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REGULARIZED ALGORITHM FOR THE PROXIMAL SPLIT FEASIBILITY PROBLEMS

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The purpose of this paper is to propose a regularized algorithm to find find common solution of proximal split feasibility problem and fixed point problems for the case of convex and nonconvex functions in real Hilbert spaces. The algorithm is motivated by the inertial method and the split proximal algorithm with self adaptive step size such that their implementation does not required any prior information about the operator norm. In addition, we give a numerical example to verify the efficiency and implementation of our scheme.

Keywords: Strong convergence, proximal Split feasibility problem, Fixed point problem.
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1. Introduction

The split feasibility problem (SFP) was introduced by Censor and Elfving [9] in a finite dimensional Hilbert space for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction, especially in intensity modulated therapy [11]. It plays key role in signal processing Byrne [8] and medical image reconstruction Byrne [7]. A more general case is the proximal split feasibility problem.

Let H_1 and H_2 be real Hilbert space, $f\colon H_1 \to \mathbb{R}\cup\{\infty\}$, $g\colon H_2 \to \mathbb{R}\cup\{\infty\}$ proper, lower semicontinuous, convex functions. Let $A\colon H_1 \to H_2$ be a bounded linear operator, then the proximal split feasibility problem is defined as below:

$$\text{find } x^* \in \arg\min f_1 \text{ such that } Ax^* \in \arg\min f_2, \tag{1.1}$$

where

$$\arg\min f = \{x \in H_1 : f(x) \le f(y), \forall y \in H_1\},$$

and

$$\arg\min g = \{x \in H_2 : g(x) \le g(y), \forall y \in H_2\}.$$

Let C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively, $f = i_C$ and $g = i_Q$ be indicator functions of C and Q, respectively, then the problem (**1**.1) reduces to the following split feasibility problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.2}$$

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A classical method to solve the SFP is Byrne's CQ algorithm ([7], [8]). Since then a number of numerical algorithms have been developed to solve the SFP; see([10, 15, 22, 23]) and the references therein.

Concerning problem (**1**.1), based on an idea of Lopez et al. [15] and with a new way of selecting the step-sizes, Moudafi and Thakur [18] introduced the following split proximal algorithm:

Set $\theta^2(x) := \|A^*(I-\text{prox}_{\lambda g})Ax\|^2 + \|(I-\text{prox}_{\lambda\mu_n f})x\|^2$, $h(x) := \frac{1}{2}\|(I-\text{prox}_{\lambda g})Ax\|^2$, and $l(x) := \frac{1}{2}\|(I-\text{prox}_{\lambda\mu_n f})x\|^2$.

For given an initial point $x_1 \in H_1$:

$$x_{n+1} = \text{prox}_{\lambda\mu_n f}(x_n - \mu_n A^*(I-\text{prox}_{\lambda g})Ax_n),\ n \geq 1, \tag{\textbf{1}.3}$$

where the stepsize is chosen as $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$ and $\text{prox}_{\lambda\mu_n f}(y) = \arg\min_{u\in H_1}\{f(u) + \frac{1}{2\lambda\mu_n}\|u-y\|^2\}$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (**1**.3) and the iterative process stops, otherwise, set $n := n+1$ and go to (**1**.3).

Moreover, Moudafi and Thakur [18] also assumed f to be convex and allowed the function g to be nonconvex and proved a weak convergence result in Hilbert spaces. They considered the more general problem of finding a minimizer $\bar{x}$ of f such that $A\bar{x}$ is a critical point of g, i.e.

$$0 \in \partial f(\bar{x}) \text{ such that } 0 \in \partial_P g(A\bar{x}),$$

where ∂_P stands for the proximal subdifferential of g.

Shehu and Ogbuisi [24] constructed the following iterative algorithm for approximating a solution of proximal split feasibility problems for the case of convex and non-convex functions and proved strong convergence in Hilbert spaces.

For given an initial point $x_1 \in H_1$ compute x_{n+1} via the following rule:

$$\begin{cases} w_n = (1-\alpha_n)x_n, \\ y_n = \text{prox}_{\lambda\mu_n f}(w_n - \mu_n A^*(I-\text{prox}_{\lambda g})Aw_n), \\ x_{n+1} = (1-\beta_n)x_n + \beta_n S y_n, n \geq 1, \end{cases} \tag{\textbf{1}.4}$$

where the stepsize $\mu_n := \rho_n \frac{h(w_n)+l(w_n)}{\|\theta^2(w_n)\|}$ with $0 < \rho_n < 4$.

If $\theta(w_n) = 0$, then $x_{n+1} = x_n$ is a solution of (**1**.4) and the iterative process stops, otherwise set $n := n+1$ and go to (**1**.4).

Khuangsatung *et al.* [14] introduced a regularized algorithm based on the viscosity method for solving the proximal split feasibility problem and the fixed point problem in Hilbert spaces as follows:

For a given initial point $x_1 \in H_1$, assume that x_n has been constructed and

$$\|A^*(I-\text{prox}_{\lambda g})Aw_n\|^2 + \|(I-\text{prox}_{\lambda\mu_n f})w_n\|^2 \neq 0,$$

then compute x_{n+1} by the following iterative scheme:

$$\begin{cases} y_n = \text{prox}_{\lambda\mu_n f}(\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I-\text{prox}_{\lambda g})Ax_n), \\ x_{n+1} = (1-\beta_n)x_n + \beta_n S y_n, n \geq 1, \end{cases} \tag{\textbf{1}.5}$$

where the stepsize $\mu_n := \rho_n \frac{h(w_n)+l(w_n)}{\|\theta^2(w_n)\|}$ with $0 < \rho_n < 4$ and $\psi, S\colon H_1 \to H_1$ is contraction and nonexpansive mappings, respectively.

Some other algorithms for proximal split feasibility problem may be found in [1, 12, 14, 19, 25, 30] and references therein.

In the context of implementation, an algorithm with a higher rate of convergence is more useful. A way to improve convergence rate is to add inertial term in the algorithm. It was first proposed by Polyak [21] as an acceleration process to solve the smooth convex minimization problem. The main feature of the inertial-type algorithms is that it uses two previous iterates to construct the next one. In recent years several convergence results were obtained using inertial type algorithms; see, for instance: [3, 4, 16, 23, 26, 28].

Inspired and motivated by the above mentioned works, in this paper propose an algorithm with inertial method to solve the proximal split feasibility problems and establish strong convergence result for by employing proposed algorithm in Hilbert spaces. We also provide a numerical example to illustrate the effectiveness of the proposed algorithm.

2. Preliminaries

Let C be a nonempty closed, convex subset of Hilbert space H. A mapping $T: C \to H$ said to be k-strictly pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \tag{2.1}$$

holds, for $0 \leq k < 1$ and for all $x, y \in C$.

It is said to be nonexpansive if $k = 0$, pseudo-contractive if $k = 1$, Strongly pseudo-contractive if there exists a positive constant $\lambda \in (0,1)$ such that $T - \lambda I$ is pseudo-contractive. The class of k-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

If $z \in F(T)$, where $F(T)$ denote the set of fixed points of T, then from (**2**.1), we have

$$(1-k)\|x - Tx\|^2 \leq 2\langle x - z, x - Tx\rangle. \tag{2.2}$$

We now recall some definitions and results:

An operator A is strong positive on H, if there exists a constant $\tau > 0$ with the property:

$$\langle Ax, x\rangle \geq \tau\|x\|^2, \qquad \forall x \in H.$$

The proximal operator $\text{prox}_{\lambda g} : H \to H$ is defined by,

$$\text{prox}_{\lambda g}(y) = \arg\min_{u\in H}\{g(u) + \frac{1}{2\lambda}\|u-y\|^2\}.$$

It is firmly nonexpansive [13], i.e.,

$$\langle \text{prox}_{\lambda g}(x) - \text{prox}_{\lambda g}(y), x-y\rangle \geq \|\text{prox}_{\lambda g}(x) - \text{prox}_{\lambda g}(y)\|^2,$$

holds, for all $x, y \in H$.

Lemma 2.1 ([12]). *In a real Hilbert space H, following hold:*

(1) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$,
(2) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$,
(3) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$,

for all $x, y \in H$ *and* $\alpha \in (0,1)$.

Lemma 2.2 ([27]). *Given* $x \in H$ *and* $y \in C$. *Then,* $P_C x = y$ *if and only if*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall\, z \in C.$$

Lemma 2.3 ([6]). *Let* C *be a nonempty closed, convex subset of a real Hilbert space* H. *If* $S\colon C \to C$ *is a nonexpansive mapping, then* $I - S$ *is demi-closed at zero.*

Lemma 2.4 ([17]). *Assume that* A *is a strong positive linear bounded operator on a Hilbert space* H *with coefficient* $\tau > 0$ *and* $0 < \rho \leq \|A\|^{-1}$, *then* $\|I - \rho A\| \leq 1 - \rho\tau$.

We now list some properties of strictly pseudo-contractive mappings from [2], [31].

Lemma 2.5. *Let* C *be a closed convex subset of a Hilbert space* H.

(i) *Let* $T : C \to C$ *be a* k*-strictly psuedo-contractive mapping, then a mapping* $S\colon C \to C$ *defined by* $Sx = \lambda x + (1-\lambda)Tx$, $x \in C$ *is nonexpansive for* $\lambda \in [k,1)$ *also* $F(S) = F(T)$.

(ii) *For an integer* $N \geq 1$, *assume for each* $1 \leq i \leq N$, $T_i\colon C \to C$ *is a* k_i*-strictly pseudo-contractive mapping for some* $0 \leq k_i \leq 1$. *Assume that* $\{\eta_i\}_{i=1}^n$ *is a positive sequence such that* $\sum_{i=1}^n \eta_i = 1$. *Then* $\sum_{i=1}^n \eta_i T_i$ *is a non-self-k-strictly pseudo-contractive mapping with* $k = \max\{k_i : 1 \leq i \leq n\}$

(iii) *If* $\{T_i\}_{i=1}^n$ *has a common fixed point in* C. *Then* $F(\sum_{i=1}^n \eta_i T_i) = \cap_{i=1}^n F(T_i)$.

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Lemma 2.6 ([29]). *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying:*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n \sigma_n + \gamma_n,\ \forall\, n \geq 1,$$

where

(1) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(2) $\limsup \sigma_n \leq 0$;
(3) $\gamma_n \geq 0\ (n \geq 1), \sum_{n=1}^\infty \gamma_n < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

3. Main Results

For rest of the paper, let

- H_1 and H_2 be two real Hilbert spaces,
- $f\colon H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g\colon H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semicontinuous convex functions,
- $B\colon H_1 \to H_1$ be a strong positive bounded linear operator with coefficient τ and $A\colon H_1 \to H_2$ be a bounded linear operator.
- $\psi\colon H_1 \to H_1$ be a contraction with $\delta \in (0,1)$ and $0 < \gamma < \frac{\tau}{\delta}$,
- $S\colon H_1 \to H_1$ be a mapping defined as $Sx = kx + (1-k)Tx$, where $T : H_1 \to H_1$ be a k-strictly pseudo-contractive mapping,
- $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty \subset (0,1)$, and $\{\theta_n\}_{n=1}^\infty \subset [0,\tilde{\theta}) \subset [0,1)$.
- the solution set of (**1**.1) and (**1**.2) is denoted by Ω and Γ, respectively.

Before describing our algorithm, the following conditions are required in convergence analysis.

(C1) The solution set $F(T) \cap \Omega \neq \emptyset$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(C3) Let $\{\sigma_n\} \subset [0, \sigma)$ with $\sigma > 0$ such that $\lim_{n\to\infty} \frac{\sigma_n}{\alpha_n} = 0$;
(C4) $\varepsilon \leq \rho_n \leq \frac{4\|(I-\text{prox}_{\lambda g})Aw_n\|^2}{\|(I-\text{prox}_{\lambda g})Aw_n\|^2 + \|(I-\text{prox}_{\lambda \mu_n f})Aw_n\|^2} - \varepsilon$ for some $\varepsilon > 0$, where $0 < \rho_n < 4$.

We now propose the a modified split proximal algorithm as follows:

Algorithm 3.1. *Given an initial start $x_1, x_0 \in H_1$, assume that x_n has been constructed and $\|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 + \|(I - \text{prox}_{\lambda \mu_n f})w_n\|^2 \neq 0$. Then compute x_{n+1} by the following iterative scheme:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \text{prox}_{\lambda \mu_n f}(w_n - \mu_n A^*(I - \text{prox}_{\lambda g})Aw_n), \\ x_{n+1} = \alpha_n \gamma \psi(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]Sy_n, \quad n \geq 1, \end{cases}$$

where the stepsize $\mu_n := \rho_n \dfrac{\frac{1}{2}\|(I - \text{prox}_{\lambda g})Aw_n\|^2 + \frac{1}{2}\|(I - \text{prox}_{\lambda \mu_n f})w_n\|^2}{\|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 + \|(I - \text{prox}_{\lambda \mu_n f})w_n\|^2}$.

Remark 3.1. *In Algorithm* **3**.*1 the inertial parameter θ_n is chosen as,*

$$\theta_n = \begin{cases} \min\{\frac{\sigma_n}{\|x_n - x_{n-1}\|}, \tilde{\theta}\}, & \text{if } x_n \neq x_{n-1}, \\ \tilde{\theta}, & \text{otherwise}. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let the conditions (C1)-(C4) hold. Then, the sequence $\{x_n\}$ generated by Algorithm* **3**.*1 converges strongly to $x^* \in F(T) \cap \Omega$, which also solves the variational inequality (VI)*

$$\langle (B - \gamma\psi)x^*, x^* - x \rangle \leq 0, \qquad \forall x \in F(T) \cap \Omega.$$

Proof. For a given $\lambda > 0$ and $x \in H_1$, set

$$h(x) := \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax\|^2,$$

$$l(x) := \frac{1}{2}\|(I - \text{prox}_{\lambda \mu_n f})x\|^2,$$

$$\theta^2(x) := \|A^*(I - \text{prox}_{\lambda g})Ax\|^2 + \|(I - \text{prox}_{\lambda \mu_n f})x\|^2,$$

then,

$$\mu_n = \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)}.$$

Let $x^* \in F(T) \cap \Omega$, then $x^* = \text{prox}_{\lambda \mu_n f} x^*$ and $Ax^* = \text{prox}_{\lambda g} Ax^*$. Since $I - \text{prox}_{\lambda g}$ is firmly nonexpansive, we have

$$\begin{aligned} \langle A^*(I - \text{prox}_{\lambda g})Ax_n, x_n - x^* \rangle &= \langle (I - \text{prox}_{\lambda g})Ax_n, Ax_n - Ax^* \rangle \\ &= \langle (I - \text{prox}_{\lambda g})Ax_n - (I - \text{prox}_{\lambda g})Ax^*, Ax_n - Ax^* \rangle \\ &\geq \|(I - \text{prox}_{\lambda g})Ax_n\|^2 \\ &= 2h(x_n). \end{aligned}$$

From the definition of y_n and the nonexpansivity of $\text{prox}_{\lambda\mu_n f}$, we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|\text{prox}_{\lambda\mu_n f}(w_n - \mu_n A^*(I - \text{prox}_{\lambda g})Aw_n) - x^*\|^2 \\
&\leq \|w_n - \mu_n A^*(I - \text{prox}_{\lambda g})Aw_n - x^*\|^2 \\
&\leq \|w_n - x^*\|^2 + \mu_n^2 \|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 - 2\mu_n \langle A^*(I - \text{prox}_{\lambda g})Aw_n, w_n - x^*\rangle \\
&\leq \|w_n - x^*\|^2 + \mu_n^2 \|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 - 4\mu_n h(w_n) \\
&\leq \|w_n - x^*\|^2 + \rho_n^2 \frac{(h(w_n) + l(w_n))^2}{\theta^4(w_n)} \|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 \\
&\quad - 4\rho_n \frac{(h(w_n) + l(w_n))}{\theta^2(w_n)} h(w_n) \\
&\leq \|w_n - x^*\|^2 + \rho_n^2 \frac{(h(w_n) + l(w_n))^2}{\theta^2(w_n)} - 4\rho_n \frac{(h(w_n) + l(w_n))^2}{\theta^2(w_n)} \frac{h(w_n)}{(h(w_n) + l(w_n))} \\
&\leq \|w_n - x^*\|^2 - \rho_n \left(\frac{4h(w_n)}{(h(w_n) + l(w_n))} - \rho_n \right) \frac{(h(w_n) + l(w_n))^2}{\theta^2(w_n)}. \qquad (\mathbf{3}.2)
\end{aligned}$$

Using condition $(C4)$, for all $n \geq 1$ we get $\frac{4h(w_n)}{(h(w_n)+l(w_n))} - \rho_n \geq 0$, and hence from (**3**.2), we have

$$\|y_n - x^*\|^2 \leq \|w_n - x^*\|^2. \qquad (\mathbf{3}.3)$$

Let $M_2 = \sup_{n\geq 1}\{\theta_n \|x_n - x_{n-1}\|, 2\|x_n - x^*\|\}$, then by the definition of w_n,

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\
&= \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - x^*, x_n - x_{n-1}\rangle \\
&\leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\| \\
&\leq \|x_n - x^*\|^2 + 2M_2 \theta_n \|x_n - x_{n-1}\|. \qquad (\mathbf{3}.4)
\end{aligned}$$

From (**3**.3) and (**3**.4), we get

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2M_2 \theta_n \|x_n - x_{n-1}\|. \qquad (\mathbf{3}.5)$$

From Remark **3**.1, we have $\theta_n \|x_n - x_{n-1}\| \leq \sigma_n$ for all $n \geq 1$. This together with (C3), yield that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\sigma_n}{\alpha_n} = 0. \qquad (\mathbf{3}.6)$$

From (**3**.3), we get

$$\begin{aligned}
\|y_n - x^*\| \leq \|w_n - x^*\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\
&\leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|. \qquad (\mathbf{3}.7)
\end{aligned}$$

Using (**3**.7), Algorithm **3**.1, and nonexpansivity of S, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| &= \|\alpha_n\gamma\psi(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n B]Sy_n-x^*\| \\
&\le \|\alpha_n\gamma\psi(x_n)-\alpha_n B(x^*)-\beta_n x^*+\beta_n x_n+[(1-\beta_n)I-\alpha_n B]Sy_n+\alpha_n Bx^*+\beta_n x^*-x^*\| \\
&\le \|\alpha_n(\gamma\psi(x_n)-B(x^*))-\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n B)Sy_n-((1-\beta_n)I-\alpha_n B)x^*\| \\
&\le \alpha_n\|\gamma\psi(x_n)-B(x^*)\|+\beta_n\|x_n-x^*\|+((1-\beta_n)I-\alpha_n\tau)\|Sy_n-x^*\| \\
&\le \alpha_n\|\gamma\psi(x_n)-B(x^*)\|+\beta_n\|x_n-x^*\|+(1-\beta_n-\alpha_n\tau)\|y_n-x^*\| \\
&\le \alpha_n\|\gamma\psi(x_n)-B(x^*)\|+\beta_n\|x_n-x^*\|+(1-\beta_n-\alpha_n\tau)\big(\|x_n-x^*\|+\theta_n\|x_n-x_{n-1}\|\big) \\
&\le \alpha_n\|\gamma\psi(x_n)-B(x^*)\|+(1-\alpha_n\tau)\|x_n-x^*\|+(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\| \\
&\le (1-\alpha_n\tau)\|x_n-x^*\|+\alpha_n\gamma\|\psi(x_n)-\psi(x^*)\|+\alpha_n\|\gamma\psi(x^*)-B(x^*)\| \\
&\qquad +(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\| \\
&\le (1-\alpha_n\tau)\|x_n-x^*\|+\alpha_n\gamma\delta\|x_n-x^*\|+\alpha_n\|\gamma\psi(x^*)-B(x^*)\| \\
&\qquad +\theta_n\|x_n-x_{n-1}\|(1-\beta_n-\alpha_n\tau) \\
&= (1-\alpha_n(\tau-\gamma\delta))\|x_n-x^*\|+\alpha_n(\tau-\gamma\delta)\Big(\frac{\|\gamma\psi(x^*)-B(x^*)\|}{\tau-\gamma\delta} \\
&\qquad +\frac{\theta_n\|x_n-x_{n-1}\|(1-\beta_n-\alpha_n\tau)}{\alpha_n(\tau-\gamma\delta)}\Big) \\
&\le \max\left\{\|x_n-x^*\|,\left(\frac{\|\gamma\psi(x^*)-B(x^*)\|}{\tau-\gamma\delta}+\frac{\theta_n\|x_n-x_{n-1}\|(1-\beta_n-\alpha_n\tau)}{\alpha_n(\tau-\gamma\delta)}\right)\right\} \\
&\ \vdots \\
&\le \max\left\{\|x_1-x^*\|,\left(\frac{\|\gamma\psi(x^*)-B(x^*)\|}{\tau-\gamma\delta}+\sup_{n\in\mathbb{N}}\phi_n\right)\right\},
\end{aligned}
$$

where, $\phi_n=\frac{\theta_n\|x_n-x_{n-1}\|(1-\beta_n-\alpha_n\tau)}{\alpha_n(\tau-\gamma\delta)}$ and by (**3**.6) we get that $\lim_{n\to\infty}\phi_n=0$. Therefore ϕ_n is bounded, which implies that the sequence $\{x_n\}$ is bounded, consequently $\{y_n\},\{Sy_n\}$ and $\{w_n\}$ are also bounded.

By Algorithm **3**.1 and Lemma **2**.1, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n\gamma\psi(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)Sy_n-x^*\|^2\\
&= \|\alpha_n(\gamma\psi(x_n)-Bx^*)+\beta_n(x_n-x^*)+[(1-\beta_n)I-\alpha_n B]Sy_n\\
&\quad -[(1-\beta_n)I-\alpha_n B]x^*\|^2\\
&= \|\alpha_n(\gamma\psi(x_n)-Bx^*)+\beta_n(x_n-x^*)+[(1-\beta_n)I-\alpha_n B](Sy_n-x^*)\|^2\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-Bx^*,x_{n+1}-x^*\rangle+\\
&\quad +\|\beta_n(x_n-x^*)+[(1-\beta_n)I-\alpha_n B](Sy_n-x^*)\|^2\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-Bx^*,x_{n+1}-x^*\rangle+2\beta_n(1-\beta_n-\alpha_n B)\langle Sy_n-x^*,x_n-x^*\rangle\\
&\quad +\|\beta_n(x_n-x^*)\|^2+\|[(1-\beta_n)I-\alpha_n B](Sy_n-x^*)\|^2\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-Bx^*,x_{n+1}-x^*\rangle+2\beta_n(1-\beta_n-\alpha_n\tau)\langle Sy_n-x^*,x_n-x^*\rangle\\
&\quad +\beta_n^2\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\tau)^2\|Sy_n-x^*\|^2\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-Bx^*,x_{n+1}-x^*\rangle+2\beta_n(1-\beta_n-\alpha_n\tau)\|Sy_n-x^*\|\|x_n-x^*\|\\
&\quad +\beta_n^2\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\tau)^2\|Sy_n-x^*\|^2\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-Bx^*,x_{n+1}-x^*\rangle+\beta_n(1-\beta_n-\alpha_n\tau)\big(\ Sy_n-x^*\|^2+\|x_n-x^*\|^2\big)\\
&\quad +\beta_n^2\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\tau)^2\|Sy_n-x^*\|^2\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-Bx^*,x_{n+1}-x^*\rangle+\beta_n\|x_n-x^*\|^2\\
&\quad +(1-\beta_n-\alpha_n\tau)\|Sy_n-x^*\|^2 \qquad (\mathbf{3}.8)\\
&\le 2\alpha_n\langle\gamma\psi(x_n)-\gamma\psi(x^*),x_{n+1}-x^*\rangle+2\alpha_n\langle\gamma\psi(x^*)-Bx^*,x_{n+1}-x^*\rangle\\
&\quad +\beta_n\|x_n-x^*\|^2+\|(1-\beta_n-\alpha_n\tau)\|y_n-x^*\|^2\\
&\le 2\alpha_n\gamma\delta\|x_n-x^*\|\|x_{n+1}-x^*\|+2\alpha_n\langle\gamma\psi(x^*)-Bx^*,x_{n+1}-x^*\rangle\\
&\quad +\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\tau)(\|x_n-x^*\|^2+2M_2\theta_n\|x_n-x_{n-1}\|)\\
&\le \alpha_n\gamma\delta(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+2\alpha_n\langle\gamma\psi(x^*)-Bx^*,x_{n+1}-x^*\rangle\\
&\quad +\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\tau)(\|x_n-x^*\|^2+2M_2\theta_n\|x_n-x_{n-1}\|),
\end{aligned}
$$

which implies that,

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{1-\alpha_n(\tau-\gamma\delta)}{1-\alpha_n\gamma\delta}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\delta}\langle\gamma\psi(x^*)-Bx^*,x_{n+1}-x^*\rangle\\
&\quad +\frac{2M_2(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\|}{1-\alpha_n\gamma\delta}\\
&\le \big(1-\frac{\alpha_n(\tau-2\gamma\delta)}{1-\alpha_n\gamma\delta}\big)\|x_n-x^*\|^2+\frac{\alpha_n(\tau-2\gamma\delta)}{1-\alpha_n\gamma\delta}\Big\{\frac{2\langle\gamma\psi(x^*)-Bx^*,x_{n+1}-x^*\rangle}{(\tau-2\gamma\delta)}\\
&\quad +\frac{2M_2\sigma_n(1-\beta_n-\alpha_n\tau)}{\alpha_n(\tau-2\gamma\delta)}\Big\}\\
&= (1-\delta_n)\|x_n-x^*\|^2+\delta_n\zeta_n, \qquad (\mathbf{3}.9)
\end{aligned}
$$

where $\delta_n=\frac{\alpha_n(\tau-2\gamma\delta)}{1-\alpha_n\gamma\delta}$ and $\zeta_n=\frac{2\langle\gamma\psi(x^*)-Bx^*,x_{n+1}-x^*\rangle}{(\tau-2\gamma\delta)}+\frac{2M_2\sigma_n(1-\beta_n-\alpha_n\tau)}{\alpha_n(\tau-2\gamma\delta)}$.

We now divide our proof into two cases:

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$, such that $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is nonincreasing. Then, $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ converges and $\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0$ as $n \to \infty$.
From (**3**.2), (**3**.8) and condition $(C4)$, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le 2\alpha_n\langle \gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\tau)\|y_n - x^*\|^2 \\
&\le 2\alpha_n\langle \gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\tau) \\
&\quad \left(\|w_n - x^*\|^2 - \rho_n\left(\frac{4h(w_n)}{(h(w_n)+l(w_n))} - \rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)}\right) \\
&\le 2\alpha_n\langle \gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n-\alpha_n\tau)\Big(\|x_n - x^*\|^2 + 2M_2\theta_n\|x_n - x_{n-1}\| \\
&\quad - \rho_n\left(\frac{4h(w_n)}{(h(w_n)+l(w_n))} - \rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)}\Big),
\end{aligned}$$

implies that

$$\begin{aligned}
&(1-\beta_n-\alpha_n\tau)\rho_n\left(\frac{4h(w_n)}{(h(w_n)+l(w_n))} - \rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)} \\
&\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n\langle \gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle \\
&\qquad - \alpha_n\tau\|x_n - x^*\|^2 + 2M_2\theta_n(1-\beta_n-\alpha_n\tau)\|x_n - x_{n-1}\| \\
&\qquad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, we have

$$\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)} \to 0 \text{ as } n \to \infty.$$

By the linearity and boundedness of A, we obtain that $\{\theta^2(w_n)\}$ is bounded. It follows that

$$\lim_{n\to\infty}((h(w_n)+l(w_n))^2) = 0,$$

which implies that

$$\lim_{n\to\infty} h(w_n) = \lim_{n\to\infty} l(w_n) = 0.$$

Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ satisfying $w_{n_j} \rightharpoonup w$. By the lower semicontinuity of h, we have

$$0 \le h(w) \le \liminf_{j\to\infty} h(w_{n_j}) = \lim_{n\to\infty} h(w_n) = 0.$$

Therefore, $h(w) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Aw\|^2 = 0$. Hence, Aw is a fixed point of the proximal mapping of g or equivalently, Aw is a minimizer of g. Similarily, from the lower semicontinuity of l, we obtain

$$0 \le l(w) \le \liminf_{j\to\infty} l(w_{n_j}) = \lim_{n\to\infty} l(w_n) = 0.$$

Therefore, $l(w) = \frac{1}{2}\|(I - \text{prox}_{\lambda\mu_n f})w\|^2 = 0$, i.e. w is a fixed point of the proximal mapping of f. In other word, w is a minimizer of f. Hence, $w \in \Omega$. Since

$$0 < \mu_n < 4\frac{h(w_n)+l(w_n)}{\theta^2(w_n)} \to 0 \text{ as } n \to \infty,$$

and hence, $\mu_n \to 0$ as $n \to \infty$.
From (**2**.2), we have

$$\begin{aligned}\|Sy_n - x^*\|^2 &= \|y_n - x^* + Sy_n - y_n\|^2\\&= \|y_n - x^*\|^2 - 2\langle y_n - z, y_n - Sy_n\rangle + \|Sy_n - y_n\|^2\\&\leq \|y_n - x^*\|^2 - (1-k)\|Sy_n - y_n\|^2 + \|Sy_n - y_n\|^2\\&\leq \|y_n - x^*\|^2 + k\|Sy_n - y_n\|^2.\end{aligned}\tag{3.10}$$

From (**3**.8) and (**3**.10), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq 2\alpha_n\langle\gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2\\&\quad + (1-\beta_n - \alpha_n\tau)\|Sy_n - x^*\|^2\\&\leq 2\alpha_n\langle\gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2\\&\quad + (1-\beta_n - \alpha_n\tau)(\|y_n - x^*\|^2 + k\|Sy_n - y_n\|^2)\\&\leq 2\alpha_n\langle\gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2\\&\quad + (1-\beta_n - \alpha_n\tau)(k\|Sy_n - y_n\|^2 + \|x_n - x^*\|^2\\&\quad + 2M_2\theta_n\|x_n - x_{n-1}\|),\end{aligned}\tag{3.11}$$

implies that

$$\begin{aligned}&-(1-\beta_n - \alpha_n\tau)k\|Sy_n - y_n\|^2\\&\leq 2\alpha_n\langle\gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \beta_n\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\&\quad + (1-\beta_n - \alpha_n\tau)(\|x_n - x^*\|^2 + 2M_2\theta_n\|x_n - x_{n-1}\|)\\&\leq 2\alpha_n\langle\gamma\psi(x_n) - Bx^*, x_{n+1} - x^*\rangle + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\&\quad + 2M_2\theta_n\|x_n - x_{n-1}\|(1-\beta_n - \alpha_n\tau),\end{aligned}$$

by condition (C2) and (**3**.6), we have

$$\lim_{n\to\infty}\|Sy_n - y_n\| = 0.\tag{3.12}$$

Since $\lim_{n\to\infty} l(w_n) = \lim_{n\to\infty}\frac{1}{2}\|(I - \text{prox}_{\lambda\mu_n f})w_n\|^2 = 0$, we get

$$\lim_{n\to\infty}\|w_n - \text{prox}_{\lambda\mu_n f}w_n\| = 0.\tag{3.13}$$

Since $\mu_n \to 0$ as $n\to\infty$, by nonexpansiveness of $\text{prox}_{\lambda\mu_n f}$, we have

$$\begin{aligned}\|y_n - \text{prox}_{\lambda\mu_n f}w_n\| &= \|\text{prox}_{\lambda\mu_n f}(w_n - \mu_n A^*(I - \text{prox}_{\lambda g})Aw_n) - \text{prox}_{\lambda\mu_n f}w_n\|\\&\leq \|w_n - \mu_n A^*(I - \text{prox}_{\lambda g})Aw_n - w_n\|\\&\leq \mu_n\|A^*(I - \text{prox}_{\lambda g})Aw_n\| \to 0 \; as \; n\to\infty.\end{aligned}$$

Therefore,

$$\|y_n - w_n\| \leq \|y_n - \text{prox}_{\lambda\mu_n f}w_n\| + \|\text{prox}_{\lambda\mu_n f}w_n - w_n\| \to 0 \; as \; n\to\infty.\tag{3.14}$$

By definition of w_n and using (**3**.6),

$$\begin{aligned}\|w_n - x_n\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x_n\|\\&= \theta_n\|x_n - x_{n-1}\| \to 0 \text{ as } n\to\infty.\end{aligned}\tag{3.15}$$

Since $w_{n_j} \rightharpoonup w \in H_1$, using (**3**.15) we conclude that $x_{n_j} \rightharpoonup w \in H_1$, similarly using (**3**.14) we get that $y_{n_j} \rightharpoonup w \in H_1$. By Lemma **2**.3, Lemma **2**.5 and (**3**.12), we have $w \in F(T)$. Hence $w \in F(T) \cap \Omega$.

Next, we show that $\limsup_{n\to\infty}\langle (B-\gamma\psi)x^*, x^*-x_n\rangle \leq 0$, where $x^* = P_{\Omega\cap F(T)}(I-B+\gamma\psi)x^*$. Then we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (B-\gamma\psi)x^*, x^*-x_n\rangle &= \lim_{j\to\infty}\langle (B-\gamma\psi)x^*, x^*-x_{n_j}\rangle \\ &= \langle (B-\gamma\psi)x^*, x^*-w\rangle \\ &\leq 0, \qquad \text{by Lemma } \mathbf{2}.2. \qquad (\mathbf{3}.16)\end{aligned}$$

Now, we prove that x_n converges strongly to x^*. From (**3**.9), we have

$$\|x_{n+1}-x^*\|^2 \leq (1-\delta_n)\|x_n-x^*\|^2 + \delta_n\zeta_n,$$

where

$$\delta_n = \frac{\alpha_n(\tau-2\gamma\delta)}{1-\alpha_n\gamma\delta} \text{ and } \zeta_n = \frac{2\langle \gamma\psi(x^*)-Bx^*, x_{n+1}-x^*\rangle}{(\tau-2\gamma\delta)} + \frac{2M_2\sigma_n(1-\beta_n-\alpha_n\tau)}{\alpha_n(\tau-2\gamma\delta)}.$$

Using $(C2)$, $(C3)$ and (**3**.16) we get, $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup\zeta_n \leq 0$. Applying Lemma **2**.6, we conclude that $\lim_{n\to\infty}\|x_n-x^*\| = 0$.

Case 2: Assume that $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ is not monotonically decreasing sequence.

Set $\Gamma_n = \|x_n-x^*\|$, $n \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}, n \geq n_0 \text{ (large enough)}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

After a similar arguments as in the Case 1, we can see that

$$\lim_{n\to\infty}\|Sy_{\tau(n)} - y_{\tau(n)}\| = 0.$$

and

$$\lim_{n\to\infty} h(w_{\tau(n)}) = \lim_{n\to\infty} l(w_{\tau(n)}) = 0.$$

Similarly from (**3**.9), we have

$$\|x_{\tau(n)+1}-x^*\|^2 \leq (1-\delta_{\tau(n)})\|x_{\tau(n)}-x^*\|^2 + \delta_{\tau(n)}\zeta_{\tau(n)},$$

implies,

$$\|x_{\tau}(n)-x^*\|^2 \leq \zeta_{\tau(n)}, \quad \text{as } \|x_{\tau(n)}-x^*\|^2 \leq \|x_{\tau(n)+1}-x^*\|^2.$$

Since $\limsup_{n\to\infty}\zeta_{\tau(n)} \leq 0$, we get that

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\| = 0.$$

Thus,

$$\lim_{n\to\infty}\|x_{\tau(n)+1}-x^*\| = 0.$$

As a consequences, we obtain for all $n \geq n_0$.

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1},$$

i.e,

$0 \leq \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \to 0$ *as* $n \to \infty$. Hence, $\lim \|x_n - x^*\| = 0$, that is, $\{x_n\}$ converges strongly to x^*. This completes the proof.

□

If we take $f = i_C$ and $g = i_Q$ in Algorithm **3**.1, then $\text{prox}_{\lambda\mu f} = P_C$ and $\text{prox}_{\lambda g} = P_Q$ for all λ, where $\arg\min f = C$ and $\arg\min g = Q$. As a direct consequence of Theorem **3**.1, we state following result for split feasibility problem.

Corollary 3.1. *Let the conditions* $(C2)$*-*$(C4)$ *hold and the solution set* $F(T) \cap \Gamma \neq \emptyset$*. Then, the sequence* $\{x_n\}$ *defined by Algorithm* **3**.*1 converges strongly to a point* $x^* \in F(T) \cap \Gamma$.

4. Finite family of pseudo-contractive mappings

Now, we propose following split proximal algorithm for finite family of pseudo-contractive mappings:

Algorithm 4.1. *Given an initial point* $x_1, x_0 \in H_1$*, assume that* x_n *has been constructed and* $\|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 + \|(I - \text{prox}_{\lambda\mu_n f})w_n\|^2 \neq 0$*, then compute* x_{n+1} *by the following iterative scheme:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \text{prox}_{\lambda\mu_n f}(w_n - \mu_n A^*(I - \text{prox}_{\lambda g})Aw_n), \\ x_{n+1} = \alpha_n \gamma \psi(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]Sy_n, n \geq 1, \end{cases}$$

where $S\colon H_1 \to H_1$ *is a mapping defined by* $Sx = kx + (1-k)\sum_{i=1}^{n} \eta_i T_i x$ *and* $k = \max\{k_i : i = 1, 2, \ldots, n\}$*, and the inertial parameter* θ_n *is chosen as Algorithm* **3**.*1.*

Theorem 4.1. *Let* $\{T_i\}_{i=1}^n : H_1 \to H_1$ *be finite family of* k_i*-strictly pseudo-contractive mappings and* $\{\eta_i\}_{i=1}^n$ *be a positive sequence such that* $\sum_{i=1}^n \eta_i = 1$*. Let* $F(T_i)_{i=1}^n \cap \Omega \neq \emptyset$ *and the conditions* $(C2)$*-*$(C4)$ *hold. Then, the sequence* $\{x_n\}$ *defined by Algorithm* **4**.*1 converges strongly to a common fixed point* x^**, i.e* $x^* \in F(\{T_i\}_{i=1}^n) \cap \Omega$ *which also solves the variational inequality:*

$$\langle (\gamma\psi - B)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in F(\{T_i\}_{i=1}^n) \cap \Omega.$$

Proof. Define $T : H_1 \to H_1$ by $Tx = \sum_{i=1}^n \eta_i T_i x$. By Lemma **2**.5, we conclude that T is k-strictly pseudo-contractive mappings and $F(T) = F(\sum_{i=1}^n \eta_i T_i) = \cap_{i=1}^n F(T_i)$. Then the conclusion follows from the Theorem **3**.1. This completes the proof. □

5. Nonconvex minimization problem

In this section, we propose an iterative algorithm and prove strong convergence theorem for common solution to nonconvex minimization feasibility problem and fixed point problem of k-strictly pseudocontractive mapping in real Hilbert spaces.

The nonconvex theory is of great practical interest, but is less developed as compared to the convex one. Moudafi and Thakur [18], studies the convergence of split proximal algorithm in which one function is allowed to be noconvex. They considered the following problem:

$$0 \in \partial f(\bar{w}) \text{ such that } 0 \in \partial_P g(A\bar{w}). \qquad (\mathbf{5}.1)$$

The solution set of Problem (**5**.1) is denoted by Ω_1. The Problem (**5**.1) includes as special cases, g convex and g lower-$\mathcal{C}^2$ function which itself is of great importance in variational analysis and optimization [18].

Poliquin–Rockafellar [20] introduced the concept of a proximal subdifferential and also investigated the limiting proximal subdifferential. Let $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ and let $\bar{w} \in \text{dom}g$, *i.e.*, $g(\bar{w}) < +\infty$. The proximal subdifferential $\partial_P g(\bar{w})$ is defined as follows:

Definition 5.1. *A vector u is in $\partial_P g(\bar{w})$ if there exist some $r > 0$ and $\varepsilon > 0$ such that for all $w \in B(\bar{w}, \varepsilon)$, then*

$$\langle u, w - \bar{w} \rangle \leq g(w) - g(\bar{w}) + \frac{r}{2} \|w - \bar{w}\|^2,$$

and $\partial_P g(\bar{w}) = \emptyset$ if $g(\bar{w}) = +\infty$.

We now recall that:

- g is locally lower semicontinuous at $\bar{w}$ if its epigraph is closed relative to a neighborhood of $(\bar{w}, g(\bar{w}))$,
- g is prox-bounded if g is minorized by a quadratic function,
- for $\varepsilon > 0$, the g-attentive ε-localisation of $\partial_P g$ around $(\bar{w}, \bar{u})$, the mapping $T_\varepsilon : H_2 \to 2^{H_2}$ is defined by

$$T_\varepsilon(w) = \begin{cases} \{u \in \partial_P g(w), \|u - \bar{u}\| < \varepsilon\}, & \text{if } \|w - \bar{w}\| < \varepsilon \text{ and } |g(w) - g(\bar{w})| < \varepsilon, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 5.2 ([20])**.** *A function g is said to be prox-regular at $\bar{w}$ for $\bar{u} \in \partial_P g(\bar{w})$, if there exist some $r > 0$ and $\varepsilon > 0$ such that for all $w, w' \in B(\bar{w}, \varepsilon)$ with $|g(w) - g(w')| \leq \varepsilon$ and all $u \in B(\bar{u}; \varepsilon)$ with $u \in \partial_P g(w)$ one has*

$$g(w') \geq g(w) + \langle u, w' - w \rangle - \frac{r}{2} \|w' - w\|^2 .$$

Lemma 5.1 ([18])**.** *Suppose that g is locally lower semicontinuous at $\bar{w}$ and prox-regular at $\bar{w}$ for $\bar{u} = 0$ with respect to r and ε. Let T_ε be the g-attentive ε-localisation of $\partial_P g$ around $(\bar{w}, \bar{u})$. Then for each $\lambda \in]0, \frac{1}{r}[$ and w_1, w_0 in a neighborhood U_λ of $\bar{w}$,*

$$\begin{aligned} &\langle (I - \text{prox}_{\lambda g})(w_1) - (I - \text{prox}_{\lambda g})(w_2), w_1 - w_2 \rangle \\ &\geq \|(I - pro_{\lambda g})(w_1) - (I - pro_{\lambda g})(w_2)\|^2 - \frac{\lambda r}{(1 - \lambda r)^2} \|w_1 - w_2\|^2. \end{aligned}$$

We now propose an algorithm to study the convergence property of problem (**5**.1).

Algorithm 5.1. *Given an initial point $x_1, x_0 \in H_1$, assume that x_n has been constructed and $\|A^*(I - \text{prox}_{\lambda g})Aw_n\|^2 + \|(I - \text{prox}_{\lambda \mu_n f})w_n\|^2 \neq 0$, then compute x_{n+1} by the following iterative scheme:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \text{prox}_{\lambda_n \mu_n f}(w_n - \mu_n A^*(I - \text{prox}_{\lambda_n g})Aw_n), \\ x_{n+1} = \alpha_n \gamma \psi(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]Sy_n, n \geq 1, \end{cases}$$

where $\lambda_n \in (0, \frac{1}{r} - \varepsilon)$ (for some $\varepsilon > 0$ small enough), stepsize

$$\mu_n := \rho_n \frac{(\frac{1}{2}\|(I - \text{prox}_{\lambda_n g})Aw_n\|^2) + (\frac{1}{2}\|(I - \text{prox}_{\lambda_n \mu_n f})w_n\|^2)}{\|A^*(I - \text{prox}_{\lambda_n g})Aw_n\|^2 + \|(I - \text{prox}_{\lambda_n \mu_n f})w_n\|^2}$$

*with $0 < \rho_n < 4$. The inertial parameter θ_n is chosen same as Algorithm **3**.1.*

We now present main result of this section.

Theorem 5.1. *Let $f\colon H_1 \to \mathbb{R}\cup\{+\infty\}$ be a proper and lower semicontinuous convex function, and $g\colon H_2 \to H_2$ be locally lower semicontinuous at $A\bar{w}$, prox-bounded and prox-regular at $A\bar{w}$ for $\bar{u}=0$ with $\bar{w}$ a point that solves (**5**.1) and A be a bounded linear operator which is surjective with a dense domain. Suppose that the conditions (C2)-(C4) holds with $\sum_{n=0}^{\infty}\lambda_n < \infty$. If $F(T)\cap\Omega_1 \neq \emptyset$ and $\|x_1-\bar{w}\|$ is small enough, then the sequence $\{x_n\}$ generated by Algorithm **5**.1 converges strongly to a point $\bar{w}\in F(T)\cap\Omega_1$ which also solves the variational inequality problem:*

$$\langle (B-\gamma\psi)\bar{w}, \bar{w}-x\rangle \le 0, \qquad \forall x \in F(T)\cap\Omega_1.$$

Proof. Since $\bar{w}\in F(T)\cap\Omega_1$, using the Lemma **5**.1 and nonexpansivity of $\text{prox}_{\lambda_n\mu_n f}$, we have

$$\begin{aligned}
\|y_n-\bar{w}\|^2 &= \|\text{prox}_{\lambda_n\mu_n f}(w_n-\mu_n A^*(I-\text{prox}_{\lambda g})Aw_n)-\bar{w}\|^2\\
&\le \|w_n-\mu_n A^*(I-\text{prox}_{\lambda_n g})Aw_n-\bar{w}\|^2\\
&\le \|w_n-\bar{w}\|^2+\mu_n^2\|A^*(I-\text{prox}_{\lambda g})Aw_n\|^2-2\mu_n\langle A^*(I-\text{prox}_{\lambda_n g})Aw_n, w_n-\bar{w}\rangle\\
&\le \|w_n-\bar{w}\|^2+\mu_n^2\|A^*(I-\text{prox}_{\lambda_n g})Aw_n\|^2-2\mu_n\big(2h(w_n)-\frac{\lambda_n r\|A\|^2}{(1-\lambda_n r)^2}\|w_n-\bar{w}\|^2\big)\\
&\le \|w_n-\bar{w}\|^2+\mu_n^2\|A^*(I-\text{prox}_{\lambda_n g})Aw_n\|^2-4\mu_n h(w_n)+2\mu_n\frac{\lambda_n r\|A\|^2}{(1-\lambda_n r)^2}\|w_n-\bar{w}\|^2\\
&\le \|w_n-\bar{w}\|^2+2\rho_n\frac{(h(w_n)+l(w_n))}{\|\nabla h(w_n)\|^2+\|\nabla h(w_n)\|^2}\frac{\lambda_n r\|A\|^2}{(1-\lambda_n r)^2}\|w_n-\bar{w}\|^2\\
&\quad -\rho_n\left(\frac{4h(w_n)}{h(w_n)+l(w_n)}-\rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)}\\
&\le \left(1+\lambda_n\rho_n\left(\frac{2h(w_n)}{\|\nabla h(w_n)\|^2}+\frac{2l(w_n)}{\|\nabla l(w_n)\|^2}\right)\frac{r\|A\|^2}{(1-\lambda_n r)^2}\right)\|w_n-\bar{w}\|^2\\
&\quad -\rho_n\left(\frac{4h(w_n)}{h(w_n)+l(w_n)}-\rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)}\\
&\le \left(1+\lambda_n\rho_n\left(1+\frac{2h(w_n)}{\|\nabla h(w_n)\|^2}\right)\frac{r\|A\|^2}{(1-\lambda_n r)^2}\right)\|w_n-\bar{w}\|^2\\
&\quad -\rho_n\left(\frac{4h(w_n)}{h(w_n)+l(w_n)}-\rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)}.
\end{aligned}\tag{5.2}$$

From Theorem II.19 of Brezis [5], we recall that A is surjective with a dense domain $\Leftrightarrow \exists\gamma>0$ such that $\|A^*x\|\ge\gamma\|x\|$. This gives that

$$\frac{2h(w_n)}{\|\nabla h(w_n)\|^2}=\frac{\|(I-\text{prox}_{\lambda_n g})A(w_n)\|^2}{\|A^*(I-\text{prox}_{\lambda_n g})A(w_n)\|^2}\le\frac{1}{\gamma^2}\frac{\|(I-\text{prox}_{\lambda_n g})A(w_n)\|^2}{\|(I-\text{prox}_{\lambda_n g})A(w_n)\|^2}=\frac{1}{\gamma^2}.$$

From conditions on the parameters λ_n and ρ_n, there exist of a positive constant M such that

$$\|y_n-\bar{w}\|^2\le(1+M\lambda_n)\|w_n-\bar{w}\|^2-\rho_n\left(\frac{4h(w_n)}{h(w_n)+l(w_n)}-\rho_n\right)\frac{(h(w_n)+l(w_n))^2}{\theta^2(w_n)}.\tag{5.3}$$

By Algorithm **5**.1 and (**5**.3), we have

$$\begin{aligned}
\|x_{n+1}-\bar{w}\| &= \|\alpha_n\gamma\psi(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n B]Sy_n-\bar{w}\| \\
&\le \|\alpha_n\gamma\psi(x_n)-\alpha_n B(\bar{w})-\beta_n\bar{w}+\beta_n x_n+[(1-\beta_n)I-\alpha_n B]Sy_n+\alpha_n B\bar{w}+\beta_n\bar{w}-\bar{w}\| \\
&\le \|\alpha_n(\gamma\psi(x_n)-B(\bar{w}))-\beta_n(x_n-\bar{w})+((1-\beta_n)I-\alpha_n B)Sy_n-((1-\beta_n)I-\alpha_n B)\bar{w}\| \\
&\le \alpha_n\|\gamma\psi(x_n)-B(\bar{w})\|+\beta_n\|x_n-\bar{w}\|+\|(1-\beta_n)I-\alpha_n B\|\|Sy_n-\bar{w}\| \\
&\le \alpha_n\|\gamma\psi(x_n)-B(\bar{w})\|+\beta_n\|x_n-\bar{w}\|+(1-\beta_n-\alpha_n\tau)\|y_n-\bar{w}\|
\end{aligned}$$

From (**5**.3) and definition of w_n, we have

$$\begin{aligned}
\|y_n-x^*\| &\le (1+M\lambda_n)^{\frac{1}{2}}\|w_n-x^*\| \\
&= (1+M\lambda_n)^{\frac{1}{2}}\left(\|x_n+\theta_n(x_n-x_{n-1})-x^*\|\right) \\
&\le (1+M\lambda_n)^{\frac{1}{2}}\left(\|x_n-x^*\|+\theta_n\|x_n-x_{n-1}\|\right).
\end{aligned} \tag{5.4}$$

From (**5**.4) and using $1+x\le e^x$, $x\ge 0$, we have

$$\begin{aligned}
\|x_{n+1}-\bar{w}\| &\le \alpha_n\|\gamma\psi(x_n)-B(\bar{w})\|+\beta_n\|x_n-\bar{w}\|+(1-\beta_n-\alpha_n\tau)\big((1+M\lambda_n)^{\frac{1}{2}}\|x_n-\bar{w}\| \\
&\quad +\theta_n\|x_n-x_{n-1}\|\big) \\
&\le \alpha_n\|\gamma\psi(x_n)-B(\bar{w})\|+(1+M\lambda_n)^{\frac{1}{2}}(1-\alpha_n\tau)\|x_n-\bar{w}\|+(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\| \\
&\le (1+M\lambda_n)^{\frac{1}{2}}(1-\alpha_n\tau)\|x_n-\bar{w}\|+\alpha_n\gamma\|\psi(x_n)-\psi(\bar{w})\|+\alpha_n\|\psi(\bar{w})-B(\bar{w})\| \\
&\quad +(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\| \\
&\le e^{(M\lambda_n)^{\frac{1}{2}}}(1-\alpha_n\tau)\|x_n-\bar{w}\|+\alpha_n\gamma\delta\|x_n-\bar{w}\|+\alpha_n\|\psi(\bar{w})-B(\bar{w})\| \\
&\quad +(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\| \\
&= e^{\frac{M}{2}\lambda_n}\left(1-\alpha_n\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)\right)\|x_n-\bar{w}\| \\
&\quad +\alpha_n\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)\left[\frac{\|\psi(\bar{w})-B(\bar{w})\|}{\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)}+\frac{(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\|}{\alpha_n\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)}\right] \\
&\le e^{\frac{M}{2}\lambda_n}\left(\max\left\{\|x_n-\bar{w}\|,\left(\frac{\|\psi(\bar{w})-B(\bar{w})\|}{\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)}+\frac{(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\|}{\alpha_n\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)}\right)\right\}\right) \\
&\ \ \vdots \\
&\le e^{\frac{M}{2}\sum_{n=1}^{\infty}\lambda_n}\left(\max\left\{\|x_1-\bar{w}\|,\left(\sup_{n\in\mathbb{N}}\psi_n+\sup_{n\in\mathbb{N}}\phi_n\right)\right\}\right),
\end{aligned}$$

where $\psi_n=\frac{\|\psi(\bar{w})-B(\bar{w})\|}{\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)}$, $\phi_n=\frac{(1-\beta_n-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\|}{\alpha_n\left(\tau-\frac{\gamma\delta}{e^{\frac{M}{2}\lambda_n}}\right)}$. Clearly, $\{\phi_n\},\{\psi_n\}$ are bounded, as $\lim_{n\to\infty}\phi_n=0$ and $\sum_{n=0}^{\infty}\lambda_n<\infty$. This implies that the sequence $\{x_n\}$ is bounded. Consequently $\{y_n\},\{Ty_n\}$ and $\{w_n\}$ are also bounded.

Now, following the methods of Theorem **3**.1, we can show that

$$\lim_{n\to\infty}(h(w_n)+l(w_n))=0,$$

i.e.,

$$\lim_{n\to\infty}(h(w_n))=0 \text{ and } \lim_{n\to\infty}(l(w_n))=0.$$

If w is a weak cluster point of $\{w_n\}$, then there exists a subsequence $\{w_{n_j}\}$ which weakly converges to w. Using similar arguments as in the proof of Theorem **3**.1, we can show that

(1) $0\in\partial f(w)$ such that $0\in\partial_P g(Aw)$,
(2) $\|Ty_n-y_n\|\to 0, n\to\infty$,
(3) $\lim_{n\to\infty}\|w_n-y_n\|=0=\lim_{n\to\infty}\|x_n-y_n\|$,
(4) $\bar{w}\in\Omega_1\cap F(T)$.

Similarly we can get that $\lim_{n\to\infty}\|x_n-\bar{w}\|=0$, i.e., $\{x_n\}$ converges strongly to $\bar{w}$. This completes the proof. □

6. Numerical results

In this section, we give a numerical example to compare Algorithm **3**.1 with Algorithm (**1**.5) of Khuangsatung *et al*. [14] in an infinite dimensional Hilbert space.

Example 6.1. *Let $E_1=E_2=L_2([0,1])$ with inner product given as $\langle x,y\rangle=\int_0^1 x(t)y(t)dt$, $\forall x,y\in L_2([0,1])$ and norm $\|x\|:=(\int_0^1|x(t)|^2dt)^{\frac{1}{2}}$, $\forall x,y\in L_2([0,1])$. Let $C=\{x\in L_2([0,1]):\langle x,a\rangle\le b\}$, where $a=e^t, b=2$ and $Q=\{x\in L_2([0,1]):\|x-d\|^2\le r\}$, where $d=t^2+1, r=2$.*
We first recall that the projections on sets C and Q are given by:

$$\operatorname{prox}_{\lambda\mu_n f}=P_C(x)=\begin{cases}x, & \text{if } x\in C,\\ \frac{b-\langle a,x\rangle}{\|a\|_2^2}a+x, & \text{otherwise},\end{cases}$$

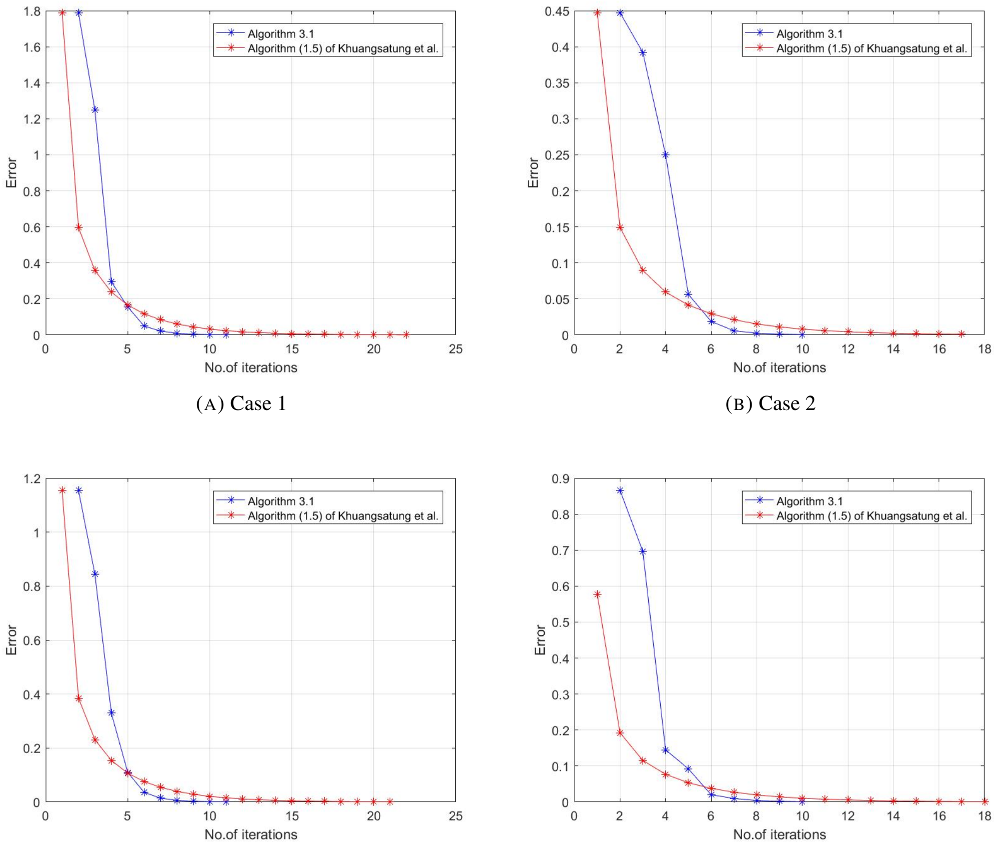
and

$$\operatorname{prox}_{\lambda g}=P_Q(x)=\begin{cases}x, & \text{if } x\in Q,\\ d+r\frac{x-d}{\|x-d\|}, & \text{otherwise},\end{cases}$$

where $f=i_C$ and $g=i_Q$, the indicator functions on C and Q respectively. Let $A\colon L_2([0,1])\to L_2([0,1])$ be a bounded linear operator defined by $Ax(t)=\frac{x(t)}{2}$. Let $S,B,\psi:L_2([0,1])\to L_2([0,1])$ be defined by $Sx(t)=\frac{x(t)}{2}$, $Bx(t)=1$, $\psi x(t)=\frac{x(t)}{3}$ for all $x\in E_1$. Take $\rho_n=1=\gamma$, $\alpha_n=\frac{1}{4n}$, $\beta_n=\frac{n}{2n+1}=\theta_n$ for $n\ge 1$, then the conditions in Theorem **3**.*1 are satisfied. We now consider the following four cases.*

Case 1: *Take $x_0(t)=3e^t$, $x_1(t)=e^t$,*
Case 2: *Take $x_0(t)=t+1$, $x_1(t)=t^2$,*
Case 3: *Take $x_0(t)=2e^t$, $x_1(t)=2t$,*
Case 4: *Take $x_0(t)=e^t+t$, $x_1(t)=\frac{2}{3}t$.*

For these cases, our Algorithm **3**.*1 is compared with Algorithm* **1**.*5 of Khuangsatung et al.* [14]. *We plot the graphs of Error $=\|x_{n+1}-x_n\|$ against number of iterations with stopping criteria $|x_{n+1}-x_n|<10^{-3}$. The graphs show that our algorithm works well and converges faster than the Algorithm* **1**.*5 and have computative advantages over the algorithm of Khuangsatung et al.* [14]. *The numerical results are displayed in Figure 1.*



(A) Case 1 (B) Case 2

(C) Case 3 (D) Case 4

FIGURE 1

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