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# ON THE INTERPOLATIVE $(\varphi, \psi)$-TYPE $\mathcal{Z}$-CONTRACTION

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*In this paper, we introduce the notions of interpolative $(\varphi, \psi)$-type $\mathcal{Z}$-contraction with respect to simulation function and quasi triangular $\theta$-orbital admissible mapping. Using these notions, some fixed point theorems are also established in the framework of metric space. An illustrative example is furnished to show that there exists a quasi triangular $\theta$-orbital admissible mapping which is not a triangular $\theta$-admissible. As an application of our result, we establish an existence of solution for a non-linear Fredholm integral equation.*

**Keywords:** interpolative $(\varphi, \psi)$-type $\mathcal{Z}$-contraction, altering distance function, comparison function, simulation function, Fredholm integral equation.

**MSC2020:** 47H10, 54H25.

## 1. Introduction and Preliminaries

The metric fixed point theory was initiated by Banach [5] with his pivotal result "contraction mapping principle" that was established for the existence and uniqueness of a solution of certain integral equations. A mapping $T : X \to X$ over a metric space $(X, d)$ is called contraction mapping, also known as Banach contraction, if there exists $k \in [0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq kd(x, y)$. A mapping that satisfies the "contraction mapping principle" is necessarily continuous and this was the main weakness of this theorem. It was considered whether the continuity condition is superfluous. An initial response to this question was given by Kannan [10] in 1968 affirmatively. Kannan [10] successfully introduced a new type of contraction mapping which is not necessarily continuous. A mapping $T : X \to X$ is called Kannan type contraction if there exists $0 \leq k < \frac{1}{2}$ such that for all $x, y \in X$, $d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty))$. In 1972, Chatterjea [8] also introduced a similar type of Kannan contraction. Mapping $T : X \to X$ is called Chatterjea type contraction if there exists $0 \leq k < \frac{1}{2}$ such that for all $x, y \in X$, $d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx))$. On the other hand, in 1976 Khan ([18], [19]) first used the idea of geometric mean of Kannan type contraction. A mapping $T : X \to X$ is called Khan type contraction if there exists $0 \leq k < 1$ such that for all $x, y \in X$, $d(Tx, Ty) \leq k(d(x, Tx).d(y, Ty))^{\frac{1}{2}}$. Recently, in 2018 Karapınar [11] revisited Kannan type contraction and introduced the concept of an interpolative Kannan type contraction, a more general form of Khan type contraction. A mapping $T : X \to X$ is said to be an interpolative Kannan contraction mapping if

$$d(Tx, Ty) \leq h[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{1-\alpha},$$

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for all $x, y \in X$ with $Tx \neq x$, where $h \in [0,1)$ and $\alpha \in (0,1)$. Note that if $\alpha = \frac{1}{2}$, then interpolative Kannan type contraction reduces to Khan type contraction.

**Theorem 1.1** ([11]). *Let $(X,d)$ be a complete metric space and $T$ be an interpolative Kannan type contraction. Then $T$ has a unique fixed point in $X$.*

Very recently Karapınar *et al.* [13] pointed out that the fixed point obtained in Theorem 1.1 [11] may not be unique. The refinement of Theorem 1.1 is stated as follows:

**Theorem 1.2** ([13]). *Let $(X,d)$ be a complete metric space. A self mapping $T : X \to X$ possesses a fixed point in $X$ if there exist constants $h \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$d(Tx,Ty) \leq h[d(x,Tx)]^{\alpha}.[d(y,Ty)]^{1-\alpha}$$

*for all $x, y \in X \setminus Fix(T)$, where $Fix(X) = \{w : Tw = w\}$.*

**Example 1.1** ([13]). *Let $X = \{0,1,2,3\}$ endow with Euclidean metric $d(x,y) = |x-y|$. Define $T : X \to X$ by $T0 = 0, T1 = 1, T2 = T3 = 1$. For all $x, y \in X \setminus Fix(T)$ $T$ satisfies Theorem 1.2, where $\alpha \in (0,1)$ and $h \in [0,1)$. Note that 0 and 1 are fixed points of $T$.*

Moreover, Karapınar *et al.* [13] also introduced the notion of interpolative Reich-Rus-Ćirić type contraction in the setting of partial metric space.

**Definition 1.1** ([13]). *Let $(X,p)$ be a partial metric space. A mappingT $: X \to X$ is called an interpolative Reich-Rus-Ćirić type contraction, if there exist $h \in [0,1)$, $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 + \alpha_2 < 1$ such that*

$$p(Tx,Ty) \leq h\Big[p(x,y)\Big]^{\alpha_1}.\Big[p(x,Tx)\Big]^{\alpha_2}.\Big[p(y,Ty)\Big]^{1-\alpha_1-\alpha_2}$$

*for all $x, y \in X \setminus Fix(T)$.*

**Theorem 1.3** ([13]). *Let $(X,p)$ be a complete partial metric space and $T : X \to X$ is an interpolative Reich-Rus-Ćirić type contraction, then $T$ has a fixed point in $X$.*

On the other hand, Karapınar *et al.*[14] introduced the notion of interpolative Hardy-Rogers type contraction by using the well known contraction of Hardy and Rogers [9].

**Definition 1.2** ([14]). *Let $(X,d)$ be a metric space. A self mapping $T : X \to X$ is said to be an interpolative Hardy-Rogers type contraction if there exist $h \in [0,1)$, $\alpha_i \in (0,1)$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that*

$$d(Tx,Ty) \leq hH(x,y), \text{ for all } x,y \in X \setminus Fix(T)$$

*where*

$$H(x,y) = [d(x,y)]^{\alpha_1}.[d(x,Tx)]^{\alpha_2}.[d(y,Ty)]^{\alpha_3}.\Big[\frac{1}{2}\Big(d(x,Ty)+d(y,Tx)\Big)\Big]^{1-\sum_{i=1}^{3}\alpha_i}.$$

In 1984, Khan *et al.* [20] introduced the notion of altering distance function.

**Definition 1.3.** [20] *A continuous function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance if it is non-decreasing and $\varphi(r) = 0$ if and only if $r = 0$.*

It is obvious that $\varphi(r) \geq 0$, for all $r \geq 0$. We denote $\Phi$, the set of all altering distance functions. We have the following examples on altering distance function.

**Example 1.2.** *Let $\varphi_i : [0,\infty) \to [0,\infty)$, where $i = 1,2$ be defined by:*
*(i) $\varphi_1(t) = e^{at} + bt - 1$;*
*(ii) $\varphi_2(t) = at^2 + \ln(bt+1)$, where $a, b > 0$.*

**Definition 1.4.** *(*[6], [7]*) A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a comparison function if it is monotonically increasing and $\psi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$.*

**Example 1.3.** *Let $\psi_i : [0, \infty) \to [0, \infty)$, where $i = 1, 2$ be defined by:*
*(i) $\psi_1(t) = kt$, where $k \in [0, 1)$, (ii) $\psi_2(t) = \frac{t}{2+t}$. Obviously $\psi_i$ is a comparison function, where $i = 1, 2$.*

If $\psi$ is comparison function, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. The symbol $\Psi$ denotes the set of all comparison functions. In 2015, Khojasteh *et al.* [21] introduced the notion of simulation function.

**Definition 1.5** ([21])**.** *A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function, if it satisfies the following conditions:*
$(\zeta_1)$ $\zeta(0, 0) = 0$;
$(\zeta_2)$ $\zeta(t, s) < s - t$, *for all* $t, s > 0$;
$(\zeta_3)$ *if* $\{t_n\}$, $\{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n > 0$, *then* $\lim_{n\to+\infty} \sup \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

**Definition 1.6** ([21])**.** *Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. If there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \text{ for all } x, y \in X,$$

*then $T$ is called $\mathcal{Z}$-contraction with respect to $\zeta$.*

In the same year, Argoubi *et al.* [4] refined the above notion by removing the first condition $(\zeta_1)$. Note that the condition $(\zeta_1)$ is indeed obtained from $(\zeta_2)$, if $T$ is a $\mathcal{Z}$-contraction with respect to $\zeta$. A basic example of $\mathcal{Z}$-contraction is Banach contraction, which is obtained by setting $\zeta(t, s) = \lambda s - t$, where $\lambda \in [0, 1)$.
In the sense of Argoubi *et al.* [4], we have the following:

**Definition 1.7** ([4])**.** *A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be simulation function if it satisfies the conditions $(\zeta_2)$ and $(\zeta_3)$.*

Clearly, any simulation function in the original Khojasteh *et al.* [21] sense is also a simulation function in sense of Argoubi *et al.* [4], but the converse is not true.

**Example 1.4** ([4])**.** *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{N}$ be a function defined by*

$$\zeta(t, s) = \begin{cases} 1, & \text{for } (t, s) = (0, 0); \\ \lambda s - t, & \text{otherwise}, \end{cases}$$

*where $\lambda \in (0, 1)$, then $\zeta$ satisfies $(\zeta_2)$ and $(\zeta_3)$ with $(\zeta(0, 0) > 0$.*

**Example 1.5.** *Let $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\psi(t) < t \leq \varphi(t)$, for all $t > 0$ and $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping defined by $\zeta(t, s) = \psi(s) - \varphi(t)$, for all $t, s \in [0, \infty)$. Then $\zeta$ is a simulation function.*

For further details and examples on simulation function, it can be found in [21, 3, 24, 16, 17, 2, 1]. Recently Karapınar[12] extended interpolative Hardy-Rogers type contraction using the simulation function $\zeta$ in the sense of Argoubi *et al.* [4] as follows:

**Definition 1.8** ([12])**.** *A mapping $T : X \to X$ is called an interpolative Hardy-Rogers type $\mathcal{Z}$-contraction with respect to $\zeta$ if there exist $\zeta \in \mathcal{Z}$, $\alpha_i \in (0, 1)$, where $i = 1, 2, 3$ such that $\alpha_1 + \alpha_2 + \alpha_3 < 1$ satisfying the inequality*

$$\zeta\Big(d(Tx, Ty), H(x, y)\Big) \geq 0, \text{ for all } x, y \in X \backslash Fix(T)$$

*where*
$$H(x, y) = [d(x, y)]^{\alpha_1}.[d(x, Tx)]^{\alpha_2}.[d(y, Ty)]^{\alpha_3}.\Big[\frac{1}{2}\Big(d(x, Ty) + d(y, Tx)\Big)\Big]^{1-\sum_{i=1}^{3}\alpha_i}.$$

**Remark 1.1.** *If $T$ is an interpolative Kannan type contraction(resp. Reich-Rus-Ćirić type contraction, Hardy-Rogers type contraction, $\mathcal{Z}$-contraction and Hardy-Rogers type $\mathcal{Z}$-contraction), then $T$ is continuous.*
*For this, consider a sequence $\{x_n\}$ in $X$ defined by $x_n = T^n x_0 = Tx_{n-1}$ for any $x_0 \in X$, where $n \geq 1$ such that $x_n \to w \in X$ as $n \to +\infty$. Suppose $T$ is an interpolative Kannan type contraction, then we have*

$$d(Tx_n, Tw) \leq h[d(x_n, Tx_n)]^{\alpha}.[d(w, Tw)]^{1-\alpha} = h[d(x_n, x_{n+1})]^{\alpha}.[d(w, Tw)]^{1-\alpha}.$$

*Letting limit as $n \to +\infty$, we have*

$$\lim_{n\to+\infty} d(Tx_n, Tw) = 0 \text{ yielding } \lim_{n\to+\infty} Tx_n = Tu.$$

*Similarly, one can prove that if $T$ is an interpolative Reich-Rus-Ćirić type(resp. Hardy-Rogers type) contraction, then $T$ is continuous. Further, suppose that $T$ is an interpolative Hardy-Rogers type $\mathcal{Z}$-contraction. By definition, we have*

$$0 \leq \zeta\Big(d(Tx_n, Tw), H(x_n, w)\Big) < H(x_n, w) - d(Tx_n, Tw),$$

*where, $x_n, w \in X \setminus Fix(T)$ and*

$$H(x_n, w) = [d(x_n, w)]^{\alpha_1}.[d(x_n, Tx_n)]^{\alpha_2}.[d(w, Tw)]^{\alpha_3}. \left[\frac{1}{2}\Big(d(x_n, Tw) + d(w, Tx_n)\Big)\right]^{1-\sum_{i=1}^{3}\alpha_i}.$$

*Letting limit as $n \to +\infty$, we obtain*

$$\lim_{n\to+\infty} d(Tx_n, Tw) < 0 \text{ yields } \lim_{n\to+\infty} Tx_n = Tw.$$

*Thus, we conclude that $T$ is continuous mapping if it is $\mathcal{Z}$-contraction or an interpolative Hardy-Rogers type $\mathcal{Z}$-contraction.*

Let $X$ be a non-empty set and $\theta : X \times X \to \mathbb{R}$. We collect the following concepts.

**Definition 1.9** ([25])**.** *A mapping $T : X \to X$ is said to be $\theta$- admissible if*

$$(\theta_1)\ \theta(x, y) \geq 1 \text{ implies } \theta(Tx, Ty) \geq 1,\ x, y \in X.$$

**Definition 1.10** ([15])**.** *A mapping $T : X \to X$ is said to be triangular $\theta$-admissible if it satisfies $(\theta_1)$ and*

$$(\theta_2)\ \theta(x, z) \geq 1 \text{ and } \theta(z, y) \geq 1 \text{ imply } \theta(x, y) \geq 1,\ x, y, z \in X.$$

Note that if $T$ fails to satisfy any one of the conditions $(\theta_1)$ and $(\theta_2)$, then $T$ is not a triangular $\theta$-orbital admissible.

**Definition 1.11** ([23])**.** *A mapping $T : X \to X$ is said to be $\theta$-orbital admissible if*

$$(\theta_3)\ \theta(x, Tx) \geq 1 \text{ implies } \theta(Tx, T^2x) \geq 1,\ x \in X.$$

**Definition 1.12** ([23])**.** *A mapping $T : X \to X$ is said to be triangular $\theta$-orbital admissible if $T$ satisfies $(\theta_3)$ and*

$$(\theta_4)\ \theta(x, y) \geq 1 \text{ and } \theta(y, Ty) \geq 1 \text{ imply } \theta(x, Ty) \geq 1,\ x, y \in X.$$

It is obvious that every triangular $\theta$-orbital admissible mapping $T$ is $\theta$-orbital admissible, but the converse is not true. For more details and examples on $\theta$-admissibility(resp. $\theta$-orbital admissibility), it may be refered to ([25], [23],[15], [22] and references therein).

**Definition 1.13** ([25],[23])**.** *Let $(X, d)$ be a metric space and $\theta : X \times X \to \mathbb{R}$ be a mapping. A sequence $\{x_n\}$ in $X$ is said to be $\theta$-regular if $\theta(x_n, x_{n+1}) \geq 1$, for each $n$ and $x_n \to w \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\theta(x_{n_k}, w) \geq 1$, for each $k \in \mathbb{N}$.*

**Remark 1.2.** *($i$) Popescu*[23] *observed that every $\theta$-admissible mapping is $\theta$-orbital admissible. Further, it has observed in Popescu*[23] *that every triangular $\theta$-admissible mapping is a triangular $\theta$-orbital admissible mapping, but the converse may not be true in general(see Example 7*[23]*).*
*($ii$) In our view, it may be observed that every $\theta$-admissible mapping $T$ is $\theta$-orbital admissible if there exist $w \in X$ and $k \in \mathbb{N} \cup \{0\}$ such that $\theta(x, y) = \theta(T^k w, T^{k+1} w) \geq 1$, otherwise $T$ is not $\theta$-orbital admissible(see Example 1.6 and Example 1.7). In fact by $(\theta_3)$, every $\theta$-orbital admissible mapping may not be $\theta$-admissible mapping(see Example 1.8).*

**Example 1.6.** *Let $X = \{0, 1, 2\}$ with usual metric $d(x, y) = |x - y|$. Let $T : X \to X$ and $\theta : X \times X \to \mathbb{R}$ be mappings defined by $T0 = 0, T1 = 2, T2 = 1$ and $\theta(x, y) = 1$ if $(x, y) \in \{(0, 0), (1, 2), (2, 1)\}$ and $\theta(x, y) = 0$, otherwise. Since $\theta(0, 0) = \theta(1, 2) = \theta(2, 1) = 1$, $\theta(T0, T0) = \theta(T1, T2) = \theta(T2, T1) = 1$. Therefore, $T$ is $\theta$-admissible mapping. On other hand, we have $\theta(0, T0) = \theta(1, T1) = \theta(2, T2) = 1$, $\theta(T0, T^2 0) = \theta(T1, T^2 1) = \theta(T2, T^2 2) = 1$, so $T$ is a $\theta$-orbital admissible. Note that $T$ satisfies neither $(\theta_2)$ nor $(\theta_4)$. Therefore, $T$ is neither triangular $\theta$-admissible nor triangular $\theta$-orbital admissible.*

**Example 1.7.** *Let $X$ and $T : X \to X$ are as in Example 1.6. Define $\theta : X \times X \to \mathbb{R}$ as: $\theta(x, y) = 1$, if $(x, y) \in \{(0, 1), (0, 2)\}$ and $\theta(x, y) = 0$, otherwise. Clearly, $T$ is a $\theta$-admissible as $\theta(0, 1) = \theta(T0, T1) = 1$, $\theta(0, 2) = \theta(T0, T2) = 1$. But, there does not exist $w \in X$ and $k \in \mathbb{N} \cup \{0\}$ such that $\theta(x, y) = \theta(T^k w, T^{k+1} w) = 1$, so $T$ is not a $\theta$-orbital admissible mapping.*

The concept of quasi triangular $\theta$-orbital admissible mapping is defined as follows:

**Definition 1.14.** *A mapping $T : X \to X$ is said to be quasi triangular $\theta$-orbital admissible if $T$ satisfies $(\theta_3)$ and*

$$(\theta_5)\; \theta(x, y) \geq 1 \text{ implies } \theta(x, Ty) \geq 1,\; x, y \in X.$$

Obviously, every triangular $\theta$-orbital admissible is quasi $\theta$-orbital admissible. In the following example we show that there exists a quasi triangular $\theta$-orbital admissible mapping, but not a triangular $\theta$-admissible mapping.

**Example 1.8.** *Let $X = \{0, 1, 2, 3\}$ with usual metric $d(x, y) = |x - y|$. Let $T : X \to X$ and $\theta : X \times X \to \mathbb{R}$ be mappings defined by*

$$T0 = 0, T1 = 2, T2 = 1, T3 = 3 \text{ and } \theta(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A, \\ 0, & \text{otherwise}, \end{cases}$$

*where, $A = \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (2, 3)\}$. Since $(1, 2), (2, 1) \in A$, then we have $\theta(1, T1) = \theta(T1, T^2 1) = \theta(2, 1) = 1$ and $\theta(2, T2) = \theta(T2, T^2 2) = \theta(1, 2) = 1$, so $T$ is $\theta$-orbital admissible mapping. Further, we have*

$$\begin{aligned} \theta(0, 1) &= \theta(0, T1) = \theta(0, 2) = 1,\; \theta(0, 2) = \theta(0, T2) = \theta(0, 1) = 1, \\ \theta(1, 2) &= \theta(1, T2) = \theta(1, 1) = 1,\; \theta(2, 0) = \theta(2, T0) = \theta(2, 0) = 1, \\ \theta(2, 1) &= \theta(2, T1) = \theta(2, 2) = 1,\; \theta(2, 3) = \theta(2, T3) = \theta(2, 3) = 1. \end{aligned}$$

*Therefore, $T$ satisfies $(\theta_5)$ and hence $T$ is quasi triangular $\theta$-orbital admissible mapping. Note that $\theta(x, y) = \theta(2, 0) = \theta(2, 3) = 1$, $\theta(y, Ty) = \theta(0, T0) = \theta(3, T3) = 0$, but $\theta(x, Ty) = \theta(2, T0) = \theta(2, T3) = 1$. It shows that the necessity of $\theta(y, Ty) \geq 1$ for $(\theta_4)$ is not required to satisfy $(\theta_5)$. On the other hand, we have $\theta(2, 0) = \theta(2, 3) = 1$, $\theta(T2, T0) = \theta(T2, T3) = 0$ as $(1, 0), (1, 3) \notin A$, so $T$ does not satisfy $(\theta_1)$. Moreover, we have $\theta(1, 2) = \theta(2, 3) = 1$, but $\theta(1, 3) = 0$, $T$ does not satisfy $(\theta_2)$ and hence $T$ is neither triangular $\theta$-admissible nor $\theta$-admissible mapping.*

**Lemma 1.1.** *Let $T : X \to X$ be a quasi triangular $\theta$-orbital admissible mapping. Assume that $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$. If there exists a sequence $\{x_n\}$ in $X$ such that $x_n = T^n x_0$, then $\theta(x_m, x_n) \geq 1$, $m > n$, for all $m, n \in \mathbb{N} \cup \{0\}$.*

*Proof.* By assumption there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$, then by $\theta$-orbital admissible of mapping $T$, we have $\theta(x_1, x_2) = \theta(Tx_0, T^2x_0) \geq 1$. By continuing in this process, we obtain $\theta(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Since $T$ is quasi triangular $\theta$-orbital admissible mapping and $\theta(x_n, x_{n+1}) \geq 1$ for $n \in \mathbb{N} \cup \{0\}$, then from $(\theta_5)$, we obtain that $\theta(x_n, x_{n+2}) = \theta(x_n, Tx_{n+1}) \geq 1$. By continuing this process repeatedly with $(\theta_5)$, we obtain that $\theta(x_n, x_m) \geq 1$, $m > n$ for all $m, n \in \mathbb{N} \cup \{0\}$. □

In this paper, using the notions of quasi triangular $\theta$-orbital admissible and interpolative $(\varphi, \psi)$-type $\mathcal{Z}$-contraction with respect to simulation function, we prove some fixed point theorems.

## 2. Interpolative $(\varphi, \psi)$-type $\mathcal{Z}$-contraction and fixed point theorems

In this section, we discuss $(\varphi, \psi)$-type $\mathcal{Z}$-contraction with respect to simulation function $\zeta$ in the sense of Argoubi *et al.*[4], using an interpolative approach in the setting of metric spaces. Let $T$ be a self mapping on a metric space $(X, d)$.

**Definition 2.1.** *A mapping $T : X \to X$ is called an interpolative $(\varphi, \psi)$-Banach-Kannan-Chatterjea type $\mathcal{Z}$- contraction with respect to $\zeta$ (in short, interpolative $(\varphi, \psi)$-BKC type $\mathcal{Z}$-contraction) if there exist $\theta : X \times X \to \mathbb{R}$, $\zeta \in \mathcal{Z}, \varphi \in \Phi, \psi \in \Psi$, $\alpha_1, \alpha_2 \in (0, 1)$ such that $\varphi(t) > \psi(t)$, for $t > 0$ and $\alpha_1 + \alpha_2 < 1$ satisfying the inequality*

$$\zeta\Big(\theta(x, y)\varphi(d(Tx, Ty)), \psi(B(x, y))\Big) \geq 0, \text{ for all } x, y \in X, \tag{1}$$

*where*

$$B(x, y) = [d(x, y)]^{\alpha_1}.\Big[\frac{1}{2}(d(x, Tx) + d(y, Ty))\Big]^{\alpha_2}.\Big[\frac{1}{2}(d(x, Ty) + d(y, Tx))\Big]^{1-\alpha_1-\alpha_2}.$$

**Definition 2.2.** *A mapping $T : X \to X$ is called an interpolative $(\varphi, \psi)$- Hardy-Rogers type $\mathcal{Z}$-contraction with respect to $\zeta$(in short, interpolative $(\varphi, \psi)$-HR type $\mathcal{Z}$-contraction) if there exist $\theta : X \times X \to \mathbb{R}$, $\zeta \in \mathcal{Z}, \varphi \in \Phi, \psi \in \Psi$, $\alpha_i \in (0, 1)$, where $i = 1, 2, 3$, such that $\varphi(t) > \psi(t)$, $t > 0$ and $\sum_{i=1}^{3} \alpha_i < 1$ satisfying the inequality*

$$\zeta\Big(\theta(x, y)\varphi(d(Tx, Ty)), \psi(H(x, y))\Big) \geq 0, \text{ for all } x, y \in X \backslash Fix(T), \tag{2}$$

*where*

$$H(x, y) = [d(x, y)]^{\alpha_1}.[d(x, Tx)]^{\alpha_2}.[d(y, Ty)]^{\alpha_3}.\Big[\tfrac{1}{2}\Big(d(x, Ty) + d(y, Tx)\Big)\Big]^{1-\sum_{i=1}^{3}\alpha_i}.$$

**Theorem 2.1.** *Let $T$ be a self-mapping on a complete metric space $(X, d)$. Suppose that $T$ is quasi triangular $\theta$-orbital admissible and forms an interpolative $(\varphi, \psi)$-BKC type $\mathcal{Z}$-contraction with respect to $\zeta$. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and $T$ is continuous, then $T$ has a fixed point in $X$.*

*Proof.* Let $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$. Consider the iterative sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$, for all $n \in \mathbb{N}$. If there exits $n_0 \in \mathbb{N}$ such that $x_{n_0-1} = x_{n_0}$, then the proof is over. Indeed, $x_{n_0-1}$ forms a fixed point since $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$. Consequently, throughout the proof we shall assume that $x_{n-1} \neq x_n$ and hence we have $d(x_{n-1}, x_n) > 0$, for all $n \in \mathbb{N}$. On the other hand, $\theta(x_0, x_1) = \theta(x_0, Tx_0) \geq 1$ and $T$ is $\theta$-orbital admissible mapping, we find $\theta(x_1, x_2) = \theta(Tx_0, T^2x_0) \geq 1$. Recursively, we derive that $\theta(x_{n-1}, x_n) \geq 1$, for all $n \in \mathbb{N}$. From (1), we obtain

$$\begin{aligned} 0 \leq\ & \zeta\Big(\theta(x_{n-1},x_n)\varphi(d(Tx_{n-1},Tx_n)),\psi(B(x_{n-1},x_n))\Big) \\ =\ & \zeta\Big(\theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})),\psi(B(x_{n-1},x_n))\Big) \\ <\ & \psi(B(x_{n-1},x_n)) - \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})), \end{aligned} \tag{3}$$

where

$$\begin{aligned} B(x_{n-1},x_n) =\ & \Big[d(x_{n-1},x_n)\Big]^{\alpha_1}.\Big[\tfrac{1}{2}\Big(d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)\Big)\Big]^{\alpha_2}. \\ & \Big[\tfrac{1}{2}\Big(d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})\Big)\Big]^{1-\alpha_1-\alpha_2} \\ =\ & \Big[d(x_{n-1},x_n)\Big]^{\alpha_1}.\Big[\tfrac{1}{2}\Big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\Big)\Big]^{\alpha_2}.\Big[\tfrac{1}{2}d(x_{n-1},x_{n+1})\Big]^{1-\alpha_1-\alpha_2}. \end{aligned} \tag{4}$$

Consequently, we arrive

$$\begin{aligned} \varphi(d(x_n,x_{n+1})) \leq\ & \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})) \\ <\ & \psi(B(x_{n-1},x_n)) \\ =\ & \psi\Big(\Big[d(x_{n-1},x_n)\Big]^{\alpha_1}.\Big[\tfrac{1}{2}\Big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\Big)\Big]^{\alpha_2}. \\ & \Big[\tfrac{1}{2}d(x_{n-1},x_{n+1})\Big]^{1-\alpha_1-\alpha_2}\Big) \\ \leq\ & \psi\Big(\Big[d(x_{n-1},x_n)\Big]^{\alpha_1}.\Big[\tfrac{1}{2}\Big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\Big)\Big]^{1-\alpha_1}\Big). \end{aligned} \tag{5}$$

Suppose $d(x_{n-1},x_n) < d(x_n,x_{n+1})$, for $n \geq 1$, then from (5), we obtain

$$\varphi(d(x_n,x_{n+1})) \leq \psi(d(x_n,x_{n+1})) < \varphi(d(x_n,x_{n+1}))$$

This is a contradiction. Accordingly, we obtain

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n), \text{ for all } n \geq 1. \tag{6}$$

Hence $\{d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of positive real numbers and bounded below by zero. So there exists $r \geq 0$ such that $\lim_{n\to+\infty} d(x_n,x_{n+1}) = r$. We claim that $r > 0$, otherwise from (3), (4) together with (6), we obtain

$$\begin{aligned} 0 \leq\ & \zeta\Big(\theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})),\psi(B(x_{n-1},x_n))\Big) \\ <\ & \psi(B(x_{n-1},x_n)) - \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \varphi(d(x_n,x_{n+1})) \leq\ & \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})) \leq \psi(B(x_{n-1},x_n)) \\ \leq\ & \varphi(B(x_{n-1},x_n)) \leq \varphi(d(x_{n-1},x_n)). \end{aligned} \tag{7}$$

Letting limit as $n \to +\infty$ in (7), we obtain

$$\lim_{n\to+\infty} \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})) = \lim_{n\to+\infty} \psi(B(x_{n-1},x_n)) = \varphi(r). \tag{8}$$

Setting $s_n = \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1}))$, $t_n = \psi(B(x_{n-1},x_n))$ in (3) and (4), then by $(\zeta_3)$ with (8), we obtain

$$0 \leq \lim_{n\to+\infty} \sup \zeta\Big(\theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})),\psi(B(x_{n-1},x_n))\Big) < 0.$$

This is a contradiction and thus we have $\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose not, there exists $\epsilon > 0$ for which we can find two sequences $\{m_k\}$ and $\{n_k\}$, for all $k \geq 1$ with $x_{m_k} > x_{n_k} \geq k$ such that $d(x_{n_k},x_{m_k}) \geq \epsilon$. Further, we assume that $m_k$ is the smallest number greater than $n_k$, then $d(x_{n_k},x_{m_k-1}) < \epsilon$. By triangular inequality, we obtain

$$\epsilon \leq d(x_{n_k},x_{m_k}) \leq d(x_{n_k},x_{m_k-1}) + d(x_{m_k-1},x_{m_k}) < \epsilon + d(x_{m_k-1},x_{m_k}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(x_{n_k}, x_{m_k}) = \epsilon. \tag{9}$$

Again by triangular inequality, we obtain

$$d(x_{n_k}, x_{m_k}) \le d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}).$$

Also we obtain

$$d(x_{n_k+1}, x_{m_k+1}) \le d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}).$$

Combining the above two inequalities and taking limit as $k \to +\infty$ together with (9), we obtain

$$\lim_{k\to+\infty} d(x_{n_k+1}, x_{m_k+1}) = \epsilon. \tag{10}$$

Furthermore, we obtain

$$d(x_{n_k}, x_{m_k}) \le d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k}) \le d(x_{n_k}, x_{m_k}) + 2d(x_{n_k}, x_{n_k+1}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(x_{n_k+1}, x_{m_k}) = \epsilon. \tag{11}$$

Similarly, we obtain

$$d(x_{n_k}, x_{m_k}) \le d(x_{n_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \le d(x_{n_k}, x_{m_k}) + 2d(x_{m_k}, x_{m_k+1}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon. \tag{12}$$

Since $T$ is quasi triangular $\theta$-orbital admissible, by Lemma 1.1, we obtain
$\alpha(x_{n_k}, x_{m_k}) \ge 1$, for all numbers $m_k, n_k$ such that $m_k > n_k$, where $k \ge 1$.
From (1), we obtain

$$\begin{aligned} 0 \le& \zeta\Big(\theta(x_{n_k}, x_{m_k})\varphi(d(Tx_{n_k}), Tx_{m_k}), \psi(B(x_{n_k}, x_{m_k}))\Big) \\ =& \zeta\Big(\theta(x_{n_k}, x_{m_k})\varphi(d(x_{n_k+1}, x_{m_k+1})), \psi(B(x_{n_k}, x_{m_k}))\Big) \\ <& \psi(B(x_{n_k}, x_{m_k})) - \theta(x_{n_k}, x_{m_k})\varphi(d(x_{n_k+1}, x_{m_k+1})). \end{aligned}$$

It follows that

$$\begin{aligned} \varphi(d(x_{n_k+1}, x_{m_k+1})) \le& \theta(x_{n_k}, x_{m_k})\varphi\Big(d(x_{n_k+1}, x_{m_k+1})\Big) \\ \le& \psi\Big(B(x_{n_k}, x_{m_k})\Big) < \varphi\Big(B(x_{n_k}, x_{m_k})\Big) \end{aligned}$$

where

$$\begin{aligned} B(x_{n_k}, x_{m_k}) =& [d(x_{n_k}, x_{m_k})]^{\alpha_1}.\Big[\frac{1}{2}\Big(d(x_{n_k}, Tx_{n_k}) + d(x_{m_k}, Tx_{m_k})\Big)\Big]^{\alpha_2}. \\ & \Big[\frac{1}{2}\Big(d(x_{n_k}, Tx_{m_k}) + d(x_{m_k}, Tx_{n_k})\Big)\Big]^{1-\alpha_1-\alpha_2} \\ =& [d(x_{n_k}, x_{m_k})]^{\alpha_1}.\Big[\frac{1}{2}\Big(d(x_{n_k}, x_{n_k+1}) + d(x_{m_k}, x_{m_k+1})\Big)\Big]^{\alpha_2}. \\ & \Big[\frac{1}{2}\Big(d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})\Big)\Big]^{1-\alpha_1-\alpha_2} \end{aligned}$$

Taking limit as $k \to +\infty$ together with (9), (10), (11) and (12), we obtain

$$0 \le \varphi(\epsilon) < \varphi(0) = 0 \Rightarrow \varphi(\epsilon) = 0 \text{ if and only if } \epsilon = 0.$$

This is a contradiction and hence $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $w \in X$ such that $\lim_{n\to+\infty} x_n = w$. On account of $T$ is continuous, we find that $Tw = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = w$. Hence, $w$ is the desired fixed point of $T$. □

**Example 2.1.** *Let $X = [0,1]$ with usual metric $d(x,y) = |x-y|$. Suppose $\theta : X \times X \to \mathbb{R}$ and $T : X \to X$ are mappings defined by*

$$\theta(x,y) = \begin{cases} 1, & 0 \le x,y \le \frac{1}{2},\ x \le y, \\ 0, & \textit{otherwise} \end{cases} \quad ;\ Tx = \begin{cases} \frac{1+x}{3}, & 0 \le x \le \frac{1}{2}, \\ x, & \frac{1}{2} < x \le 1. \end{cases}$$

*Since $n^{th}$-iterate of $T$ is either $T^n x = \frac{1}{3}\sum_{i=0}^{n-2}(\frac{1}{3})^i + \frac{1+x}{3^n}$, $x \in [0,\frac{1}{2}]$ or, $T^n x = x$, $x \in (\frac{1}{2},1]$, for all $n \in \mathbb{N}$. Also, we have $\frac{1}{3} \le T^n x \le \frac{1}{2}$, for all $x \in [0,\frac{1}{2}]$. It is obvious that $T$ is quasi triangular $\theta$-orbital admissible in $X$. Let $x_0 \in X$ such that $\theta(x_0,Tx_0) \ge 1$. Define a sequence $\{x_n\}$ in $X$ such that $x_n = T^n x_0$, for all $n \in \mathbb{N}$. By $\theta$-orbital admissibility of $T$, we have $\theta(Tx_0,T^2x_0) \ge 1$. Recursively, one may obtain that $\theta(T^{n-1}x_0,T^n x_0) \ge 1$, where $T^n x_0 = \frac{1}{3}\sum_{i=0}^{n-2}(\frac{1}{3})^i + \frac{1+x_0}{3^n}$.*
*Taking $\zeta(t,s) = \psi(s) - t$, for all $s,t > 0$ in Theorem 2.1, we obtain*

$$\theta(x,y)\varphi(d(Tx,Ty)) \le \psi(B(x,y)),\ \textit{for all } x,y \in X.$$

*Setting $\varphi(t) = t, \psi(t) = ht$, $t > 0$, where $h = \frac{1}{\sqrt{3}}$, $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{3}$, then $\varphi(t) > \psi(t)$. Since $0 \le x,y \le \frac{1}{2}$, we obtain*

$$0 \le |x-y| \le \frac{1}{2} \Rightarrow 0 \le |x-y|^{\frac{1}{2}} \le \frac{1}{\sqrt{2}},$$

$$0 \le \Big[\frac{1}{2}(|x-Tx| + |y-Ty|)\Big]^{\alpha_2} = \Big[\frac{1}{6}(|1-2x| + |1-2y|)\Big]^{\frac{1}{3}} \le \Big(\frac{1}{3}\Big)^{\frac{1}{3}} \textit{ and}$$

$$\Big(\frac{1}{6}\Big)^{\frac{1}{6}} \le \Big[\frac{1}{2}(|x-Ty| + |y-Tx|)\Big]^{\frac{1}{6}} = \Big[\frac{1}{6}(|1+y-3x| + |1+x-3y|)\Big]^{\frac{1}{6}} \le \Big(\frac{1}{2}\Big)^{\frac{1}{6}}.$$

*By simple calculation for all $x,y \in X$, we obtain*

$$\begin{aligned}\theta(x,y)\varphi(d(Tx,Ty)) =& \theta(x,y)|Tx-Ty| = \frac{1}{3}|x-y| \\ \le& \frac{1}{\sqrt{3}}|x-y|^{\frac{1}{2}}.\Big[\frac{1}{6}(|1-2x|+|1-2y|)\Big]^{\frac{1}{3}}.\Big[\frac{1}{6}(|1+y-3x|+|1+x-3y|)\Big]^{\frac{1}{6}} \\ =& \frac{1}{\sqrt{3}}|x-y|^{\frac{1}{2}}.\Big[\frac{1}{2}(|x-Tx|+|y-Ty|)\Big]^{\frac{1}{3}}.\Big[\frac{1}{2}(|x-Ty|+|y-Tx|)\Big]^{\frac{1}{6}} = \psi(B(x,y)).\end{aligned}$$

*Thus, all the conditions of Theorem 2.1 are satisfied and on account of continuity of $T$, we obtain*

$$\begin{aligned} T\frac{1}{2} &= \lim_{n\to+\infty} T(T^n x_0) = \lim_{n\to+\infty} T^{n+1}x_0 = \lim_{n\to+\infty}\Big(\frac{1}{3}\sum_{i=0}^{n-1}\Big(\frac{1}{3}\Big)^i + \frac{1+x_0}{3^n}\Big) \\ &= \lim_{n\to+\infty}\Big(\frac{1}{3} \times \frac{1}{1-\frac{1}{3}} + \frac{1+x_0}{3^n}\Big) = \frac{1}{2}.\end{aligned}$$

*Thus $T$ possess a fixed point in $X$. Note that $Fix(T) = \{\frac{1}{2}\} \cup \{x : x \in (\frac{1}{2},1]\}$.*

In the following theorem, we replace one of the heavy conditions of Theorem 2.1 i.e., continuity of $T$ with the notion of $\theta$-regularity as follows:

**Theorem 2.2.** *Let $T$ be a self-mapping on a complete metric space $(X,d)$. Suppose that $T$ is quasi triangular $\theta$-orbital admissible and forms an interpolative $(\varphi,\psi)$-BKC type $\mathcal{Z}$-contraction with respect to $\zeta$. If there exists $x_0 \in X$ such that $\theta(x_0,Tx_0) \geq 1$ and $\{x_n\}$ in $X$ is $\theta$-regular, where $x_n = T^n x_0$, $n \in \mathbb{N}$, then $T$ has a fixed point in $X$.*

*Proof.* By given assumption, there exists $x_0 \in X$ such that $\theta(x_0,Tx_0) \geq 1$. Consider the iterative sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Since $T$ is $\theta$-orbital admissible, we obtain recursively that $\theta(x_{n-1},x_n) \geq 1$, for all $n \in \mathbb{N}$. Without lost of generality, we shall assume that $x_{n-1} \neq x_n$ and hence we have $d(x_{n-1},x_n) > 0$, for all $n \in \mathbb{N}$. By repeating the same steps as in the proof of Theorem 2.1, we derive that $\{x_n\}$ converges to $w$. Since $\{x_n\}$ is $\theta$-regular in $X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\theta(x_{n_k},w) \geq 1$, for each $k \in \mathbb{N}\cup\{0\}$. From (1), we obtain

$$\begin{aligned}0 \leq& \zeta\Big(\theta(x_{n_k},w)\varphi(d(Tx_{n_k},Tw)),\psi(B(x_{n_k},Tw))\Big)\\ =&\zeta\Big(\theta(x_{n_k},w)\varphi(d(x_{n_k+1},Tw)),\psi(B(x_{n_k},Tw))\Big) < \psi(B(x_{n_k},Tw)) - \theta(x_{n_k},w)\varphi(d(x_{n_k+1},Tw))\end{aligned}$$

which is equivalent to

$$\varphi(d(x_{n_k+1},Tw)) \leq \theta(x_{n_k},w)\varphi(d(x_{n_k+1},Tw)) < \psi(B(x_{n_k},Tw)) < \varphi(B(x_{n_k},Tw))$$

where

$$\begin{aligned}B(x_{n_k},Tw) =&[d(x_{n_k},w)]^{\alpha_1}.\Big[\frac{1}{2}\Big(d(x_{n_k},Tx_{n_k}) + d(w,Tw)\Big)\Big]^{\alpha_2}.\\ &\Big[\frac{1}{2}\Big(d(x_{n_k},Tw) + d(w,Tx_{n_k})\Big)\Big]^{1-\alpha_1-\alpha_2}\\ =&[d(x_{n_k},w)]^{\alpha_1}.\Big[\frac{1}{2}\Big(d(x_{n_k},x_{n_k+1}) + d(w,Tw)\Big)\Big]^{\alpha_2}.\\ &\Big[\frac{1}{2}\Big(d(x_{n_k},Tw) + d(w,x_{n_k+1})\Big)\Big]^{1-\alpha_1-\alpha_2}.\end{aligned}$$

Taking $k \to +\infty$, we obtain $\varphi(d(w,Tw)) = 0$ implies $d(w,Tw) = 0$. This shows that $w$ is a fixed point of $T$. □

Let $(U)$ be the uniqueness condition which is given as: For any distinct fixed points $w, w^* \in Fix(T) \neq \phi$, $\theta(w,w^*) \geq 1$, where $Fix(T) = \{x : Tx = x\}$.

**Theorem 2.3.** *In addition to the assumption of Theorem 2.1 (or Theorem 2.2), we suppose the condition $(U)$ holds. Then the observed fixed point is unique.*

*Proof.* Taking $w, w^* \in X$, $w \neq w^*$ such that $Tw = w$ and $Tw^* = w^*$ in (1), we obtain

$$\begin{aligned}0 \leq& \zeta\Big(\theta(Tw,Tw^*)\varphi(d(Tw,Tw^*)),\psi(B(w,w^*))\Big)\\ =&\zeta\Big(\theta(w,w^*)\varphi(d(w,w^*)),\psi(B(w,w^*))\Big)\\ <&\psi(B(w,w^*)) - \theta(w,w^*)\varphi(d(w,w^*)) = -\theta(w,w^*)\varphi(d(w,w^*)\end{aligned}$$

This is contradiction and hence $T$ has a unique fixed point in $X$. □

**Remark 2.1.** *In Example 2.1, $Fix(T) = \{\frac{1}{2}\} \cup \{x : x \in (\frac{1}{2},1]\}$, so the condition $(U)$ does not hold and hence Theorem 2.3 is not applicable in Example 2.1.*

**Theorem 2.4.** *Let $T : X \to X$ be a self mapping on a complete metric space $(X,d)$. If there exist $\theta : X \times X \to \mathbb{R}$, $\varphi \in \Phi, \psi \in \Psi$, $\alpha_1,\alpha_2 \in (0,1)$ such that $\varphi(t) > \psi(t)$, for $t > 0$ and $\alpha_1 + \alpha_2 < 1$ satisfying the inequality*

$$\theta(x,y)\varphi(d(Tx,Ty)) \leq \psi(B(x,y))$$

*for all $x, y \in X$. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and $T$ is continuous, or $\{x_n\}$ in $X$ is $\theta$-regular, where $x_n = T^n x_0$, $n \in \mathbb{N}$, then $T$ has a fixed point in $X$. Further suppose the condition $(U)$ holds, then $Fix(T)$ is singleton.*

*Proof.* Setting $\zeta(t, s) = \psi(s) - t$, for all $s, t > 0$ in Theorem 2.3. □

**Corollary 2.1.** *Let $T$ be a self mapping on a complete metric space $(X, d)$. If there exist $\psi \in \Psi$, $\alpha_1, \alpha_2 \in (0, 1)$ such that $\alpha_1 + \alpha_2 < 1$ satisfying the inequality*

$$d(Tx, Ty) \leq \psi(B(x, y))$$

*for all $x, y \in X$, then $T$ has a unique fixed point in $X$.*

**Example 2.2.** *Let $X = \{0, 1, 2, 3\}$ endow with Euclidean metric $d(x, y) = |x - y|$. Define $T : X \to X$ by*

$$Tx = \begin{cases} 0, & x \neq 3 \\ 1, & x = 3. \end{cases}$$

*Setting $\psi(t) = \frac{9}{10}t$, $t > 0$ and $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}$. Then $T$ satisfies all the conditions of Corollary 2.1 and $0$ is the unique fixed point of $T$.*

**Theorem 2.5.** *Let $T : X \to X$ be a self mapping on a complete metric space $(X, d)$. Suppose that $T$ is quasi triangular $\theta$-orbital admissible and forms an interpolative $(\varphi, \psi)$-HR type $\mathcal{Z}$-contraction with respect to $\zeta$. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and $T$ is continuous, then $T$ has a fixed point in $X$.*

*Proof.* Let $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and consider the iterative sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Following the proof of Theorem 2.1, we shall consider $x_n \neq x_{n-1}$ and hence we have $d(x_{n-1}, x_n) > 0$, for all $n \in \mathbb{N}$. On the other hand, $T$ is $\theta$-orbital admissible, we obtain recursively that $\theta(x_{n-1}, x_n) \geq 1$, for all $n \in \mathbb{N}$. From (2.2), we obtain

$$\begin{aligned} 0 \leq\ & \zeta(\theta(x_{n-1}, x_n)\varphi(d(Tx_{n-1}, Tx_n)), \psi(H(x_{n-1}, x_n))) \\ =\ & \zeta(\theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})), \psi(H(x_{n-1}, x_n))) \\ <\ & \psi(H(x_{n-1}, x_n)) - \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})) \end{aligned} \tag{13}$$

where

$$\begin{aligned} H(x_{n-1}, x_n) =\ & [d(x_{n-1}, x_n)]^{\alpha_1}.[d(x_{n-1}, Tx_{n-1})]^{\alpha_2}.[d(x_n, Tx_n)]^{\alpha_3}. \\ & \left[\tfrac{1}{2}(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}))\right]^{1-\sum_{i=1}^{3}\alpha_i} \\ =\ & [d(x_{n-1}, x_n)]^{\alpha_1+\alpha_2}.[d(x_n, x_{n+1})]^{\alpha_3}.\left[\tfrac{1}{2}d(x_{n-1}, x_{n+1})\right]^{1-\sum_{i=1}^{3}\alpha_i}. \end{aligned} \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} \varphi(d(x_n, x_{n+1})) \leq\ & \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})) < \psi(H(x_{n-1}, x_n)) \\ =\ & \psi([d(x_{n-1}, x_n)]^{\alpha_1+\alpha_2}.[d(x_n, x_{n+1})]^{\alpha_3}.\left[\tfrac{1}{2}d(x_{n-1}, x_{n+1})\right]^{1-\sum_{i=1}^{3}\alpha_i}) \\ \leq\ & \psi([d(x_{n-1}, x_n)]^{\alpha_1+\alpha_2}.[d(x_n, x_{n+1})]^{\alpha_3}. \\ & \left[\tfrac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))\right]^{1-\sum_{i=1}^{3}\alpha_i}), \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{15}$$

Assume that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, for all $n \in \mathbb{N}$, then from (15), we obtain

$$\varphi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) < \varphi d(x_n, x_{n+1})), \text{ for all } n \in \mathbb{N}.$$

This is a contradiction. Therefore,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \tag{16}$$

Thus, as in Theorem 2.1, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Now, we claim that $l > 0$, otherwise from (15) together with (16), we obtain

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) \leq &\quad \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})) < \psi(H(x_{n-1}, x_n)) \\ \leq &\quad \psi(d(x_{n-1}, x_n)) < \varphi(d(x_{n-1}, x_n)).\end{aligned}$$

Letting limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})) = \lim_{n\to+\infty} \psi(H(x_{n-1}, x_n)) = \varphi(l). \tag{17}$$

Note that $\varphi(l) > 0$, for all $l > 0$. Letting $s_n = \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1}))$ and $t_n = \psi(H(x_{n-1}, x_n))$ and by condition $(\zeta_3)$ with (17), we obtain

$$0 \leq \quad \lim_{n\to+\infty} \sup \zeta\Big(\theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})), \psi(H(x_{n-1}, x_n))\Big) < 0.$$

This is a contradiction and hence $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$, for all $n \in \mathbb{N}\cup\{0\}$. Moreover, $T$ is quasi triangular $\theta$-orbital admissible mapping, by Lemma 1.1, we obtain $\alpha(x_{n_k}, x_{m_k}) \geq 1$ for all numbers $m_k, n_k \in \mathbb{N} \cup \{0\}$ such that $m_k > n_k \geq k$. Following the same steps as in Theorem 2.1, we can prove that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $w \in X$ such that $\lim_{n\to\infty} x_n = w$. On account of $T$ is continuous, immediately we find that $Tw = w$. Hence $w$ is the desired fixed point of $T$. □

Next, we replace the continuity condition of $T$ in Theorem 2.5 by $\theta$-regularity.

**Theorem 2.6.** *Let $T : X \to X$ be a self mapping on a complete metric space $(X, d)$. Suppose that $T$ is quasi triangular $\theta$-orbital admissible and forms an interpolative $(\varphi, \psi)$-HR type $\mathcal{Z}$-contraction with respect to $\zeta$. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and $\{x_n\}$ in $X$ is $\theta$-regular, where $x_n = T^n x_0$, $n \in \mathbb{N}$, then $T$ has a fixed point in $X$.*

From Theorem 2.5, by letting $\theta(x, y) = 1$ for all $x, y \in X$ and $\varphi = \psi = I_X$(identity mapping) we get the following:

**Corollary 2.2** ([12]). *Let $T : X \to X$ be a self mapping on a complete metric space $(X, d)$. Suppose that $T$ is an interpolative Hardy-Rogers type $\mathcal{Z}$-contraction with respect to $\zeta$, then $T$ has a fixed point in $X$.*

For $\zeta(t, s) = hs - t$, where $h \in [0, 1)$ and $s, t > 0$ in Corollary 2.2, we find the following:

**Corollary 2.3** ([14]). *Let $(X, d)$ be a complete metric space and $T$ be an interpolative Hardy-Rogers type contraction. Then $T$ has a fixed point in $X$.*

## 3. Application

We apply our result to establish an existence theorem for non-linear Fredholm integral equation. Let $X = C[a, b]$ be a set of all real continuous functions on $[a, b]$ equipped with metric $d(f, g) = |f - g| = \max_{t\in[a,b]} |f(t) - g(t)|$, for all $f, g \in C[a, b]$. Then, $(X, d)$ is a complete metric space.

Now, we consider non-linear Fredholm integral equation

$$x(t) = v(t) + \frac{1}{b-a}\int_a^b K(t, s, x(s))ds, \tag{18}$$

where $t, s \in [a, b]$. Assume that $K : [a, b] \times [a, b] \times X \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ continuous, where $v(t)$ is a given function in $X$.

**Theorem 3.1.** *Suppose $(X, d)$ be a metric space equipped with metric $d(f, g) = |f - g| = \max_{t\in[a,b]} |f(t) - g(t)|$, for all $f, g \in X$ and $T : X \to X$ be an operator on $X$ defined by*

$$Tx(t) = v(t) + \frac{1}{b-a}\int_a^b K(t, s, x(s))ds. \tag{19}$$

*If there exist $k \in [0,1), \alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 + \alpha_2 < 1$ such that for all $x, y \in X,\ s, t \in [a,b]$ satisfying the following inequality*

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq kM(x(s),y(s)) \tag{20}$$

$$\text{where, } M(x(s),y(s)) = |x(s)-y(s)|^{\alpha_1}.[\frac{1}{2}(|x(s)-Tx(s)|+|y(s)-Ty(s)|)]^{\alpha_2}. \\ [\frac{1}{2}(|x(s)-Ty(s)|+|y(s)-Tx(s)|)]^{1-\alpha_1-\alpha_2}.$$

*Then, the integral equation (19) has a unique solution in $X$.*

*Proof.* From (19) and (20), we obtain

$$\begin{aligned} |Tx(t)-Ty(t)| =& \frac{1}{|b-a|}|\int_a^b K(t,s,x(s))ds - \int_a^b K(t,s,y(s))ds| \\ \leq & \frac{1}{|b-a|}\int_a^b |K(t,s,x(s)) - K(t,s,y(s))|ds \leq \frac{k}{|b-a|}\int_a^b M(x(s),y(s))ds \\ \leq & \frac{k}{|b-a|}\int_a^b (|x(s)-y(s)|^{\alpha_1}.\Big[\frac{1}{2}(|x(s)-Tx(s)|+|y(s)-Ty(s)|)]^{\alpha_2} \\ & .[\frac{1}{2}(|x(s)-Ty(s)|+|y(s)-Tx(s)|\Big)]^{1-\alpha_1-\alpha_2})ds. \end{aligned}$$

Taking maximum on both sides for all $t \in [a,b]$, we obtain

$$\begin{aligned} d(Tx,Ty) = \max_{t\in[0,1]} |Tx(t)-Ty(t)| \leq & \frac{k}{|b-a|}\max_{t\in[a,b]}\int_a^b (|x(s)-y(s)|^{\alpha_1}.[\frac{1}{2}(|x(s)-Tx(s)| \\ & + |y(s)-Ty(s)|)]^{\alpha_2}.[\frac{1}{2}(|x(s)-Ty(s)|+|y(s)-Tx(s)|)]^{1-\alpha_1-\alpha_2})ds \\ \leq & \frac{k}{|b-a|}(\max_{t\in[a,b]}(|x(s)-y(s)|^{\alpha_1}.[\frac{1}{2}(|x(s)-Tx(s)| \\ & + |y(s)-Ty(s)|)]^{\alpha_2}.[\frac{1}{2}(|x(s)-Ty(s)|+|y(s)-Tx(s)|)]^{1-\alpha_1-\alpha_2}))\int_a^b ds \\ =& k[d(x,y)]^{\alpha_1}.[\frac{1}{2}(d(x,Tx)+d(yTy))]^{\alpha_2}.[\frac{1}{2}(d(x,Ty)+d(y,Tx))]^{1-\alpha_1-\alpha_2} = kB(x,y). \end{aligned}$$

Since $X = C[a,b]$ is complete metric space. Therefore, all the conditions of Corollary 2.1 are satisfied by setting $\psi(t) = kt$ for all $t > 0$, where $k \in [0,1)$ and hence the integral equation (19) has a unique solution in $X$. □

## REFERENCES


[1] *M.A. Alghamdi, S. Gulyaz-Ozyurt, E. Karapınar*, A Note on Extended Z-Contraction. Mathematics 2020, **8**, 195.

[2] *O.Alqahtani, E.Karapinar,* A Bilateral Contraction via Simulation Function, Filomat **33(15)** (2019), 4837.4843

[3] *H. H. Alsulami, E. Karapınar, F. Khojasteh, A. F. Roldán-Léopez-de-Hierro,* A proposal to the study of contractions in quasi-metric spaces, Discrete Dynamics in Nature and Soc., **2014**, Article ID 269286, 10 pages

[4] *H. Argoubi, B. Samet, C. Vetro*, Nonlinear contractions involving simu- lation functions in a metric space with a partial order, J. Nonlinear Sci. Appl., **8** (2015), 1082-1094.

[5] *S. Banach,* Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., 3, (1922), 133-181.

[6] *V. Berinde,* Generalized contractions in quasimetric spaces, Seminar on fixed point theory, Babeş-Bolyai Unversity, **3**(1993), 3-9.

[7] *V. Berinde,* Sequences of operators and fixed points in quasimetric spaces, Studia Univ. Babeş-Bolyai Math., 41(4), (1996), 23-27.

[8] *S. K. Chatterjea,* Fixed point theorems, C. R. Acad. Bulgare Sci., **25**, (1972), 727-730.

[9] *G.E Hardy, T. D. Rogers,* A generalization of a fixed point theorem of Reich, Can. Math. Bull. 1973, **16**, 201-206.

[10] *R. Kannan*, Some results on fixed point, Bull. Cal. Math. Soc., **60**, (1968), 71-76.

[11] *E. Karapınar*, Revisiting the Kannan type contraction via interpolation, Adv. Theory Nonlinear Anal. Appl., **2(2)**,(2018), 85-87.

[12] *E. Karapınar*, Revisiting simulation functions via interpolative contrations, Appl. Anal. Discrete Math., **13** (2019), 859-870.

[13] *E. Karapınar, R. P. Agarwal, H. Aydi*, Interpolative Reich-Rus-Ćirić type contraction on partial metric spaces, Mathematics, 2018, **6**, 256, 1-7.

[14] *E. Karapınar, O. Alqahtani, H. Aydi*, On interpolative Hardy-Rogers type contractions, Symmetry, 2018, **11**, 8, 1-7.

[15] *E. Karapınar, P. Kumam,P. Salimi*, On $\alpha$-$\psi$-Meir-Keeler contractive mappings, Fixed Point Theory Appl., **2013**, Article ID 94, (2013), 12 pages.

[16] *E. Karapınar, F. Khojasteh,* An approach to best proximity points results via simulation functions, J. Fixed Point Theory Appl., **19(3)**, 1983-1995 , 2017 (doi:10.1007/s11784-016-0380-2)

[17] *E. Karapınar,* Fixed points results via simulation functions, Filomat, **30(8)**, 2016, 2343-2350

[18] *M. S. Khan,* A theorem on fixed points, Math. Seminar Notes, Kobe University, 4(2), (1976), 227-228.

[19] *M. S. Khan,* Some fixed point theorems in metric metric and Banach space, Indian J. Pure. appl. Math., **11 (4)**, (1980), 413-421.

[20] *M. S. Khan, M. Swaleh, S. Sessa,* Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc., **30** (1984), 1-9.

[21] *F. Khojasteh, S. Shukla, S. Radenović,* A new approach to the study of fixed point theorems via simulation functions, Filomat, **29(6)**, (2015), 1189-1194.

[22] *Y. Mahendra Singh, M. S. Khan, S. M. Kang, F*-Convex contraction via admissible mapping and related fixed point theorems with an application, Mathematics, 2018, **6**,105, 1-15.

[23] *O. Popescu,* Some new fixed point theorems for $\alpha$-Geraghty contractive type maps in metric spaces. Fixed Point Theory Appl., **2014:190**, (2014), 1-12.

[24] *A. F. Roldán-Léopez-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, J. Martínez-Morenoa*, Coincidence point theorems on metric spaces via simulation functions, J. Computational and Appl. Math., **257**(2015), 345-355.

[25] *B. Samet, C.Vetro, P. Vetro*, Fixed point theorems for $\alpha - \psi$-contractive type mappings, Nonlinear Anal. **75** (2012), 2154-2165.