

$$\frac{zf'(z)}{f^{1-\mu}(z) g^{\mu}(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - (1-\mu) \frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)} \right) \prec \phi(z^n), \tag{20}$$

for some $n \in \mathbb{N}, \mu \geq 0$ and for some $g \in \mathcal{S}^*$ with $f'(z)\frac{f(z)}{z} \neq 0$ in $\mathbb{U}$, where only principal values of the powers are considered and $\phi \in \mathcal{P}$ is of the form

$$\phi(z) = 1 + \frac{Hz}{1+Bz} \quad (-1 \leq B < 1,\ 0 < H \leq 1 - B)\,. \tag{21}$$

Obviously, for $\phi(z)$ given by (21), we have

$$\phi(z^n) \ \prec \phi(z) \quad (-1 \leq B < 1, 0 < H \leq 1 - B, n \in \mathbb{N}; z \in \mathbb{U})\,, \tag{22}$$

and we also note that $\mathcal{L}\,(n, \alpha, \mu, [\phi]) \subset \mathcal{L}\,(1, \alpha, \mu, [\phi])\,.$ We need the following results which are special cases of the results mentioned in [14, Th. 3.2a, p.81 and Th. 3.2d, p. 86].

Lemma 3.1. *Let h be convex in $\mathbb{U}$ with $\Re\,(\beta h(z)) > 0$. If q is analytic in $\mathbb{U}$ with $q(0) = h(0)$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z)} \prec h(z) \Rightarrow q(z) \prec h(z).$$

Lemma 3.2. *Let h be analytic in $\mathbb{U}$ with $h(0) = 1$, and let $\alpha > 0$ be finite. If*

$$h(z) \prec \alpha \mathcal{R}_{1/\alpha,1}(z) = \frac{1+z}{1-z} + \frac{2\alpha z}{1-z^2}, \tag{23}$$

then the solution q of the Briot-Bouquet differential equation:

$$q(z) + \frac{\alpha z q'(z)}{q(z)} = h(z) \ (z \in \mathbb{U}) \tag{24}$$

with $q(0) = 1$ is analytic in $\mathbb{U}$ such that $\Re\,(q(z)) > 0$, and is given by

$$q(z) = \frac{\alpha F^{1/\alpha}(z)}{\int\limits_0^z \frac{F^{1/\alpha}(t)}{t}\mathrm{d}t}, \tag{25}$$

where

$$F(z) = z \exp \int\limits_0^z \frac{h(t)-1}{t}\mathrm{d}t.$$

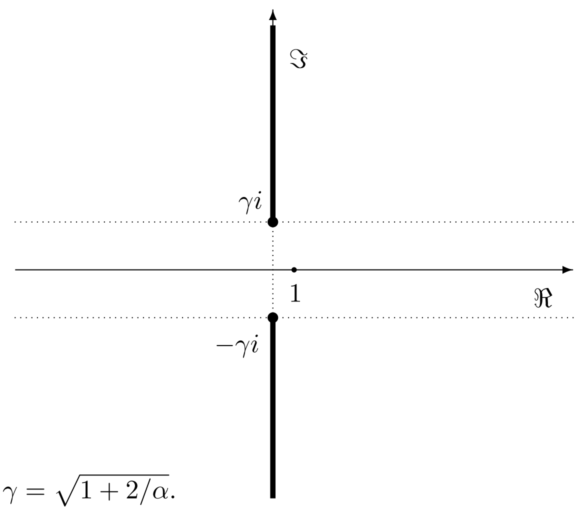

Fig.1. $\mathcal{R}_{1/\alpha,1}(\mathbb{U}),\ \ \gamma = \sqrt{1+2/\alpha}.$

Theorem 3.1. *If $f \in \mathcal{L}(n, \alpha, \mu, [\phi])$, then $f \in \mathcal{M}(1, \mu, [\phi])$. Furthermore, $f(z)$ is given by the representation* (9) *with $\frac{f(z)}{z} \neq 0$ in $\mathbb{U}$, where for some Schwarz function $w(z)$:*

$$p(z) = \frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} = \frac{\alpha z^{1/\alpha} \exp \frac{1}{\alpha} \int\limits_0^z (\phi(w^n(t) - 1) \frac{\mathrm{d}t}{t}}{\int\limits_0^z t^{\frac{1}{\alpha}-1} \exp\left(\frac{1}{\alpha} \int\limits_0^t (\phi(w^n(u) - 1) \frac{du}{u}\right) \mathrm{d}t} \quad (z \in \mathbb{U}). \tag{26}$$

Proof. Let $f \in \mathcal{L}(n, \alpha, \mu, [\phi])$ and let $p(z)$ be defined by (10). Then the condition (20) becomes

$$p(z) + \frac{\alpha z p'(z)}{p(z)} \prec \phi(z^n).$$

By putting

$$p(z) + \frac{\alpha z p'(z)}{p(z)} = h(z) \ \ (z \in \mathbb{U}), \tag{27}$$

and using the class conditions (20) and (22), we get

$$h(z) \prec \phi(z^n) \ \prec \phi(z) \ \ (z \in \mathbb{U}), \tag{28}$$

where $\phi(z)$ is given by (21), which by Lemma 3.1 implies that $p(z) \prec \phi(z), z \in \mathbb{U}$ and hence, $f \in \mathcal{M}(1, \mu, [\phi])$. Also, from (28), we have for some Schwarz function $w(z)$ that

$$\Re\{h(z)\} = \Re\{\phi(w^n(z))\} > 0 \ \ (z \in \mathbb{U}). \tag{29}$$

Since $\mathcal{R}_{\beta,n}(z) = \beta\frac{1+z}{1-z} + \frac{2nz}{1-z^2}, z \in \mathbb{U}$ is the open door function and is univalent in $\mathbb{U}$ for every $\beta > 0$ and $n \in \mathbb{N}$, hence the set $\mathcal{R}_{\beta,n}(\mathbb{U})$ is the complex plane with slits along the half-lines $\Re\{w\} = 0$ and $|\Im\{w\}| \geq \gamma = n\sqrt{1 + 2\beta/n}$; (see Fig. 1 above). We note that the condition (29) verifies the subordinate condition $h(z) \prec \alpha\mathcal{R}_{1/\alpha,1}(z), z \in \mathbb{U}$ of Lemma 3.2, since $h(\mathbb{U}) \subset \alpha\mathcal{R}_{1/\alpha,1}(\mathbb{U})$. Moreover, $h(\mathbb{U})$ is contained in a circle in the right half-plane when $B \neq -1$. Again, if $h(z)$ satisfies the Briot-Bouquet differential equation (27), then by Lemma 3.2, $p(z)$ is the solution of (27) with $\Re\{p(z)\} > 0$ and is given by (26). Therefore for this $p(z)$, the representation of $f(z)$ is given by (9) for the cases when $\mu > 0$ and when $\mu = 0$. This proves Theorem 3.1. □

Before concluding this paper, we deem it worthwhile to consider some applications of the result Theorem 3.1. If we set $\phi(z) = \frac{1+z}{1-z}$ $(z \in \mathbb{U})$ and choose $\mu = \alpha = 1$ in Theorem 3.1, we then arrive at the following consequences of Theorem 3.1 by specializing, respectively, the function $g(z)$ as $g(z) = \frac{z}{1-z}$ and $g(z) = z$ (which are starlike in $\mathbb{U}$).

Corollary 3.1. *If $f \in \mathcal{A}$ satisfies the condition that*

$$(1 - z) f'(z) + \frac{zf''(z)}{f'(z)} - \frac{z}{1-z} \prec \frac{1 + z^n}{1 - z^n} \quad (n \in \mathbb{N}; z \in \mathbb{U}),$$

then

$$(1 - z) f'(z) \prec \frac{1 + z^n}{1 - z^n} \ \prec \frac{1 + z}{1 - z} \ \ (n \in \mathbb{N}; z \in \mathbb{U}) \tag{30}$$

and the function f satisfying (30) *is of the form:*

$$f(z) = \int\limits_0^z \frac{t}{(1 - t)(1 - t^n)^{2/n} I_{n,t}} \mathrm{d}t,$$

where

$$I_{n,t} = \int_0^t \frac{\mathrm{d}u}{(1-u^n)^{2/n}}. \tag{31}$$

In particular, if $n = 1$, *then* $f(z) = \frac{z}{1-z}$.

Corollary 3.2. *If* $f \in \mathcal{A}$ *satisfies the condition that*

$$f'(z) + \frac{zf''(z)}{f'(z)} \prec \frac{1+z^n}{1-z^n} \quad (n \in \mathbb{N}; z \in \mathbb{U}),$$

then

$$f'(z) \prec \frac{1+z^n}{1-z^n} \prec \frac{1+z}{1-z} \quad (n \in \mathbb{N}; z \in \mathbb{U}) \tag{32}$$

and the function f *satisfying* (32) *is of the form:*

$$f(z) = \int_0^z \frac{t}{(1-t^n)^{2/n} I_{n,t}} \mathrm{d}t,$$

where $I_{n,t}$ *is given by* (31) *and in particular, if* $n = 1$, *then* $f(z) = \log \frac{1}{1-z}$ $(z \in \mathbb{U})$.

Lastly, if we set $\phi(z) = \frac{1+z}{1-z}$ $(z \in \mathbb{U})$ and choose $\mu = 0$ in Theorem 3.1, then we get the following obvious result showing that every convex function is starlike in $\mathbb{U}$.

Corollary 3.3. *If* $f \in \mathcal{A}$ *satisfies the condition that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z^n}{1-z^n} \quad (n \in \mathbb{N}; z \in \mathbb{U}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z^n}{1-z^n} \prec \frac{1+z}{1-z} \quad (n \in \mathbb{N}; z \in \mathbb{U}) \tag{33}$$

and the function f *satisfying* (15) *is of the form:*

$$f(z) = z \exp\left\{ \int_0^z \left(\frac{t}{(1-t^n)^{2/n} I_{n,t}} - 1 \right) \frac{\mathrm{d}t}{t} \right\},$$

where $I_{n,t}$ *is given by* (31) *and in particular, if* $n = 1$, *then* $f(z) = \frac{z}{1-z}$.

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