A TRANSFER THEOREM OF THE CONTINUOUS TRACE AND TYPE I CROSSED PRODUCT OF A GROUPOID BY A BUNDLE OF C^*-ALGEBRAS

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The purpose of this paper is to give equivalence conditions of the groupoid dynamical systems $(\mathcal{A}, G, \alpha)$ *and* $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ *where G is a topological, locally compact, second countable groupoid with a Haar measures system,* $\mathcal{A}$ *is a bundle of C*-algebras indexed by the unit space of G,* α *is a continuous homomorphism from G to Iso(*$\mathcal{A}$*),* Γ *is the subgrupoid of stabilizers of G, and* $\alpha/_{\Gamma}$ *is the restriction of* α *to* Γ. *This equivalence is then used to transfer the property of being continuous trace or type I C*-algebra between* $C^*(G, \mathcal{A})$ *and* $C^*(\Gamma, \mathcal{A})$.

Keywords: C*-algebras, crossed product algebras, topological groupoids, Morita equivalence

1. Introduction

Known for their frequent use in quantum mechanics, C^*-algebras are an important tool in describing physical systems and the possible states of these systems. Special cases of C^*-algebras are type I C^*-algebra and continuous trace C^*-algebra. Part of this large research domain of C^*-algebras is the study of different kind of crossed products associated to a group, group transformation or groupoid and the study of the conditions under which these crossed products are type I or continuous trace C^*-algebra. Since Morita equivalence preserves the property of being type I or continuous trace C^*-algebra a tool in establishing such conditions as mentioned above is to transfer the property of being type I or continuous trace C^*-algebra between a smaller crossed product or group (groupoid) algebra and the entire crossed product or group (groupoid) algebra. For example, in [1, Proposition 7.29] D.Williams shows that in the hypothesis of regularity of the dynamical system $(C_0(X), G, lt)$, where $lt : G \to Aut(C_0(X))$ is $lt_g f(x) = f(g^{-1}x)$, the crossed product $C_0(X) \times_{lt} G$ is a type I C^*-algebra if and only if the group algebra associated to every stability group G_x is a type I C^*-algebra. Also, in [2, Theorem 2.7] it is shown that, if the stability groups vary

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continuously, every compact set of base space X is G-wandering and the C^*-algebra associated to all stability groups is a continuous trace C^*-algebra, then the C^*-algebra associated to a group transformation, (G, X), is a continuous trace C^*-algebra.

In this context, the author of this paper showed in [3, Theorem 4] that in the hypothesis that the stability groups vary continuously, and that if g_1 and g_2^{-1} are composable then $g_1 g_2^{-1}$ is an element of the subgroupoid of stability groups and the grupoid algebra associated to the subgroupoid of stability groups is a continuous trace C^*-algebra, it follows that the entire groupoid algebra is a continuous C^*-algebra. The main purpose of this paper is to extend this result to the case of the crossed product of a locally compact groupoid by a bundle of C^*-algebras. The construction of this crossed product is described by Renault in [4]. For this purpose, we will use the notion of equivalent groupoid dynamical systems, notion described in [4, Definition 5.3] and the fact that the equivalent groupoid dynamical systems determine Morita equivalent crossed products, [4, Corollaire 5.4]. We will show in Theorem 3.1.1 that in the first and second hypothesis described in [3, Theorem 4] the groupoid dynamical systems $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$, where G is a topological, locally compact, second countable groupoid with Haar measure system $\{\lambda^u\}_{u \in G^{(0)}}$, Γ is the subgroupoid of stability groups, $\mathcal{A}$ is a bundle of separable C^*-algebras indexed by unit space $G^{(0)}$, $\alpha : G \to Iso(\mathcal{A})$ is a continuous homomorphism and $\alpha/_{\Gamma}$ is the restriction of α on Γ, are equivalent. Moreover, in Corollary 3.1.2, we show that the crossed products $C^*(G, \mathcal{A})$ and $C^*(\Gamma, \mathcal{A})$ are Morita equivalent, and the properties of being type I or continuous trace C^*-algebra can be transfered from $C^*(\Gamma, \mathcal{A})$ to $C^*(G, \mathcal{A})$.

2. Preliminaries

In this paper, we used the general notions concerning groupoids and groupoid dynamical systems as given in [4], [5] and [6]. We also assume that all groupoids have a Haar measures system. The important notion of equivalence of groupoid dynamical systems is used as in [4, Definition 5.3] and the Morita equivalence of the crossed products obtained from two equivalent groupoid dynamical systems is given by Renault's Equivalence Theorem [4, Corrolary 5.4]. Moreover, we used as in [3, Theorem 4] the topologically equivalence of a groupoid G and the subgroupoid of stability groups Γ.

3. The main results

THEOREM 3.1.1 *Let* $(\mathcal{A}, G, \alpha)$ *be a groupoid dynamical system. If the following conditions hold:*

a) *the stability groups vary continously;*
b) *for every pair* $(g_1, g_2) \in G \times G$ *such that* $(g_1, g_2^{-1}) \in G^{(2)}$ *it follows that* $g_1 g_2^{-1} \in \Gamma$,

then we can form the groupoid dynamical system $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ *where* Γ *is the subgroupoid of stability groups and* $\alpha/_{\Gamma}$ *the restriction of* α *to* Γ. *Moreover,* $(\mathcal{A}, G, \alpha)$ *and* $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ *are equivalent as groupoid dynamical systems.*

Proof The condition that the stability groups vary continously insures that Γ has its own Haar measures system $\{\nu^u\}_{u \in \Gamma^{(0)}}$ (in fact for every $u \in \Gamma^{(0)}$, ν^u is the Haar measure of the stability group $G/\{u\}$) and, because the unit space of Γ coincides with the unit space of $G^{(0)}$, it makes sense to form the groupoid dynamical system $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$. In the following sentences, for brevity, $\alpha/_{\Gamma}$ will be simply denoted by α, and the elements from Γ by γ, hence $\alpha/_{\Gamma}(\gamma)$ will be $\alpha(\gamma)$. Let us recall here that in [3, Theorem 4], the second condition from this Theorem was used to obtain a topologically equivalence between Γ and *G,* via the space *G*. This equivalence has been obtained with respect to the following left action of Γ and right action of *G* on *G*. We have considered the surjections $p : G \to \Gamma^{(0)} = G^{(0)}$, $p(g) := r(g)$ (*r* the range map of *G*), $\sigma : G \to G^{(0)}$, $\sigma(g) := s(g)$ (*s* the source map of *G*), and the actions $\gamma \cdot g = \gamma g \in G$, $\gamma \in \Gamma, g \in G$, $s(\gamma) = p(g)$ and $g_1 \cdot g_2 = g_1 g_2$ for $g_1, g_2 \in G, \sigma(g_1) = r(g_2)$.

Since the conditions from the hypothesis of the theorem insure there exists a topologically equivalence between groupoids *G* and Γ via *G*, we will obtain the equivalence from [4, Definition 5.3] of groupoid dynamical systems $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ in the following way:

We consider that the Banach bundle $\mathcal{V}$ indexed by *G* will be the bundle $r^*(\mathcal{A}) = \{(g, a) / a \in A(r(g))\}$ with the canonical surjection $t : r^*(\mathcal{A}) \to G$, $t(g, a) = g$. A fibre of $r^*(\mathcal{A})$ has the form $(g, A(r(g))$ and a structure of

$A(p(g))-A(\sigma(g))$-imprimitivity bimodule, or taking into account above considerations $A(r(g))-A(s(g))$-imprimitivity bimodule, will be defined as follows:

- the left action of $A(r(g))$ on $(g,A(r(g))$ will be $b\cdot(g,a):=(g,ba)$.

Since b and a are in the fiber $A(r(g))$, this action is corectly defined.

- the right action of $A(s(g))$ on $(g,A(r(g))$ will be

$(g,a)\cdot c:=(g,a\alpha_g(c))$. Since $\alpha_g:A(s(g))\to A(r(g))$ is an isomorphism of C^*-algebras, $\alpha_g(c)$ denotes the image of c in $A(r(g))$ and this action is corectly defined.

- the inner product $_{A(r(g))}\langle(g,a);(g,b)\rangle:=ab^*$. Since $a,b\in A(r(g))$ implies $ab^*\in A(r(g))$, this inner product is corectly defined as an element of $A(r(g))$.

- the inner product $\langle(g,a);(g,b)\rangle_{A(s(g))}:=\alpha_g^{-1}(a^*b)$. Since $\alpha_g^{-1}(a^*b)$ is the image in $A(s(g))$ of the element a^*b from $A(r(g))$, this inner product is corectly defined as an element of $A(r(g))$.

Let us check now that $(g,A(r(g))$ is a left $A(r(g))$-Hilbert module, and a right $A(s(g))$-Hilbert module respectively, with respect to the vector space structure induced by $A(r(g))$:
$(g,a)+(g,b)=(g,a+b), \lambda(g,a)=(g,\lambda a)$, for every $g\in G,\ a,b\in A(r(g)),\lambda$ scalar.

For $c\in A(r(g))$ we have to show that: $c\,{}_{A(r(g))}\langle(g,a);(g,b)\rangle={}_{A(r(g))}\langle c\cdot(g,a);(g,b)\rangle$, $\forall a,b\in A(r(g)),\ g\in G$.

Indeed, $c\,{}_{A(r(g))}\langle(g,a);(g,b)\rangle=cab^*$ and ${}_{A(r(g))}\langle c\cdot(g,a);(g,b)\rangle=$
$={}_{A(r(g))}\langle(g,ca);(g,b)\rangle=cab^*$.

We show now that ${}_{A(r(g))}\langle(g,a);(g,b)\rangle={}_{A(r(g))}\langle(g,b);(g,a)\rangle^*$.

Indeed ${}_{A(r(g))}\langle(g,a);(g,b)\rangle=ab^*$ and ${}_{A(r(g))}\langle(g,b);(g,a)\rangle^*=(ba^*)^*=$
$=(a^*)^*b^*=ab^*$.

Moreover ${}_{A(r(g))}\langle(g,a);(g,a)\rangle=aa^*$ is a positive element of $A(r(g))$, and $aa^*=0$ implies $//aa^*//=//a//^2=0$, hence $a=0$.The liniarity in the first argument of inner product follows from the properties of addition and scalar multiplication of the elements of the C^*-algebra $A(r(g))$. Indeed: ${}_{A(r(g))}\langle \lambda(g,a)+\mu(g,b);(g,c)\rangle = {}_{A(r(g))}\langle (g,\lambda a+\mu b);(g,c)\rangle = (\lambda a+\mu b)c^* =$

$$= \lambda ac^* + \mu bc^* = \lambda_{A(r(g))}\langle (g,a);(g,c)\rangle + \mu_{A(r(g))}\langle (g,b);(g,c)\rangle .$$

Let us check that $(g, A(r(g))$ is a right $A(s(g))$-Hilbert module.

We have to show that $\langle (g,a);(g,b)\rangle_{A(s(g))} c = \langle (g,a);(g,b)\cdot c\rangle_{A(s(g))}$, for every $g \in G, a,b \in A(r(g)), c \in A(s(g))$.

Indeed $\langle (g,a);(g,b)\rangle_{A(s(g))} c = \alpha_g^{-1}(a^* b)c$ and

$$\langle (g,a);(g,b)\cdot c\rangle_{A(s(g))} = \langle (g,a);(g,b\alpha_g(c)\rangle_{A(s(g))} = \alpha_g^{-1}(a^* b\alpha_g(c)) =$$

$$= \alpha_g^{-1}(a^* b)\alpha_g^{-1}\alpha_g(c) = \alpha_g^{-1}(a^* b)c .$$

Then $\langle (g,a);(g,b)\rangle_{A(s(g))} = \langle (g,b);(g,a)\rangle^*_{A(s(g))}$.

$\langle (g,a);(g,b)\rangle_{A(s(g))} = \alpha_g^{-1}(a^* b)$;

$\langle (g,b);(g,a)\rangle^*_{A(s(g))} = (\alpha_g^{-1}(b^* a))^* = \alpha_g^{-1}((b^* a)^*) = \alpha_g^{-1}(a^*(b^*)^*) = \alpha_g^{-1}(a^* b)$.

$\langle (g,a);(g,a)\rangle_{A(s(g))} = \alpha_g^{-1}(a^* a) = \alpha_g^{-1}(a)^*\alpha_g^{-1}(a)$ and $\alpha_g^{-1}(a)^*\alpha_g^{-1}(a)$ is a positive element of $A(s(g))$. Moreover, since α_g is an isomorphism of C^*-algebras, we deduce in a similar way as in the case of the left module that $\alpha_g^{-1}(a^* a) = 0$ implies $a = 0$. The liniarity the second argument of this inner product can be proved in similar way as in the case of the left module.

We show now that $(g, A(r(g))$ is a $A(r(g))$- $A(s(g))$-imprimitivity bimodule.

We have:

$\langle a\cdot(g,b);(g,c)\rangle_{A(s(g))} = \langle (g,ab);(g,c)\rangle_{A(s(g))} = \alpha_g^{-1}((ab)^* c) = \alpha_g^{-1}(b^* a^* c)$ and

$\left\langle (g,b);a^*\cdot(g,c)\right\rangle_{A(s(g))} = \left\langle (g,b);(g,a^* c)\right\rangle_{A(s(g))} = \alpha_g^{-1}(b^* a^* c)$.

Hence $\langle a\cdot(g,b);(g,c)\rangle_{A(s(g))} = \left\langle (g,b);a^*\cdot(g,c)\right\rangle_{A(s(g))}$ for every $a,b,c \in A(r(g))$, $g \in G$. (1)

We have ${}_{A(r(g))}\langle (g,b)\cdot a;(g,c)\rangle = {}_{A(r(g))}\langle (g,b\alpha_g(a));(g,c)\rangle = b\alpha_g(a)c^*$ and

$${}_{A(r(g))}\left\langle (g,b);(g,c)\cdot a^{*}\right\rangle = {}_{A(r(g))}\left\langle (g,b);(g,c\alpha_{g}(a^{*}))\right\rangle = b(c\alpha_{g}(a^{*}))^{*} = b\alpha_{g}(a)c^{*}.$$

Hence ${}_{A(r(g))}\langle (g,b)\cdot a;(g,c)\rangle = {}_{A(r(g))}\left\langle (g,b);(g,c)\cdot a^{*}\right\rangle$, for every $a \in A(s(g))$, $g \in G,\ b,c \in A(r(g))$. (2)

We have ${}_{A(r(g))}\langle (g,a);(g,b)\rangle \cdot (g,c) = (ab^{*})\cdot (g,c) = (g,ab^{*}c)$ and

$(g,a)\cdot \langle (g,b);(g,c)\rangle_{A(s(g))} = (g,a)\cdot \alpha_{g}^{-1}(b^{*}c) = (g,a\alpha_{g}(\alpha_{g}^{-1}(b^{*}c))) = (g,ab^{*}c)$.

Hence ${}_{A(r(g))}\langle (g,a);(g,b)\rangle \cdot (g,c) = (g,a)\cdot \langle (g,b);(g,c)\rangle_{A(s(g))}$, for every $g \in G,\ a,b,\ c \in A(r(g))$. (3)

From (1), (2) and (3) we conclude that $(g,A(r(g))$ is a $A(r(g))$-$A(s(g))$-imprimitivity bimodule.

In the following sentences, we define a left action of Γ and a right action of G on $r^{*}(\mathcal{A})$ and we show that these actions comute and fulfill conditions from [4, Definition 5.3].

Considering the surjection from [5, Definition 2.12], namely $p_{\mathcal{A}} : r^{*}(\mathcal{A}) \to \Gamma^{(0)} = G^{(0)}$, $p_{\mathcal{A}}(g,a) := r(g)$, the set of composable elements will be $\Gamma * r^{*}(\mathcal{A}) = \{(\gamma,(g,a)) / p_{\mathcal{A}}(g,a) = s_{\Gamma}(\gamma)\}$, the left action of Γ on $r^{*}(\mathcal{A})$ will be $\gamma\cdot (g,a) = (\gamma\cdot g,\alpha_{\gamma}(a))$. But, from the considerations from the beginning of this proof, $\gamma\cdot g = \gamma g$ (the multiplication from *G*), hence $(\gamma\cdot g,\alpha_{\gamma}(a)) = (\gamma g,\alpha_{\gamma}(a))$. Since $p_{\mathcal{A}}(g,a) = r(g) = s_{\Gamma}(\gamma)$, γ and g are composable in G, and since $\alpha_{\gamma}(a)$ denotes an element from $A(r_{\Gamma}(\gamma)) = A(r_{\Gamma}(\gamma g))$, the above definition makes sense.

Considering the surjection $\sigma_{\mathcal{A}} : r^{*}(\mathcal{A}) \to G^{(0)}$, $\sigma_{\mathcal{A}}(g,a) := s(g)$, $r^{*}(\mathcal{A}) * G = \{((g_1,a),g_2) / \sigma_{\mathcal{A}}(g_1,a) = r(g_2)\}$, we define $(g_1,a)\cdot g_2 =$ $= (g_1\cdot g_2,a) = (g_1 g_2,a)$. Since $\sigma_{\mathcal{A}}(g_1,a) = s(g_1) = r(g_2)$ and $a \in A(r(g_1 g_2)) =$ $= A(r(g_1))$, the definition makes sense.

We show that these actions commute. That means: $(\gamma\cdot (g_1,a))\cdot g_2 = \gamma\cdot ((g_1,a)\cdot g_2)$, for every $\gamma \in \Gamma, g_1,g_2 \in G, a \in A(r(g_1))$, such that $p_{\mathcal{A}}(g_1,a) = r(g_1) = s_{\Gamma}(\gamma)$, $\sigma_{\mathcal{A}}(g_1,a) = s(g_1) = r(g_2)$.

We have $(\gamma\cdot (g_1,a))\cdot g_2 = (\gamma g_1,\alpha_{\gamma}(a))\cdot g_2$. Since $\sigma_{\mathcal{A}}(\gamma g_1,\alpha_{\gamma}(a)) = s(\gamma g_1) =$

$= s(g_1) = r(g_2)$, the elements $(\gamma g_1, \alpha_\gamma(a))$ and g_2 are composable and $(\gamma g_1, \alpha_\gamma(a)) \cdot g_2 = (\gamma g_1 g_2, \alpha_\gamma(a))$. On the other hand, $\gamma \cdot ((g_1, a) \cdot g_2) =$ $= \gamma \cdot (g_1 g_2, a)$. Since $p_{\mathcal{A}}(g_1 g_2, a) = r(g_1 g_2) = r(g_1) = s_\Gamma(\gamma)$, the elements γ and $(g_1 g_2, a)$ are composable, and $\gamma \cdot (g_1 g_2, a) = (\gamma g_1 g_2, \alpha_\gamma(a))$. Hence the actions commute. Since these actions are given as multiplications of elements from topological groupoids and α_g is an isomorphism of C^*-algebras, they are continuous. The similar arguments, adding that multiplication of elements from a C^*-algebra is continuous, prove the first condition from [4, Definition 5.3].

Now we prove the next conditions from [4, Definition 5.3]:

- the equivariance of the bundle map *t* at the groupoid actions:

We have to show $t(\gamma \cdot (g,a)) = \gamma \cdot t(g,a)$, for every $\gamma \in \Gamma$, $g \in G, s_\Gamma(\gamma) = r(g)$, $a \in A(r(g))$ and $t((g_1,a) \cdot g_2) = t(g_1,a) \cdot g_2$, for every $g_1, g_2 \in G, s(g_1) = r(g_2)$, $a \in A(r(g_1))$.

Indeed $t(\gamma \cdot (g,a)) = t(\gamma g, \alpha_\gamma(a)) = \gamma g = \gamma \cdot g = \gamma \cdot t(g,a)$ and

$t((g_1,a) \cdot g_2) = t(g_1 g_2, a) = g_1 g_2 = g_1 \cdot g_2 = t(g_1,a) \cdot g_2$.

- the compatibility of the groupoid actions with the inner products of imprimitivity bimodule structure:

${}_{A(r(g))}\langle \gamma \cdot (g,a); \gamma \cdot (g,b) \rangle = \alpha_\gamma({}_{A(r(g))}\langle (g,a);(g,b) \rangle)$, for every $\gamma \in \Gamma, g \in G$, $s(\gamma) = r(g)$, $a, b \in A(r(g))$ (4);

$\langle (g_1,a) \cdot g_2; (g_1,b) \cdot g_2 \rangle_{A(s(g_1))} = \alpha_{g_2}^{-1}(\langle (g_1,a);(g_1,b) \rangle_{A(s(g_1))})$, for every $g_1, g_2 \in G,\ s(g_1) = r(g_2),\ a, b \in A(r(g_1))$ (5).

Concerning the computations from (4) and (5), some remarks have to be made. Since $\gamma \cdot (g,a) = (\gamma g, \alpha_\gamma(a))$, respectively $\gamma \cdot (g,b) = (\gamma g, \alpha_\gamma(b))$, they are elements contained in the fibre $(\gamma g, A(r(\gamma g))) = (\gamma g, A(r(\gamma)))$. But, since $\gamma \in \Gamma$, $r(\gamma) = s(\gamma)$. Moreover $r(\gamma) = s(\gamma) = r(g)$, and we conclude that $(\gamma g, \alpha_\gamma(a))$ and $(\gamma g, \alpha_\gamma(b))$ are in the fibre $(\gamma g, A(r(g)))$. It makes sense that the inner product ${}_{A(r(g))}\langle \gamma \cdot (g,a); \gamma \cdot (g,b) \rangle$ is associated to fibre $(\gamma g, A(r(g)))$. On the right hand side of the equality (4), the inner product ${}_{A(r(g))}\langle (g,a);(g,b) \rangle$ is in $A(r(g) = A(s(\gamma))$, and its image through α_γ is contained in $A(r(\gamma)) = A(s(\gamma)) = A(r(g))$.

In the right hand side of the equality (5), the inner product is an element of $A(s(g_1)) = A(r(g_2))$ (g_1 and g_2 are composable) and its image through $\alpha_{g_2}^{-1}$ will be in $A(s(g_1))$, similar to the inner product from left hand side.

We have ${}_{A(r(g))}\langle \gamma \cdot (g,a); \gamma \cdot (g,b) \rangle = {}_{A(r(g))}\langle (\gamma g, \alpha_\gamma(a)); (\gamma g, \alpha_\gamma(b)) \rangle =$
$\alpha_\gamma(a)\alpha_\gamma(b)^* = \alpha_\gamma(ab^*)$;

$\alpha_\gamma({}_{A(r(g))}\langle (g,a);(g,b) \rangle) = \alpha_\gamma(ab^*)$, and it results (4).

We have

$\langle (g_1,a)\cdot g_2; (g_1,b)\cdot g_2 \rangle_{A(s(g_1))} = \langle (g_1 g_2, a); (g_1 g_2, b) \rangle_{A(s(g_1))} = \alpha_{g_1 g_2}^{-1}(a^* b)$;

$\alpha_{g_2}^{-1}(\langle (g_1,a);(g_1,b) \rangle_{A(s(g_1))}) = \alpha_{g_2}^{-1}(\alpha_{g_1}^{-1}(a^* b)) = \alpha_{g_2^{-1} g_1^{-1}}(a^* b) = \alpha_{g_1 g_2}^{-1}(a^* b)$

and it results (5).

- the compatibility of the groupoid actions with the actions of imprimitivity bimodule structure.

We have to prove: $\gamma \cdot (a \cdot (g,b)) = \alpha_\gamma(a) \cdot (\gamma \cdot (g,b))$, for every $\gamma \in \Gamma, g \in G$, $s(\gamma) = r(g), a, b \in A(r(g))$ (6)

and

$((g_1,a)\cdot b)\cdot g_2 = ((g_1,a)\cdot g_2)\cdot \alpha_{g_2}^{-1}(b), g_1, g_2 \in G, s(g_1) = r(g_2), a \in A(r(g_1)),$
$b \in A(s(g_1))$ (7)

Indeed, $\gamma \cdot (a \cdot (g,b)) = \gamma \cdot (g, ab) = (\gamma g, \alpha_\gamma(ab))$;

$\alpha_\gamma(a) \cdot (\gamma \cdot (g,b)) = \alpha_\gamma(a) \cdot (\gamma g, \alpha_\gamma(b)) = (\gamma g, \alpha_\gamma(a)\alpha_\gamma(b)) = (\gamma g, \alpha_\gamma(ab))$ and it results (6).

We have $((g_1,a)\cdot b)\cdot g_2 = (g_1, a\alpha_{g_1}(b))\cdot g_2 = (g_1 g_2, a\alpha_{g_1}(b))$;

$((g_1,a)\cdot g_2)\cdot \alpha_{g_2}^{-1}(b) = (g_1 g_2, a)\cdot \alpha_{g_2}^{-1}(b) = (g_1 g_2, a\alpha_{g_1 g_2}(\alpha_{g_2}^{-1}(b))) =$
$= (g_1 g_2, a\alpha_{(g_1 g_2)g_2^{-1}}(b)) = (g_1 g_2, a\alpha_{g_1 r(g_2)}(b)) = (g_1 g_2, a\alpha_{g_1}(b))$ and it results (7).

- the comutativity of the groupoid actions with the actions of imprimitivity bimodule structure

We have to prove:

$\gamma \cdot ((g,a)\cdot b) = (\gamma \cdot (g,a))\cdot b$; for every $\gamma \in \Gamma, g \in G, s(\gamma) = r(g), a \in A\,(r(g)),$
$b \in A(s(g))$ (8)

$(a \cdot (g_1,b))\cdot g_2 = a \cdot ((g_1,b)\cdot g_2)$, for every $g_1, g_2 \in G, s(g_1) = r(g_2),$

$a,b \in A(r(g_1))$ (9).

Indeed

$\gamma \cdot ((g,a)\cdot b) = \gamma \cdot (g, a\alpha_g(b)) = (\gamma g, \alpha_\gamma(a\alpha_g(b)) = (\gamma g, \alpha_\gamma(a)\alpha_\gamma(\alpha_g(b))) =$

$= (\gamma g, \alpha_\gamma(a)\alpha_{\gamma g}(b))$;

$(\gamma \cdot (g,a))\cdot b = (\gamma g, \alpha_\gamma(a))\cdot b = (\gamma g, \alpha_\gamma(a)\alpha_{\gamma g}(b))$ and it results (8).

We have $(a\cdot(g_1,b))\cdot g_2 = (g_1, ab)\cdot g_2 = (g_1 g_2, ab)$;

$a\cdot((g_1,b)\cdot g_2) = a\cdot(g_1 g_2, b) = (g_1 g_2, ab)$ and it results (9).

Since we have checked all conditions from [4, Definition 5.3], the groupoid dynamical systems $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ are equivalent.

COROLLARY 3.1.2 *Let* $(\mathcal{A}, G, \alpha)$ *be a groupoid dynamical system and* Γ *the stabilizers subgroupoid of G. If the following conditions are satisfied:*

a) the stability groups vary continously;

b) for every pair $(g_1, g_2) \in G\times G$ *such that* $(g_1, g_2^{-1}) \in G^{(2)}$ *it follows that* $g_1 g_2^{-1} \in \Gamma$,

then $C^*(G,\mathcal{A})$ *şi* $C^*(\Gamma,\mathcal{A})$ *are Morita equivalent. Moreover, if* $C^*(\Gamma,\mathcal{A})$ *is a continous trace* C^**-algebra or type I* C^**-algebra, then* $C^*(G,\mathcal{A})$ *will also be a continous trace* C^**-algebra or type I* C^**-algebra, respectively.*

Proof . By Theorem 3.1.1 $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ are equivalent dynamical systems, and by Renault's Equivalence Theorem [4, Corrolary 5.4] $C^*(G,\mathcal{A})$ şi $C^*(\Gamma,\mathcal{A})$ are Morita equivalent. The posibility to transfer the properties of being continous trace C^*-algebra or type I C^*-algebra between $C^*(\Gamma,\mathcal{A})$ and $C^*(G,\mathcal{A})$ is given by [1, Proposition I.42] and [2, Theorem 2.15].

PROPOSITION 3.1.3 *Let* $(\mathcal{A}, G, \alpha)$ *be a groupoid dynamical system such that G is a transitive groupoid. We consider* $G/\{u\}$ *the stability group of an element* $u \in G^{(0)}$ *and* $A(u)$ *the* C^**-algebra from* $\mathcal{A}$ *with index* $u \in G^{(0)}$. *If the group crossed product* $C^*(G/\{u\}, A(u))$ *obtained from the group dynamical system* $(A(u), G/\{u\}, \alpha/_{G/\{u\}})$ *is a continous trace* C^**-algebra or type I* C^**-algebra, then* $C^*(G,\mathcal{A})$ *will also be a continous trace* C^**-algebra or type I* C^**-algebra, respectively.*

Proof According to [5, Proposition 2.19], the condition of transitivity of groupoid G insures that for any unit $u \in G^{(0)}$, G and $G/\{u\}$ are topologically equivalent. In the same manner as in Theorem 3.1.1, we can show that $(A(u), G/\{u\}, \alpha/_{G/\{u\}})$ is equivalent to $(\mathcal{A}, G, \alpha)$, and we have the posibility to transfer the properties of being continous trace C^*-algebra or type I C^*-algebra between $C^*(G/\{u\}, A(u))$ and $C^*(G, \mathcal{A})$.

4. Conclusions

In this paper, we have studied the equivalence of certain groupoid dynamical systems. We have offered in Theorem 3.1.1 an equivalence of the groupoid dynamical systems $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}, \Gamma, \alpha/_{\Gamma})$ where G is a topological, locally compact, second countable groupoid G with the Haar measures system $\{\lambda^u\}_{u \in G^{(0)}}$, Γ is the subgroupoid of stability groups, $\mathcal{A}$ is a bundle of separable C^*-algebras indexed by $G^{(0)}$, $\alpha : G \to Iso(\mathcal{A})$ is a continuous homomorphism and $\alpha/_{\Gamma}$ is the restriction of α to Γ. In Corollary 3.1.2, using Renault's Equivalence Theorem we have showed that $C^*(G, \mathcal{A})$ and $C^*(\Gamma, \mathcal{A})$ are Morita equivalent and that we have the posibility to transfer the properties of being a continous trace C^*-algebra or type I C^*-algebra from $C^*(\Gamma, \mathcal{A})$ to $C^*(G, \mathcal{A})$.

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