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ON APPROXIMATION OF BASKAKOV-DURRMEYER TYPE OPERATORS OF TWO VARIABLES

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In this study, we have constructed a sequence of positive linear operators with two variables by using Baskakov-Durrmeyer type operators. We study approximation these operators and give a Voronovskaja type theorem. Furthermore, we study of the linear positive operators in a weighted space of functions of two variables and find the rate of these convergence using weighted modulus of continuity.

Keywords: Baskakov-Durrmeyer operators, weighted approximation, rates of approximation.

MSC2010: 41A25, 41A36.

1. Introduction

The well known Baskakov-Durrmeyer type operators of one variable are defined as

$$M_n(f;x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\,dt$$

and

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

The approximation of these operators were studied in [10, 11, 15]. We generalize these operators to the operators of two variables. Now we give necessary notations and definitions. Let

$$D = \left\{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1, 0 \le x_2 \right\},$$

$C(D)$ be the set of real valued continuous functions on D and

$$D_a = \left\{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le a, 0 \le x_2 \le a \right\}.$$

Let us define two variables of Baskakov-Durrmeyer operators M_n defined on subset of all continuous functions on D for which the following double integral exists finitely.

$$M_n(f;(x_1,x_2)) = (n-1)^2\sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2)\int_0^{\infty}\int_0^{\infty} b_{n,k,l}(y_1,y_2)f(y_1,y_2)\,dy_1dy_2, \quad (1)$$

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where

$$b_{n,k,l}(y_1, y_2) = p_{n,k}(y_1)p_{n,l}(y_2).$$

Approximation functions one or two variables by certain positive linear operators in weighted spaces can be found in [2, 4, 5, 6, 7, 8, 9, 13, 14, 16, 17, 18, 19]. We need Bohman-Korovkin's Theorem in [1, 3, 12] for giving our Remark 2.1.

Theorem 1.1 (Bohman-Korovkin)**.** *Let K be a compact Hausdorff topological space which contains at least two distinct points and let $2m$ functions $f_1, f_2, ..., f_m$, $a_1, a_2, ..., a_m$ in $C(K, \mathbb{R})$ be such that*

$$\begin{aligned} P(x_1, x_2) \quad &: \quad = \sum_{j=1}^{m} f_j(x_1)a_j(x_2) \geq 0, \ \forall (x_1, x_2) \in K, \\ P(x_1, x_2) \quad &= \quad 0 \text{ if and only if } x_1 = x_2. \end{aligned}$$

If (H_n), $H_n : C(K, \mathbb{R}) \to C(K, \mathbb{R})$, is a sequence of linear positive operators such that

$$H_n(f_j) \to f_j, \ n \to \infty, \ j = 1, ..., m,$$

then we have $H_n(f) \to f$, $f \in C(K, \mathbb{R})$.

2. Generalization of Baskakov-Durrmeyer type operators

In this section, we give some classical approximation properties of these operators. We use the following notations, for brevity,

$$\begin{aligned} e_{00}(x_1, x_2) &= 1, e_{10}(x_1, x_2) = x_1, e_{01}(x_1, x_2) = x_2, e_{11}(x_1, x_2) = x_1 + x_2, \\ e_{20}(x_1, x_2) &= x_1^2, e_{02}(x_1, x_2) = x_2^2, e_{22}(x_1, x_2) = x_1^2 + x_2^2, \\ e_{30}(x_1, x_2) &= x_1^3, e_{03}(x_1, x_2) = x_2^3, e_{40}(x_1, x_2) = x_1^4, e_{04}(x_1, x_2) = x_2^4 \\ \text{and } F_s(n) &= \prod_{i=2}^{s} (n - i). \end{aligned}$$

We are ready to give some properties of the Baskakov-Durrmeyer operators of two variables given in (1) which we apply to the proofs of the main results. By simple calculations, we get the following lemma;

Lemma 2.1. *The following statements hold;*

$$
\begin{aligned}
(i)\ M_n(e_{00};(x_1,x_2)) &= 1,\ (ii)\ M_n(e_{10};x)=\frac{n}{n-2}x_1+\frac{1}{n-2},\\
(iii)\ M_n(e_{01};(x_1,x_2)) &= \frac{n}{n-2}x_2+\frac{1}{n-2},\\
(iv)\ M_n(e_{20};(x_1,x_2)) &= \frac{1}{F_3(n)}\left\{n(n+1)x_1^2+4nx_1+2\right\},\\
(v)\ M_n(e_{02};(x_1,x_2)) &= \frac{1}{F_3(n)}\left\{n(n+1)x_2^2+4nx_2+2\right\},\\
(vi)\ M_n(e_{22};(x_1,x_2)) &= \frac{1}{F_3(n)}\left\{n(n+1)\left(x_1^2+x_2^2\right)+4n\left(x_1+x_2\right)+4\right\},\\
(vii)\ M_n(e_{30};(x_1,x_2)) &= \frac{1}{F_4(n)}\left\{n(n+1)(n+2)x_1^3+9n(n+1)x_1^2+18nx_1+6\right\},\\
(viii)\ M_n(e_{03};(x_1,x_2)) &= \frac{1}{F_4(n)}\left\{n(n+1)(n+2)x_2^3+9n(n+1)x_2^2+18nx_2+6\right\},\\
(ix)\ M_n(e_{40};(x_1,x_2)) &= \frac{1}{F_5(n)}\left\{n(n+1)(n+2)(n+3)x_1^4+16n(n+1)(n+2)x_1^3\right.\\
&\quad \left.+72n(n+1)x_1^2+96nx_1+24\right\},\\
(x)\ M_n(e_{04};(x_1,x_2)) &= \frac{1}{F_5(n)}\left\{n(n+1)(n+2)(n+3)x_2^4+16n(n+1)(n+2)x_2^3\right.\\
&\quad \left.+72n(n+1)x_2^2+96nx_2+24\right\}.
\end{aligned}
$$

Proof. We obtain the estimate

$$
\int_0^\infty p_{n,k}(y_1)(e_{10}(y_1,y_2))^m dy_1=\frac{(k+m)!(n-m-2)!}{k!(n-1)!}. \tag{2}
$$

Then using (2) we get

$$
\begin{aligned}
M_n(e_{00};(x_1,x_2)) &= (n-1)^2\sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2)\left\{\int_0^\infty \begin{pmatrix} n+k-1\\ k\end{pmatrix}\frac{y_1^k}{(1+y_1)^{n+k}}dy_1\right.\\
&\quad\left.\times\int_0^\infty \begin{pmatrix} n+l-1\\ l\end{pmatrix}\frac{y_2^l}{(1+y_2)^{n+l}}dy_2\right\}=1,
\end{aligned}
$$

$$
\begin{aligned}
M_n(e_{10};(x_1,x_2)) &= (n-1)^2\sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2)\left\{\int_0^\infty \begin{pmatrix} n+k-1\\ k\end{pmatrix}\frac{y_1^{k+1}}{(1+y_1)^{n+k}}dy_1\right.\\
&\quad\left.\times\int_0^\infty \begin{pmatrix} n+l-1\\ l\end{pmatrix}\frac{y_2^l}{(1+y_2)^{n+l}}dy_2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= (n-1)^2 \sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2)\frac{k+1}{(n-1)^2(n-2)} \\
&= \frac{1}{n-2}\left\{\sum_{k=1}^{\infty}\frac{(n+k-1)!}{(k-1)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}+1\right\} \\
&= \frac{n}{n-2}x_1+\frac{1}{n-2},
\end{aligned}
$$

$$
\begin{aligned}
M_n(e_{01};(x_1,x_2)) &= (n-1)^2\sum_{l=0}^{\infty} b_{n,k,l}(x_1,x_2)\frac{l+1}{(n-1)^2(n-2)} \\
&= \frac{n}{n-2}x_2+\frac{1}{n-2},
\end{aligned}
$$

$$
\begin{aligned}
M_n(e_{20};(x_1,x_2)) &= \frac{1}{F_3(n)}\sum_{k=0}^{\infty} b_{n,k,l}(x_1,x_2)(k+1)(k+2) \\
&= \frac{1}{F_3(n)}\left\{\sum_{k=2}^{\infty}\frac{(n+k-1)!}{(k-2)!\,(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}\right. \\
&\quad \left. +4\sum_{k=1}^{\infty}\frac{(n+k-1)!}{(k-1)!\,(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}+2\right\} \\
&= \frac{1}{F_3(n)}\left\{n(n+1)x_1^2+4nx_1+2\right\},
\end{aligned}
$$

$$
M_n(e_{02};(x_1,x_2)) = \frac{1}{F_3(n)}\left\{n(n+1)x_2^2+4nx_2+2\right\},
$$

summing of (iv) and (v) we get (vi).

$$
\begin{aligned}
M_n(e_{30};(x_1,x_2)) &= \frac{1}{F_4(n)}\sum_{k,l\in\mathbb{N}} b_{n,k,l}(x)\left\{k(k-1)(k-2)\right. \\
&\quad \left. +9k(k-1)+18k+6\right\} \\
&= \frac{1}{F_4(n)}\left\{\sum_{k=3}^{\infty}\frac{(n+k-1)!}{(k-3)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}\right. \\
&\quad +9\sum_{k=2}^{\infty}\frac{(n+k-1)!}{(k-2)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}} \\
&\quad \left. +18\sum_{k=1}^{\infty}\frac{(n+k-1)!}{(k-1)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}+6\right\} \\
&= \frac{1}{F_4(n)}\left\{n(n+1)(n+2)x_1^3+9n(n+1)x_1^2+18nx_1+6\right\},
\end{aligned}
$$

using the same processes we have the equality in $(viii)$.

$$\begin{aligned} M_n(e_{40};(x_1,x_2)) &= \frac{1}{F_5(n)}\sum_{k,l\in\mathbb{N}} b_{n,k,l}(x)\left\{k(k-1)(k-2)(k-3)\right.\\ &\left.+16k(k-1)(k-2)+72k(k-1)+96k+24\right\}\\ &= \frac{1}{F_5(n)}\left\{\sum_{k=4}^{\infty}\frac{(n+k-1)!}{(k-4)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}\right.\\ &+16\sum_{k=3}^{\infty}\frac{(n+k-1)!}{(k-3)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}\\ &+72\sum_{k=2}^{\infty}\frac{(n+k-1)!}{(k-2)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}\\ &\left.+96\sum_{k=1}^{\infty}\frac{(n+k-1)!}{(k-1)!(n-1)!}\frac{x_1^k}{(1+x_1)^{n+k}}+24\right\}, \end{aligned}$$

again using the same processes we obtain the equality in (x).

□

Let's define the weighted space $C_\rho^*(D)$ consisting of all functions f in $C_\rho(D)$ for which $\lim\limits_{|(y_1,y_2)|\to\infty}\frac{|f(y_1,y_2)|}{1+|(y_1,y_2)|^2}$ exists finitely, where $|(y_1,y_2)| = \sqrt{y_1^2+y_2^2}$ and $|f(y_1,y_2)|\leq K_f(1+y_1^2+y_2^2)$, $K_f>0$.

The following theorem gives us the Baskakov type theorem (see [4]) to get the uniform approximation to the functions in $C_\rho^*(D)$ by the sequence of the positive linear operators M_n.

Theorem 2.1. *Let f be belong to $C_\rho^*(D)$. Then*

$$\lim_n \|M_n(f)-f\|_{C_\rho^*(D)} = 0$$

if and only if the following statements are satisfied

$$(i)\lim_n \|M_n(e_{00})-e_{00}\|_{C_\rho^*(D)} = 0$$

$$(ii)\lim_n \|M_n(e_{10})-e_{10}\|_{C_\rho^*(D)} = 0$$

$$(iii)\lim_n \|M_n(e_{01})-e_{01}\|_{C_\rho^*(D)} = 0$$

$$(iv)\lim_n \|M_n(e_{20}+e_{02})-(e_{20}+e_{02})\|_{C_\rho^*(D)} = 0.$$

Proof. The necessity part is trivial. The sufficient part needs proof: Let $(x_1,x_2)\in D_a$ and $f\in C_\rho^*(D)$. Since for each $f\in C_\rho^*(D)$ is uniform continuous on D_r, $(r>a)$, for each $\varepsilon>0$ there exists some $\delta>0$, such that for each $(y_1,y_2)\in D_r$ with $\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}<\delta$ implies $|f(x_1,x_2)-f(y_1,y_2)|<\varepsilon$. On the other hand, if $\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\geq\delta$, we have

$$|f(x_1,x_2)-f(y_1,y_2)|\leq 2K_f\frac{(x_1-y_1)^2+(x_2-y_2)^2}{\delta^2},\quad \text{for}\quad (y_1,y_2)\in D.$$

So, we get the following inequality

$$|f(x_1,x_2)-f(y_1,y_2)|\le\varepsilon+2K_f\frac{(x_1-y_1)^2+(x_2-y_2)^2}{\delta^2}. \tag{3}$$

Using positivity of the operators M_n for each n and the equality

$$f(y_1,y_2)=(f-f(x_1,x_2)e_{00})(y_1,y_2)+f(x_1,x_2),$$

we obtain the following inequality

$$\begin{aligned} & |M_n(f;(x_1,x_2))-f(x_1,x_2)| \\ = & |M_n(f-f(x_1,x_2)e_{00}+f(x_1,x_2)e_{00};(x_1,x_2))-f(x_1,x_2)| \\ = & |M_n(f-f(x_1,x_2)e_{00};(x_1,x_2))+(f(x_1,x_2)(M_n(e_{00};(x_1,x_2))-1)| \\ \le & M_n(|f-f(x_1,x_2)e_{00}|;x)+|f(x_1,x_2)|\,|M_n(e_{00};(x_1,x_2))-1)| \\ \le & M_n(|f-f(x_1,x_2)e_{00}|;x)+\|f\|_{C^*_\rho(D)}\,|M_n(e_{00};(x_1,x_2))-1)|. \end{aligned} \tag{4}$$

Now applying the operators M_n to (3) and using (4) we get

$$\begin{aligned} |M_n(f;(x_1,x_2))-f(x_1,x_2)| &\le \varepsilon+\frac{2K_f}{\delta^2}\left\{M_n\left((y_1-x_1)^2;x\right)\right. \\ &\left.+M_n\left((y_2-x_2)^2;x\right)\right\}+\|f\|_{C^*_\rho(D)}\,|M_n(e_{00};(x_1,x_2))-e_{00}(x_1,x_2)| \\ &=\varepsilon+\frac{2K_f}{\delta^2}\left\{\frac{1}{F_3(n)}\left\{n(n+1)\left(x_1^2+x_2^2\right)+4n\left(x_1+x_2\right)+4\right\}\right. \\ &\left.-\frac{2n}{n-2}\left(x_1^2+x_2^2\right)-\frac{2}{n-2}\left(x_1+x_2\right)+x_1^2+x_2^2\right\} \\ &+\|f\|_{C^*_\rho(D)}\,|M_n(e_{00};(x_1,x_2))-e_{00}(x_1,x_2)|, \end{aligned}$$

hence the proof of Theorem is finished. □

Remark 2.1. *If we choose $f\in C(D_a)$ instead of $f\in C^*_\rho(D)$ in the above theorem, then we obtain alternative proof of the Theorem 2.1 by means of Bohman-Korovkin's result, Indeed, take $K=D_a$, $f_1=e_{00}$, $f_2=e_{10}$, $f_3=e_{01}$, $f_4=e_{22}, a_1=e_{22}, a_2=-2e_{10}, a_3=-2e_{01}$ and $a_4=e_{00}$, then $P((x_1,x_2),(y_1,y_2))=(x_1-y_1)^2+(x_2-y_2)^2$.*

Full modulus of continuity of $f\in C(D_a)$ is denoted by $w(f;\delta)$ and defined as follows :

$$w(f;\delta):=\max\left\{\left|f(b^1)-f(b^2)\right|:b^1,b^2\in D_a \text{ and } \left\|b^1-b^2\right\|\le\delta\right\}.$$

Partial modulus of continuity with respect to x_1 and x_2 are given by

$$w^1(f;\delta):=\max_{0\le x_2\le a}\max_{|x_1-x_3|\le\delta}|f(x_1,x_2)-f(x_3,x_2)|$$

and

$$w^2(f;\delta):=\max_{0\le y_1\le a}\max_{|y_2-y_3|\le\delta}|f(y_1,y_2)-f(y_1,y_3)|,$$

respectively. We shall need some well known properties of full and partial modulus of continuity :

$$w(f;\lambda\delta)\le(1+\lambda)\,w(f;\delta)$$

for any $\lambda\ge 0$ and $\lim\limits_{\delta\to 0}w(f;\delta)=0$.

The following theorem gives the rate of convergence of the sequence of linear positive operators (M_n) to f, by means of partial and full modulus of continuity.

Theorem 2.2. *For $f \in C(D_a)$ the following inequalities are satisfied*

$$\|M_n(f) - f\|_{C(D_a)} \leq 4\sum_{i=1}^{2} w^i(f;\delta_{n,i})$$

and

$$\|M_n(f) - f\|_{C(D_a)} \leq 4w(f;\delta_n)$$

where $\delta_{n,i} = \sqrt{\frac{2n+6}{(n-2)(n-3)}}\,(a+1)$, $i = 1,2$, $\delta_n = \sqrt{\frac{2(2n+6)}{(n-2)(n-3)}}\,(a+1)$.

Proof. For $(x_1, x_2) \in D_a$

$$\begin{aligned}
&|M_n f;(x_1,x_2) - f(x_1,x_2)| \\
\leq\ & (n-1)^2 \sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2) \int_0^\infty\int_0^\infty b_{n,k,l}(y_1,y_2)\,|f(y_1,y_2) - f(x_1,y_2)|\,dy_1dy_2 \\
\leq\ & (n-1)^2 \sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2) \int_0^\infty\int_0^\infty b_{n,k,l}(y_1,y_2)\left|1 + \frac{|y_1-x_1|}{\delta_{n,1}}\right| w(f,\delta_{n,1})\,dy_1dy_2.
\end{aligned}$$

Using the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ by Cauchy-Schwarz inequality,

$$\begin{aligned}
&|M_n f;(x_1,x_2) - f(x_1,x_2)| \\
\leq\ & 2\left\{1 + \frac{(n-1)^2}{\delta_{n,1}^2} \sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2) \int_0^\infty\int_0^\infty b_{n,k,l}(y)\left(y_1^2 - 2y_1x_1 + x_1^2\right)dy_1dy_2\right\} \\
&\times w(f,\delta_{n,1}) \\
\leq\ & 2\left\{1 + \frac{1}{\delta_{n,1}^2}\frac{(2n+6)}{F_3(n)}(x_1+1)^2\right\} w(f,\delta_{n,1}), \text{ for the sufficiently large } n\in\mathbb{N} \text{ we get} \\
\leq\ & 4w(f,\delta_{n,1}).
\end{aligned}$$

By using the same processes above we get the following inequality

$$|M_n(f;(x_1,x_2)) - f(x_1,x_2)| \leq 4w(f,\delta_{n,2}).$$

Using above inequalities and full modulus of continuity

$$\begin{aligned}
&|M_n(f;(x_1,x_2)) - f(x_1,x_2)| \\
\leq\ & (n-1)^2 \sum_{k,l\in\mathbb{N}} b_{n,k,l}(x_1,x_2) \\
&\times\left\{\int_0^\infty\int_0^\infty b_{n,k,l}(y)\left|1 + \frac{\sqrt{(y_1-x_1)^2 + (y_2-x_2)^2}}{\delta_n}\right| w(f,\delta_n)\,dy_1dy_2\right\} \\
\leq\ & 2\left\{1 + \frac{1}{\delta^2}\frac{(2n+6)}{F_3(n)}\left\{(x_1+1)^2 + (x_2+1)^2\right\}\right\} w(f,\delta_n)
\end{aligned}$$

and then we take supremum on last inequality we get the desired result. □

For $M>0$ and $0<\alpha\leq 1$, the class of the functions $f\in C(D_a)$ satisfying the following relation

$$w(f;\delta)\leq M\delta^{\alpha},\ \text{for all } \delta\geq 0,$$

is called a Lipschitz class and denoted by $Lip_M(\alpha)$. The following Corollary is routine and its proof is omitted.

Corollary 2.1. *Let $f\in C(D_a)$. If $f\in Lip_M(\alpha)$, for $0<\alpha\leq 1$, then the following inequality*

$$\|M_n(f)-f\|_{C(D_a)}\leq 4M\delta_n^{\alpha}$$

holds, where $M>0$ and δ_n is given in the above theorem.

3. **Voronovskaja-Type Theorem**

Let us define the moment functions $V_{i,j}$ by

$$V_{i,j}(y_1,y_2)=(y_1-x_1)^i(y_2-x_2)^j,\ i,j\in\mathbb{N}_0.$$

By simple calculations, the following lemma can be obtained easily.

Lemma 3.1. *For each $(x,y)\in D$ and $n\in\mathbb{N}$ the following equations are hold*

$$\begin{aligned}
(i)\ M_n(V_{2,0};(x_1,x_2)) &= \frac{2n+6}{F_3(n)}x_1^2+\frac{2n+6}{F_3(n)}x_1+\frac{2}{F_3(n)},\\
(ii)\ M_n(V_{0,2};(x_1,x_2)) &= \frac{2n+6}{F_3(n)}x_2^2+\frac{2n+6}{F_3(n)}x_2+\frac{2}{F_3(n)},\\
(iii)\ M_n(V_{1,1};(x_1,x_2)) &= \frac{(nx_1+1)(nx_2+1)}{(n-2)^2}-\frac{(nx_1+1)}{(n-2)}x_2\\
&\quad -\frac{(nx_2+1)}{(n-2)}x_1+x_1x_2,
\end{aligned}$$

$$
\begin{aligned}
(iv)\ M_n(V_{2,2};(x_1,x_2)) &= \frac{n(n+1)\left(x_1^2+x_2^2\right)+4n\left(x_1+x_2\right)+4}{\left(F_3(n)\right)^2} \\
&-\frac{2\left(n(n+1)x_1^2+4nx_1+2\right)\left(nx_2^2+x_2\right)}{(n-2)F_3(n)} \\
&+\frac{\left(n(n+1)x_1^2+4nx_1+2\right)x_2^2}{F_3(n)} \\
&-\frac{2\left(nx_1^2+x_1\right)\left(n(n+1)x_2^2+4nx_2+2\right)}{(n-2)F_3(n)} \\
&+\frac{4(nx_1^2+x_1)(nx_2^2+x_2)}{(n-2)^2}-\frac{2(nx_1^2+x_1)x_2^2}{(n-2)} \\
&+\frac{\left(n(n+1)x_2^2+4nx_2+2\right)x_1^2}{F_3(n)}-\frac{2x_1^2(nx_2^2+x_2)}{(n-2)}+x_1^2x_2^2, \\
(v)\ M_n(V_{4,0};(x_1,x_2)) &= \frac{12n^2+252n+120}{F_5(n)}x_1^4+\frac{24n^2+504n+240}{F_5(n)}x_1^3 \\
&+\frac{12n^2+324n+240}{F_5(n)}x_1^2+\frac{72n+120}{F_5(n)}x_1+\frac{24}{F_5(n)}, \\
(vi)\ M_n(V_{0,4};(x_1,x_2)) &= \frac{12n^2+252n+120}{F_5(n)}x_2^4+\frac{24n^2+504n+240}{F_5(n)}x_2^3 \\
&+\frac{12n^2+324n+240}{F_5(n)}x_2^2+\frac{72n+120}{F_5(n)}x_2+\frac{24}{F_5(n)}.
\end{aligned}
$$

The following corollary is clear and its proof is omitted.

Corollary 3.1. *If $(x_1,x_2)\in D$, then we have*

$$\lim_{n\to\infty} n^2 M_n(V_{4,0}+V_{0,4};(x_1,x_2)) = 12x_1^2\left(x_1+1\right)^2+12x_2^2\left(x_2+1\right)^2.$$

We now give a Voronovskaja type theorem for M_n operators.

Theorem 3.1. *Suppose that $f\in C^2(D)$. Then for each (x_1,x_2) in D, we have*

$$
\begin{aligned}
\lim_{n\to\infty} n(M_n\left(f;(x_1,x_2)\right)-f(x_1,x_2)) &= (2x_1+1)f_{x_1}(x_1,x_2)+(2x_2+1)f_{x_2}(x_1,x_2) \\
&+x_1(x_1+1)f_{x_1x_1}(x_1,x_2) \\
&+x_2(x_2+1)f_{x_2x_2}(x_1,x_2).
\end{aligned}
$$

Proof. Let (x_1,x_2) be a fixed point in D and $f\in C^2(D)$. By Taylor formula for f we get

$$
\begin{aligned}
f(y_1,y_2) &= f(x_1,x_2)+f_{x_1}(x_1,x_2)(y_1-x_1) \\
&+f_{x_2}(x_1,x_2)(y_2-x_2)+\frac{1}{2}f_{x_1x_2}(x_1,x_2)\left(y_1-x_1\right)^2 \\
&+f_{x_1x_2}(x_1,x_2)(y_1-x_1)(y_2-x_2)+\frac{1}{2}f_{x_2x_2}(x_1,x_2)(y_2-x_2)^2 \\
&+\psi((y_1,y_2);(x_1,x_2))\sqrt{(y_1-x_1)^4+(y_2-x_2)^4}. \qquad (5)
\end{aligned}
$$

Now applying the operators M_n to the equation (5) we obtain

$$M_n\left(f;(x_1,x_2)\right)-f(x_1,x_2)$$

$$\begin{aligned}
&= f_{x_1}(x_1,x_2)M_n\left((y_1-x_1);(x_1,x_2)\right)+f_{x_2}(x_1,x_2)M_n\left((y_2-x_2);(x_1,x_2)\right)\\
&\quad+\frac{1}{2}f_{x_1x_1}(x_1,x_2)M_n\left((y_1-x_1)^2;(x_1,x_2)\right)\\
&\quad+f_{x_1x_2}(x_1,x_2)M_n\left((y_1-x_1)(y_2-x_2);(x_1,x_2)\right)\\
&\quad+\frac{1}{2}f_{x_2x_2}(x_1,x_2)M_n\left((y_2-x_2)^2;(x_1,x_2)\right)\\
&\quad+M_n\left(\psi((y_1,y_2);(x_1,x_2))\sqrt{(y_1-x_1)^4+(y_2-x_2)^4}\right)\\
&= f_{x_1}(x_1,x_2)\left\{\frac{nx_1}{n-2}+\frac{1}{n-2}-x_1\right\}+f_{x_2}(x_1,x_2)\left\{\frac{nx_2}{n-2}+\frac{1}{n-2}-x_2\right\}\\
&\quad+\frac{1}{2}f_{x_1x_2}(x_1,x_2)\left\{\frac{2n+6}{F_3(n)}x_1^2+\frac{2n+6}{F_3(n)}x_2+\frac{2}{F_3(n)}\right\}\\
&\quad+f_{x_1x_2}(x_1,x_2)\left\{\frac{(nx_2+1)(nx_1+1)}{(n-2)^2}-\frac{(nx_1+1)}{(n-2)}x_2-\frac{(nx_2+1)}{(n-2)}x_1+x_1x_2\right\}\\
&\quad+\frac{1}{2}f_{x_1x_2}(x_1,x_2)\left\{\frac{2n+6}{F_3(n)}x_2^2+\frac{2n+6}{F_3(n)}x_2+\frac{2}{F_3(n)}\right\}\\
&\quad+M_n\left(\psi\sqrt{V_{4,0}+V_{0,4}};(x_1,x_2)\right). \qquad (6)
\end{aligned}$$

From Theorem 2.2 we have

$$\lim_{n\to\infty} M_n\left(\psi^2;(x_1,x_2)\right)=0. \qquad (7)$$

Combining Corollary 3.1 and (7) yield

$$\lim_{n\to\infty} nM_n\left(\psi\sqrt{V_{4,0}+V_{0,4}};(x_1,x_2)\right)=0. \qquad (8)$$

The proof of the Theorem is completed by using (6) and (8). □

Let's define the operators L_n and T_n for preliminary to the following Theorem.

$$\begin{aligned}
L_n(f;(x_1,x_2)) &= n(n-1)\sum_{k=0}^{\infty}P_{n,k}(x_1)\sum_{l=0}^{\infty}P_{n+1,l}(x_2)\\
&\quad\times\int_0^{\infty}\int_0^{\infty}P_{n,k}(y_1)\,P_{n-1,l+1}(y_2)\,f(y_1,y_2)\,dy_1dy_2,
\end{aligned}$$

$$\begin{aligned}
T_n(f;(x_1,x_2)) &= n(n-1)\sum_{k=0}^{\infty}p_{n+1,k}(x_1)\sum_{l=0}^{\infty}p_{n,l}(x_2)\\
&\quad\times\int_0^{\infty}\int_0^{\infty}p_{n-1,k+1}(y_1)\,p_{n,l}(y_2)\,f(y_1,y_2)\,dy_1dy_2.
\end{aligned}$$

The following Theorem gives us derivative of the operator M_n with respect to x_1 and x_2.

Theorem 3.2. *Let $f \in C^1(D)$ and $f(y_1, y_2) = O(y_2^{n-2})$, then the following equalities are satisfied for each $(x_1, x_2) \in D$*

$$(i)\ \frac{\partial}{\partial x_2} M_n(f; (x_1, x_2)) = L_n(f_{y_2}; (x_1, x_2)).$$

$$(ii)\ \frac{\partial}{\partial x_1} M_n(f; (x_1, x_2)) = T_n(f_{y_1}; (x_1, x_2))$$

Proof. Now we only prove the part (i) and the proof of the part (ii) is similar and omitted.

$$\begin{aligned}
& M_n(f; (x_1, x_2)) \\
= & (n-1)^2 \sum_{k=0}^{\infty} p_{n,k}(x_1) \sum_{l=0}^{\infty} p_{n,l}(x_2) \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) p_{n,l}(y_2) f(y_1, y_2)\, dy_1 dy_2 \\
= & (n-1)^2 \sum_{k=0}^{\infty} p_{n,k}(x_1) \Bigg\{ p_{n,0}(x_2) \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) p_{n,0}(y_2) f(y_1, y_2)\, dy_1 dy_2 \\
& + \sum_{l=1}^{\infty} p_{n,l}(x_2) \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) p_{n,l}(y_2) f(y_1, y_2)\, dy_1 dy_2 \Bigg\}. \qquad (9)
\end{aligned}$$

Differentiating both sides of the equality (9) with respect to x_2 and using the equality $p'_{n,l}(x_2) = n\left(p_{n+1,l-1}(x_2) - p_{n+1,l}(x_2)\right)$ we get

$$\begin{aligned}
& \frac{\partial}{\partial x_2} M_n(f; (x_1, x_2) \\
= & (n-1)^2 \sum_{k=0}^{\infty} p_{n,k}(x_1) \Bigg\{ -\frac{n}{(1+x_2)^{n+1}} \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) p_{n,0}(y_2) f(y_1, y_2)\, dy_1 dy_2 \\
& + \sum_{l=1}^{\infty} n\left(p_{n+1,l-1}(x_2) - p_{n+1,l}(x_2)\right) \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) p_{n,l}(y_2) f(y_1, y_2)\, dy_1 dy_2 \Bigg\} \\
= & n(n-1) \sum_{k=0}^{\infty} p_{n,k}(x_1) \sum_{l=0}^{\infty} p_{n+1,l}(x_2) \\
& \times \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) \left(-p'_{n-1,l+1}(y_2)\right) f(y_1, y_2)\, dy_1 dy_2.
\end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned}
& \frac{\partial}{\partial x_2} M_n(f; (x_1, x_2)) \\
= & n(n-1) \sum_{k=0}^{\infty} p_{n,k}(x_1) \sum_{l=0}^{\infty} p_{n+1,l}(x_2) \int_0^{\infty}\int_0^{\infty} p_{n,k}(y_1) p_{n-1,l+1}(y_2) f_{y_2}(y_1, y_2)\, dy_1 dy_2,
\end{aligned}$$

as desired. $\square$

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