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ON THE GENERALIZED TZITZEICA CURVE EQUATION

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Dedicated to the memory of Prof. Emeritus Dr. Constantin Udrişte

The aim of this paper is to present a generalization of the Tzitzeica curve equation based on a new centro-affine invariant. The third order nonlinear ordinary differential equation of the generalized Tzitzeica curves is derived and expressed in the curve's defining functions. A few intriguing solutions are discussed in connection with Tzitzeica curves.

Keywords: Tzitzeica curves, Nonlinear differential equations, Affine invariants

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1. Introduction

Introduced at the beginning of 1900s, Tzitzeica's curves [6] and surfaces [7] are still considered nowadays topics of analysis, especially due to the their geometrical importance as being the first examples of centro-affine invariants. In his paper [8], Professor Emeritus Dr. Constantin Udrişte stated "*The work of Gheorghe Tzitzeica is a permanent incitation to mathematical reflection*" as himself was a promoter of affine differential geometry and inspired young generations to pursue this direction of research "*Tzitzeica theory can be considered as essential part of variational principles on differential manifolds. It was a matter of course since the real world is governed, among other things, by optimum principles, and Gheorghe Tzitzeica realized indirectly that this is the clue of real problems*". In 1911, Tzitzeica introduced [7] the centro-affine invariant τ/d^2, where τ is the torsion of a smooth, regular skew curve and d denotes the distance from the origin to the osculating plane at an arbitrary point of the curve. The curves that have this property are called *Tzitzeica curves*. The ordinary differential equation (ODE) resulting from this condition determined by the curve's defining functions has been studied sporadically in the mathematics literature, and, hence, at the moment, there are known only a few Tzitzeica curves that can be expressed in terms of the elementary or special functions (see, for instance, [1], [2], [4], and [9])

In this paper, we propose the generalization of the Tzitzeica curve equation given by the ODE (7). To the best of our knowledge, the proposed differential equation does not appear in the literature. In [1], it was shown that if the defining functions of a Tzitzeica curve satisfy a specific auxiliary third order linear homogeneous ODE with constant coefficients, then the Tzitzeica curve equation can be reduced to a linear equation for the curve's constant. Although the generalized Tzitzeica curves share the same property, we show that the Wronskian of the curve's defining functions will play the main role to differentiate among these types of curves as the Wronskian itself is a centro-affine invariant.

The paper is structured as follows. In Section 2, we recall the basic theoretical concepts needed to derive the Tzitzeica curve equation (4). The proposed generalization (7) is introduced in Section 3. Particular cases for to the generalized Tzitzeica curve equation are discussed. Section 4 is reserved for a few remarks.

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2. The Tzitzeica curve equation

Let us consider a parametrically-defined smooth, regular skew curve

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \quad t \in I, \tag{1}$$

whose curvature $k(t)$ and torsion $\tau(t)$ defined by

$$k(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3} \quad \text{and} \quad \tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2} \tag{2}$$

are nonzero on the interval $I \subset \mathbf{R}$, where $||\mathbf{r}'(t) \times \mathbf{r}''(t)||$ is the magnitude of the vector cross product of the tangent vector $\mathbf{r}'(t)$ and the acceleration vector $\mathbf{r}''(t)$ and

$$\left(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\right) = \begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}.$$

denotes the scalar triple product of the vectors $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$.

Definition 2.1. The space curve (1) is a *Tzitzeica curve* if the ratio of its torsion τ and the square of the distance d from the origin to its osculating plane at an arbitrary point is constant, i.e.,

$$\frac{\tau(t)}{d^2(t)} = \alpha, \quad \text{for all} \quad t \in I, \tag{3}$$

where α is an arbitrary constant.

Proposition 2.1. [9] *The component functions x, y and z of a Tzitzeica curve satisfy the nonlinear third order ODE*

$$az''' - a'z'' + bz' = \alpha \left(cz'' - c'z' + az\right)^2, \tag{4}$$

where

$$a = x'y'' - x''y', \quad b = x''y''' - x'''y'', \quad \textit{and} \quad c = xy' - x'y. \tag{5}$$

Proof. The osculating plane at an arbitrary point of the curve (1) is described by the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ (or simply by $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$) and has the equation given by

$$\begin{vmatrix} x - x(t) & y - y(t) & z - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$$

Since the square of the distance from the origin to the osculating plane is

$$d^2(t) = \frac{1}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2} \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2,$$

the substitution of this expression along with the torsion τ from (2) into the condition (3) yields the equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2.$$

By expanding the above determinants along their last columns, this equation may be rewritten explicitly in the form (4). □

Definition 2.2. The nonlinear ODE (4) is called the *Tzitzeica curve equation*, and the constant α is referred to as the *curve's constant*.

3. Generalized Tzitzeica curve equation

In this section, we consider the smooth, regular skew curves that satisfy the condition

$$\frac{\tau(t)}{d(t)^{2r}||\mathbf{r}'(t)\times\mathbf{r}''(t)||^{2r-2}}=\mu, \quad t\in I, \tag{6}$$

where r is a rational number, and μ is a nonzero constant. The expression defined above generalizes Tzitzeica's centro-affine invariant (3) and, as it is shown in this section, (6) is a centro-affine invariant as well.

Definition 3.1. A smooth, regular skew curve (1) with nonzero curvature $k(t)$ for which the condition (6) holds is called a *generalized Tzitzeica curve.*

Proposition 3.1. *The smooth, regular skew curve (1) is a generalized Tzitzeica curve if and only if the functions x, y, and z are solutions to the following third order nonlinear ODE*

$$az''' - a'z'' + bz' = \mu\left(cz'' - c'z' + az\right)^{2r}, \tag{7}$$

where the functions a, b, and c are given by (5).

Proof. Similarly to the proof of Proposition 2.1, it may be shown that the curve's defining functions x, y, and z satisfy the equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \mu \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^{2r} \tag{8}$$

that may also be rewritten in terms of Wronskians as follows

$$W(x',y',z')(t) = \mu\,[W(x,y,z)(t)]^{2r}, \tag{9}$$

where

$$W(x,y,z)(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}$$

denotes the Wronskian of the functions x, y, and z. By expanding the determinants across the last columns in (8), the ODE (9) becomes (7). □

Definition 3.2. The nonlinear third order ODE (7) is called the *generalized Tzitzeica curve equation*.

According to Tzitzeica's theory, a curve satisfying (3) is centro-affine invariant. Next, we show that the generalized Tzitzeica curves share the same property.

Theorem 3.1. *A generalized Tzitzeica curve is a centro-affine invariant.*

Proof. Let us consider the centro-affine transformation

$$\begin{cases} \tilde{x} = a_{11}x + a_{12}y + a_{13}z \\ \tilde{y} = a_{21}x + a_{22}y + a_{23}z \\ \tilde{z} = a_{31}x + a_{32}y + a_{33}z \end{cases} \tag{10}$$

on the space of the dependent variables associated with the ODE (7); here the 3×3 matrix $A = (a_{ij}) \in SL(3,\mathbb{R})$ belongs to the special linear group and has $\det(A) = 1$. We may regard (10) as a change of the dependent variables x, y, and z. By using the property that the Wronskian is linear in its arguments, after solving (10) for x, y, and z, it can be shown that

$$W(x,y,z)(t) = \det(A^{-1})W(\tilde{x},\tilde{y},\tilde{z})(\tilde{t})$$

and

$$W(x',y',z')(t) = \det(A^{-1})W(\tilde{x}',\tilde{y}',\tilde{z}')(\tilde{t}).$$

Consequently, the generalized Tzitzeica curve equation becomes

$$W(\tilde{x}',\tilde{y}',\tilde{z}') = \mu[\det(A^{-1})]^{2r-1}\,[W(\tilde{x},\tilde{y},\tilde{z})]^{2r}.$$

Since $\det(A^{-1}) = \det(A) = 1$, it follows that

$$W(\tilde{x}',\tilde{y}',\tilde{z}')(\tilde{t}) = \mu\,[W(\tilde{x},\tilde{y},\tilde{z})(\tilde{t})]^{2r},$$

and this concludes the proof that a generalized Tzitzeica curve is invariant under a centro-affine transformation (10), and, hence, (6) is a centro-affine invariant. □

Remark 3.1. The expression (6) maybe also written as follows

$$\tau(t)d(t)^{-2} = \mu G(t), \quad t \in \mathbb{R},$$

where

$$G(t) = \left[d(t)||\mathbf{r}'(t)\times\mathbf{r}''(t)||\right]^{2r-2}. \tag{11}$$

Since $\tau(t)d(t)^{-2}$ is a centro-affine invariant, then $G(t)$ is also a centro-invariant. Notice that $G(t)$ may be expressed in terms of the Wronskian of the curve's defining functions. Indeed, by substituting $d(t)$ into (11), we get

$$G(t) = [W(x,y,z)(t)]^{2r-2} = [W(\tilde{x},\tilde{y},\tilde{z})(t)]^{2r-2}$$

which shows that $G(t)$ remains invariant under the centro-affine transformation (10). Notice that any centro-affine transformation maps a Tzitzeica curve to another Tzitzeica curve. Same property is shared now by the generalized Tzitzeica curves.

Proposition 3.2. *Any three linearly independent solutions of the third order homogeneous linear ODE with constant coefficients*

$$u''' + \gamma u' + \delta u = 0, \tag{12}$$

where γ and $\delta \neq 0$ are real numbers, define a generalized Tzitzeica curve for which the curve's constant μ is given by

$$\mu = -\frac{\delta}{[W(x,y,z)(t)]^{2r-1}}. \tag{13}$$

Proof. Let us consider a generalized Tzitzeica curve whose defining functions x, y, and z satisfy the homogeneous third order linear ODE

$$u''' + \beta u'' + \gamma u' + \delta u = 0. \tag{14}$$

The solutions x, y, and z of the above equation are considered to be linearly independent. Otherwise, the torsion of the curve is zero. Notice that the constant δ in (14) is assumed nonzero because the curve's torsion does not vanish. Thus, the general solution to the above ODE is given by

$$u(t) = C_1x(t) + C_2y(t) + C_3z(t),$$

where C_i are real constants, $i = 1,2,3$. Since each of the curve's defining functions satisfies the equation (14), we have

$$\begin{aligned} x''' + \beta x'' + \gamma x' + \delta x &= 0,\\ y''' + \beta y'' + \gamma y' + \delta y &= 0,\\ z''' + \beta z'' + \gamma z' + \delta z &= 0. \end{aligned} \tag{15}$$

After substituting the third order derivatives x''', y''' and z''' from (15) into (8), we obtain the following relation

$$-\delta W(x,y,z)(t) = \mu\,[W(x,y,z)(t)]^{2r}$$

which may be rewritten as

$$[W(x,y,z)(t)]^{2r-1} = -\frac{\delta}{\mu}, \tag{16}$$

and leads to the condition (13). Next, Abel's identity [3] given by

$$\frac{d}{dt}W(x,y,z)(t) = -\beta W(x,y,z)(t) \tag{17}$$

is used. From (16) and (17), we obtain $\beta = 0$ (here $W(x,y,z)(t)$ is nonzero because the functions are assumed linearly independent). Letting $\beta = 0$ in (14) yields (12). □

Corollary 3.1. *Any Tzitzeica curve whose defining functions x, y, and z satisfy the the third order homogeneous linear ODE with constant coefficients (12) is also a generalized Tzitzeica curve and vice versa. In this case, the constants of the curves, α and, respectively, μ, satisfy the relation*

$$\alpha = \mu\,[W(x,y,z)(t)]^{2r-2}. \tag{18}$$

Proof. Indeed, the same solutions x, y, and z of (12) may be used to construct a Tzitzeica curve (for which the curve's constant is α) and also a generalized Tzitzeica curve (with the constant denoted by μ). Since the Wronskian of the solutions of the equation (12) is constant, from the curve's equations, we obtain (18). □

In the paper [1], the equation (12) and its solutions have been discussed in detail. In what follows, we present briefly only a few examples of curves that satisfy both the Tzitzeica curve equation and the Tzitzeica generalized equation with distinct constants, α and μ, related by (18) (the constants are equal for x,y, and z having a unit Wronskian). The characteristic equation associated with (12) is given by the depressed cubic equation

$$v^3 + \gamma v + \delta = 0 \tag{19}$$

whose associated determinant is

$$D = -4\gamma^3 - 27\delta^2.$$

If $D > 0$, then the roots of (19) are real and denoted by v_1, v_2 and $v_3 = -(v_1 + v_2)$. The associated (generalized) Tzitzeica curve is

$$x(t) = \exp(v_1 t), \quad y(t) = \exp(v_2 t), \quad z(t) = \exp[-(v_1 + v_2)t], \quad t \in \mathbb{R}.$$

The case $D = 0$ yields the real roots $v_1 = v_2$ and $v_3 = -2v_1$ for which the related (generalized) Tzitzeica curve is given by

$$x(t) = \exp(v_1 t), \quad y(t) = t\exp(v_1 t), \quad z(t) = \exp(-2v_1 t), \quad t \in \mathbb{R}.$$

For $D < 0$, the equation has two complex roots $v_{1,2} = m \pm +in$ and a real root $v_3 = -2m$, where m and n are arbitrary nonzero constants. The corresponding (generalized) Tzitzeica curve is

$$x(t) = \exp(mt)\cos(nt), \quad y(t) = \exp(mt)\sin(nt), \quad z(t) = \exp(-2mt), \quad t \in \mathbb{R}.$$

We notice that if the Wronskian of the curve's defining functions x, y, and z of a generalized Tzitzeica curve is not constant (and nonzero) or if not all curve's component functions satisfy the linear ODE (12), the function G defined in (11) is not constant, and then the generalized Tzitzeica curve is not a Tzitzeica curve. For instance, for $r = 1/2$, it may be shown that

$$x(t) = \cos(t), \quad y(t) = \sin(t), \quad z(t) = \exp(\mu t), \quad t \in \mathbb{R}, \tag{20}$$

is a generalized Tzitzeica curve. The Tzitzeica curve with $x(t) = \cos(t)$ and $y(t) = \sin(t)$ has been given by Crâşmăreanu in [4]. The third component of the curve is

$$z(t) = \frac{1}{\alpha}\int_0^t \frac{\sin(s-t)}{s+l}\,ds, \tag{21}$$

where l is a constant. Another example is the generalized Tzitzeica curve

$$x(t) = \cosh(t), \quad y(t) = \sinh(t), \quad z(t) = \exp(-\mu t), \quad t \in \mathbb{R}, \tag{22}$$

with $r = 1/2$ and $\mu \neq \pm 1$. The Tzitzeica curve satisfying $x(t) = \cosh(t)$ and $y(t) = \sinh(t)$ has the third component [4] given by

$$z(t) = \frac{1}{\alpha} \int_0^t \frac{\sinh(t-s)}{s+l} ds, \tag{23}$$

here l is a constant. For $r = 1/2$ and $\mu = -1$, the generalized Tzitzeica curve is

$$x(t) = \cosh(t), \quad y(t) = \sinh(t), \quad z(t) = t \exp(t), \quad t \in \mathbb{R}, \tag{24}$$

while for $r = 1/2$ and $\mu = 1$, we have

$$x(t) = \cosh(t), \quad y(t) = \sinh(t), \quad z(t) = t \exp(-t), \quad t \in \mathbb{R}. \tag{25}$$

4. Conclusions

In this work, a nonlinear ODE has been proposed in the context of skew curves invariant under centro-affine transformations. The Tzitzeica curve equation (4) has been generalized as (7). The function G introduced in (11) is also a centro-affine invariant as it is written as a function of the Wronskian of the curve's defining functions (that remains unchanged under these type of transformations). Rewriting $G(t) = \left[k(t)d(t)||\mathbf{r}'(t)||^3\right]^{2r-2}$ in terms of the curvature of the curve k yields $k(t)d(t)||\mathbf{r}'(t)||^3$ is a cento-affine invariant. Although many other intriguing properties related to the Tzitzeica and generalized Tzitzeica curves may be discussed, we resumed to point out only a few that would make them distinct of each other.

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