SOLUTIONS OF A COUPLED WAVE EQUATION

E. V. Krishnan[1], Ghodrat Ebadi[2], Anjan Biswas[3]

This paper obtains the traveling wave solution to a nonlinearly coupled wave equation. Subsequently, the mapping method and its extended and modified versions are also employed to obtain additional solutions to this equation. Finally, the exponential function method and the G'/G-expansion method are employed to extract more solutions to this coupled system of equations. The parameter constraints are also given in order for the solutions to exist.

MSC2010: 35Q51, 35Q53, 37K10.

1. Introduction

The traveling wave solution is one of the most fundamental approaches to solving nonlinear evolutions equations (NLEEs). This method is also applicable to nonlinearly coupled NLEEs as well as to complex valued NLEEs. However, there are various other approaches to handling NLEEs [1-10]. One of them is the most powerful method that is known as the Inverse Scattering Transform (IST) that obtains soliton solutions as well as soliton radiations. In the last couple of decades, there was an abundance of new mathematical approaches that was available to deal with NLEEs. Some of these methods are Adomian decomposition method, Riccati equation method, Fan's F-expansion method, G'/G expansion method, He's semi-inverse variational principle and many others. In this paper, besides the traveling wave hypothesis, the mapping method, variational iteration method (VIM) as well as the homotopy perturbation method (HPM) are all going to be employed to study the coupled nonlinear wave equation.

The dimensionless form of the coupled nonlinear wave equation that is going to be studied in this paper is given by

$$q_t + ar^2 r_x + bq^2 q_x + cq_x + dq_{xxx} = 0, \tag{1}$$

$$r_t + \alpha(qr)_x + \beta r^3 r_x = 0. \tag{2}$$

In (1), the first term is the evolution term, while the second and third terms are the nonlinear terms, the fourth term is the advection term and finally the last term is the dispersion term. In (2), the first term is the evolution term while the second term is the nonlinear coupled term and finally the third term is the nonlinear term. The coefficients a, b, c, d, α and β are all constants.

[1] Department of Mathematics and Statistics, Sultan Qaboos University, P. O. Box 36, Al Khod 123, Muscat, Oman

[2] Faculty of Mathematical Sciences, University of Tabriz, Tabriz, 51666-14766, Iran

[3] Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA

2. **Traveling wave solution**

In order to solve this coupled system of equations given by (1) and (2), for solitary waves, the traveling wave hypothesis is taken to be

$$q(x,t) = g(x - vt) = g(s) \tag{3}$$

and

$$r(x,t) = h(x - vt) = h(s) \tag{4}$$

where

$$s = x - vt \tag{5}$$

while g and h are the solitary wave profiles of (1) and (2) respectively moving with a velocity v. Now substituting (3) and (4) into (1) and (2) respectively gives

$$-vg' + ah^2h' + bg^2g' + cg' + dg''' = 0 \tag{6}$$

and

$$-vh' + \alpha(gh)' + \beta h^3h' = 0 \tag{7}$$

Integrating (6) and (7) once, with respect to s and taking the integration constants to be zero, since the search is for a soliton solution, gives

$$-vg + \frac{ah^3}{3} + \frac{bg^3}{3} + cg + dg'' = 0 \tag{8}$$

and

$$-v + \alpha g + \beta\frac{h^3}{4} = 0 \tag{9}$$

respectively. Now, eliminating h between (8) and (9) and simplifying gives

$$\frac{d\beta}{4}g'' = \left(\frac{a\alpha}{3} + \frac{\beta v}{4} - \frac{c\beta}{4}\right)g - \frac{b\beta}{12}g^3 - \frac{av}{3} \tag{10}$$

Multiplying both sides of (10) by g' and integrating gives, after simplification

$$\left(\frac{dg}{ds}\right)^2 = -\frac{b}{6d}\left\{g^4 - \left(\frac{8a\alpha}{\beta b} + \frac{6v}{b} - \frac{6c}{b}\right)g^2 + \frac{16av}{\beta b}g\right\} \tag{11}$$

For integrability, it is necessary to choose

$$a = 0 \tag{12}$$

which simplifies the above equation further. Thus separating variables and integrating gives

$$\int \frac{dg}{g\sqrt{\lambda^2 - g^2}} = \sqrt{\frac{b}{6d}}s \tag{13}$$

which gives

$$-\frac{1}{\lambda}\text{sech}^{-1}\left(\frac{g}{\lambda}\right) = \sqrt{\frac{b}{6d}}s \tag{14}$$

where

$$\lambda = \sqrt{\frac{6(v-c)}{b}} \tag{15}$$

and (14) imposes the restriction

$$bd > 0.$$

Equation (14) then finally yields

$$g(x - vt) = g(s) = A\text{sech}[B(x - vt)] \tag{16}$$

where the amplitude of the soliton is given by

$$A = \lambda = \sqrt{\frac{6(v - c)}{b}} \tag{17}$$

and the inverse width of the soliton is

$$B = \sqrt{\frac{v - c}{d}} \tag{18}$$

Finally the soliton profile can be obtained from

$$h(s) = h(x - vt) = \left[\frac{4(v - \alpha g)}{\beta}\right]^{\frac{1}{3}} \tag{19}$$

which is obtained from (9). Thus equations (15) and (18) pose constraint conditions given by

$$d(v - c) > 0 \tag{20}$$

and

$$b(v - c) > 0 \tag{21}$$

Thus, finally, the 1-soliton solution to (1) and (2) is given by (16) and (19) where the amplitude and width of the g profile are given by (17) and (18) respectively. These relations introduce the integrability conditions given by (20) and (21) that needs to hold in order for the soliton solutions to exist.

3. Mapping methods

Now, we solve eq. (10) by a mapping method, a modified mapping method and an extended mapping method [1-4] which generate a variety of periodic wave solutions (PWSs) in terms of Jacobi elliptic functions and we subsequently derive their infinite period counterparts in terms of hyperbolic functions which are either solitary wave solutions (SWSs) or explode decay mode solutions.

Now, eq. (10) can be written as

$$A\,g'' + B\,g + C\,g^3 + K = 0, \tag{22}$$

where

$$A = \frac{d\,\beta}{4},\ B = \frac{c\,\beta}{4} - \frac{v\,\beta}{4} - \frac{a\,v}{3}\,\beta,\ C = \frac{b\,\beta}{12},\ K = \frac{a\,v}{3}. \tag{23}$$

3.1. Mapping method

We assume that eq. (22) has a solution in the form

(24) $$g = A_0 + A_1 f,$$

where

(25) $$f'' = p f + q f^3, \; f'^2 = p f^2 + \frac{1}{2} q f^4 + r.$$

Eq. (24) is the mapping relation between the solution to eq. (25) and that of eq. (22).

We substitute eq. (24) into eq. (22), use eq. (25) and equate the coefficients of like powers of f to zero to arrive at the set of equations

(26) $$q A A_1 + C A_1^3 = 0,$$

(27) $$3 C A_0 A_1^2 = 0,$$

(28) $$(p A + B) A_1 + 3 C A_0^2 A_1 = 0,$$

(29) $$B A_0 + C A_0^3 + K = 0,$$

from which we obtain

(30) $$A_0 = 0, \;\; A_1 = \pm \sqrt{\frac{q B}{p C}}, \;\; p A + B = 0, \, K = 0.$$

Since $K = 0$, a automatically becomes 0.

Using eq. (23), we obtain the exact solution of eq. (22) as

(31) $$g(s) = \pm \sqrt{\frac{3 q (c - v)}{4 b p}} f(s)$$

Now, we consider the specific expressions of f according to eq. (25).

Case 1. $p = -2, \; q = 2, \; r = 1.$

In this case, the solution of eq. (25) is $f(s) = \tanh(s)$.

So, we have the shock wave solution of eq. (22) as

(32) $$g(s) = \pm \sqrt{- \frac{3 (c - v)}{4 b}} \tanh(s).$$

Case 2. $p = 1, \; q = -2, \; r = 0.$

Here, the solution of eq. (25) is $f(s) = \operatorname{sech}(s)$.

Now, we have the SWS of eq. (22) as

(33) $$g(s) = \pm\sqrt{-\frac{3\,(c-v)}{2\,b}}\,\text{sech}(s).$$

Case 3. $p = -(1+m^2),\ q = 2\,m^2,\ r = 1.$

The solution of eq. (25) is $f(s) = \text{sn}(s)$ or $f(s) = \text{cd}(s)$.

Thus, we have the PWSs of eq. (22) as

(34) $$g(s) = \pm\sqrt{-\frac{3\,(c-v)}{2\,b\,(1+m^2)}}\,m\,\text{sn}(s)$$

and

(35) $$g(s) = \pm\sqrt{-\frac{3\,(c-v)}{2\,b\,(1+m^2)}}\,m\,\text{cd}(s)$$

As $m \to 1$, the shock wave solution (32) is recovered from eq. (34).

Case 4. $p = 2-m^2,\ q = -2,\ r = m^2-1.$

Now, the solution of eq. (25) is $f(s) = \text{dn}(s)$

So, the PWSs of eq. (22) is

(36) $$g(s) = \pm\sqrt{-\frac{3\,(c-v)}{2\,b\,(2-m^2)}}\,\text{dn}(s)$$

As $m \to 1$, the SWS (33) is recovered from eq. (36).

Case 5. $p = -(1+m^2),\ q = 2,\ r = m^2.$

Here, the solution of eq. (25) is $f(s) = \text{ns}(s)$ or $f(s) = \text{dc}(s)$

So, the PWSs of eq. (22) are

(37) $$g(s) = \pm\sqrt{-\frac{3\,(c-v)}{4\,b\,(1+m^2)}}\,\text{ns}(s)$$

and

$$g(s) = \pm\sqrt{-\frac{3(c-v)}{4b(1+m^2)}}\,\mathrm{dc}(s) \tag{38}$$

As $m \to 0$, eqs. (37) and (38) degenerate as

$$g(s) = \pm\sqrt{-\frac{3(c-v)}{4b}}\,\mathrm{cosec}(s) \tag{39}$$

and

$$g(s) = \pm\sqrt{-\frac{3(c-v)}{4b}}\,\sec(s) \tag{40}$$

When $m \to 1$, we obtain from eq. (37) the solution in the form

$$g(s) = \pm\sqrt{-\frac{3(c-v)}{8b}}\,\coth(s) \tag{41}$$

3.2. Modified mapping method

Now, we use the modified mapping method in which we assume a solution of eq. (22) in the form

$$g = A_0 + A_1 f + B_1 f^{-1} \tag{42}$$

where f satisfies eq. (25).

We substitute eq. (42) into eq. (22), use eq. (25) and equate the coefficients of like powers of f to zero to arrive at a set of equations from which it can be found that

$$A_0 = 0, \quad A_1 = \pm\sqrt{-\frac{qA}{C}} \tag{43}$$

$$B_1 = \pm\sqrt{-\frac{2rA}{C}}, \quad pA + B + 3CA_1B_1 = 0. \tag{44}$$

Thus for real solutions of eq. (22) to exist, when q and r are both positive, A and C should be of opposite signs and when q and r are both negative, A and C should be of same signs.

So, we have another set of new exact solutions of eq. (22) which is given by

$$g(s) = \pm\sqrt{-\frac{3qd}{b}}\,f(s) \pm \sqrt{-\frac{6rd}{b}}\,f^{-1}(s). \tag{45}$$

Case 1. $p = -2,\ q = 2,\ r = 1.$

Here, the solution of eq. (25) is $f(s) = \tanh(s)$.

So, the solution of eq. (22) is

$$g(s) = \sqrt{-\frac{6\,d}{b}}\,\{\pm\tanh(s) \pm \coth(s)\}\,. \tag{46}$$

Case 2. $p = -(1 + m^2),\ q = 2\,m^2,\ r = 1.$

So, the solution of eq. (25) is $f(s) = \mathrm{sn}(s)$ or $f(s) = \mathrm{cd}(s)$.

Thus, the PWSs of eq. (22) are

$$g(s) = \sqrt{-\frac{6\,d}{b}}\,\{\pm m\,\mathrm{sn}(s) \pm \mathrm{ns}(s)\}\,. \tag{47}$$

and

$$g(s) = \sqrt{-\frac{6\,d}{b}}\,\{\pm m\,\mathrm{cd}(s) \pm \mathrm{dc}(s)\}\,. \tag{48}$$

When $m \to 0$, eqs. (47) and (48) will give rise respectively to the solutions

$$g(s) = \pm\sqrt{-\frac{6\,d}{b}}\,\mathrm{cosec}(s). \tag{49}$$

and

$$g(s) = \pm\sqrt{-\frac{6\,d}{b}}\,\sec(s). \tag{50}$$

and, when $m \to 1$, eq. (47) degenerates to eq. (46).

Case 3. $p = 2 - m^2,\ q = 2,\ r = 1 - m^2.$

So, the solution of eq. (25) is $f(s) = \mathrm{cs}(s)$.

Hence, the PWS of eq. (22) is

$$g(s) = \sqrt{-\frac{6\,d}{b}}\,\left\{\pm\,\mathrm{cs}(s) \pm \sqrt{1 - m^2}\,\mathrm{sc}(s)\right\}. \tag{51}$$

When $m \to 0$, eq. (51) will reduce to

$$g(s) = \sqrt{-\frac{6\,d}{b}}\,\{\pm\tan(s) \pm \cot(s)\}\,, \tag{52}$$

and when $m \to 1$, eq. (51) will give rise to the explode decay mode solution

(53) $$g(s) = \pm\sqrt{-\frac{6\,d}{b}}\,\mathrm{cosech}(s).$$

Case 4. $p = 2 - m^2,\ q = -2,\ r = -(1 - m^2).$

Here, the solution of eq. (25) is $f(s) = \mathrm{dn}(s)$.

So, the PWS of eq. (22) is

(54) $$g(s) = \sqrt{-\frac{6\,d}{b}}\,\left\{\pm\,\mathrm{dn}(s) \pm \sqrt{1-m^2}\,\mathrm{nd}(s)\right\}.$$

When $m \to 1$, eq. (54) will give rise to the SWS

(55) $$g(s) = \pm\sqrt{-\frac{6\,d}{b}}\,\mathrm{sech}(s).$$

3.3. Extended mapping method

Now, we use the extended mapping method in which we assume a solution of equation (22) in the form

(56) $$g = A_0 + A_1\,f + B_1\,f^*$$

where f satisfies eq. (25) and f^* satisfies

(57) $$f^{*\prime\prime} = f^*\,(c_1 + c_2\,f^2),\quad f^{*2} = c_3 + c_4\,f^2.$$

We substitute eq. (56) into eq. (22), use eqs. (25) and (57) and equate the coefficients of like powers of f to zero to arrive at a set of equations from which we obtain

(58) $$A_0 = 0,$$

(59) $$A_1 = \pm\sqrt{\frac{3\,c_4\,(p\,d + c - v) - 3\,c_3\,q\,d}{c_3\,b}},$$

(60) $$B_1 = \pm\sqrt{\frac{v - p\,d - c}{c_3\,b}}.$$

By this method, the new exact solution of eq. (22) is given by

(61) $$g(s) = \pm\sqrt{\frac{3\,c_4\,(p\,d + c - v) - 3\,c_3\,q\,d}{c_3\,b}}\,f(s) \pm \sqrt{\frac{v - p\,d - c}{c_3\,b}}\,f^*(s).$$

Case 1.

$$p = 2m^2 - 1,\ q = -2m^2,\ r = 1 - m^2,$$
$$c_1 = m^2,\ c_2 = -2m^2,\ c_3 = 1 - m^2,\ c_4 = m^2.$$

In this case, $f(s) = \mathrm{cn}(s)$ and $f^*(s) = \mathrm{dn}(s)$. Thus the PWS of eq. (22) is

$$g(s) = \pm\sqrt{\frac{3(d + c - v)}{(1-m^2)\,b}}\, m\,\mathrm{cn}(s) \pm \sqrt{\frac{v - c - (2m^2-1)d}{(1-m^2)\,b}}\,\mathrm{dn}(s). \tag{62}$$

Case 2.

$$p = 2 - m^2,\ q = -2(1 - m^2),\ r = -1$$
$$c_1 = 1,\ c_2 = -2(1 - m^2),\ c_3 = -\frac{1}{m^2},\ c_4 = \frac{1}{m^2}.$$

In this case, $f(s) = \mathrm{nd}(s)$ and $f^*(s) = \mathrm{sd}(s)$. So, the PWS of eq. (22) is

$$g(s) = \pm\sqrt{\frac{3(v - c - 2dm^2)}{b}}\,\mathrm{nd}(s) \pm \sqrt{\frac{(2-m^2)d + c - v}{b}}\, m\,\mathrm{sd}(s). \tag{63}$$

Case 3.

$$p = 2 - m^2,\ q = 2,\ r = 1 - m^2,$$
$$c_1 = 1,\ c_2 = 2,\ c_3 = 1 - m^2,\ c_4 = 1.$$

In this case, $f(s) = \mathrm{cs}(s)$ and $f^*(s) = \mathrm{ds}(s)$. Thus the PWS of eq. (22) is

$$g(s) = \pm\sqrt{\frac{3(m^2 d + c - v)}{(1-m^2)\,b}}\,\mathrm{cs}(s) \pm \sqrt{\frac{v - c - (2 - m^2)\,d}{(1 - m^2)\,b}}\,\mathrm{ds}(s). \tag{64}$$

When $m \to 0$, eq. (64) will reduce to the solution

$$g(s) = \pm\sqrt{\frac{3(c - v)}{b}}\cot(s) \pm \sqrt{\frac{v - c - 2d}{b}}\csc(s). \tag{65}$$

4. G'/G-expansion method

In this section, we carry out integration of the eqs. (1) and (2) with G'/G method which provides hyperbolic, trigonometric and rational function traveling wave solutions. For solving eqs (1) and (2) here, we consider eq. (11) as the converted form of them. We assume that eq.(11) has a solution in the form of

$$g(s) = \sum_{l=0}^{m} a_l \left(\frac{G'}{G}\right)^l,\ a_m \neq 0, \tag{66}$$

where

$$G''(s) + \lambda G'(s) + \mu G(s) = 0 \tag{67}$$

and a_l, $l = 0, 1, \cdots, m, \lambda$ and μ are unknown constants to be determined later, and to obtain the integer m in eq. (66), we balance $(g'(s))^2$ with $g^4(4)$. So we have

$2m+2=4m$, hence $m=1$. Therefore we can suppose that eq. (11) has a solution in the form

$$g(s)=a_0+a_l\left(\frac{G'}{G}\right),\ a_1\neq 0. \tag{68}$$

Substituting eq. (68) with (67) in the eq. (11) and collecting all terms of the same powers of $\frac{G'}{G}$ to gether gets a polynomial in $\frac{G'}{G}$. Then setting each coefficient of its to zero gives a system of algebraic equations for a_0, a_1, λ, μ and v that solving it by Maple gets the following solutions:

$$v=c,\mu=\frac{\lambda^2}{4},a_0=\frac{-3\lambda d}{\pm\sqrt{\frac{-6d}{b}}},a_1=\pm\sqrt{\frac{-6d}{b}}. \tag{69}$$

Since in eq. (69), $\Delta=\lambda^2-4\mu=0$, so we obtain only the following traveling wave solution of rational function type.

$$q(x,t)=g(s)=\pm\frac{d\sqrt(6)(\lambda c_1+\lambda c_2 s-c_2)}{b(c_1+c_2 s)\sqrt{\frac{-b}{d}}} \tag{70}$$

and

$$r(x,t)=h(s)=f(s),\ r(x,t)=h(s)=\frac{-1}{2}f(s)\pm\frac{\sqrt{3}}{2}f(s)I, \tag{71}$$

where $f(s)=\frac{((4v-4\alpha g(s))\beta^2)^{\frac{1}{3}}}{\beta}$ and $s=x-ct$.

5. EXP-function method

In this section, we want to solve eqs. (1) and (2) with the exp-function method. For this, we consider eq. (11) as their converted form. Now, we solve eq. (11) with exp-function method which provides traveling wave solutions of the form

$$g(s)=\frac{\sum_{n=-p_1}^{p_2}a_n e^{ns}}{\sum_{m=-p_3}^{p_4}b_m e^{ms}} \tag{72}$$

where p_1, p_2, p_3 and p_4 are unknown positive integers. Using the ansatz (72) for $g'(s)^2$ and $g^4(s)$ gets

$$(g')^2(s)=\frac{c_1 e^{-2(p_1+p_3)s}+\cdots+c_2 e^{2(p_2+p_4)s}}{c_3 e^{-4p_3 s}+\cdots+c_4 e^{4p_4 s}} \tag{73}$$

and

$$g^4(s)=\frac{d_1 e^{-4p_1 s}+\cdots+d_2 e^{4p_2 s}}{d_3 e^{-4p_3 s}+\cdots+d_4 e^{4p_4 s}} \tag{74}$$

where $c_i, d_i,\ i=1,\cdots,4$ are obtained easily by calculations.
Balancing the highest order of exp-function in the eqs. (73) and (74) gives

$$2p_2+2p_4-4p_4=4p_2-4p_4$$

which leads

$$p_2=p_4.$$

Similarly, for determining p_1 and p_3 in eq.(72), balancing the lowest order of exp-function in the eqs. (73) and (74) gives

$$-2p_1 - 2p_3 + 4p_3 = -4p_1 + 4p_3$$

which leads

$$p_1 = p_3.$$

Since the final solution dose not depend on the values of p_1, p_2, p_3 and p_4, so for simplicity we let $p_1 = p_3 = 1$ and $p_2 = p_4 = 1$. Therefore eq. (74) becomes

$$g(s) = \frac{a_{-1}e^{-s} + a_0 + a_1 e^s}{b_{-1}e^{-s} + b_0 + b_1 e^s}. \tag{75}$$

Substituting eq. (75) into eq. (11), and equating the coefficients of all powers of e^{ns} to zero yields a system of algebraic equations for $a_{-1}, a_0, a_1, b_{-1}, b_0, b_1$ and v. Solving the system with the Maple gives the following solutions.

$$a_1 = a_{-1} = b_0 = 0,\ b_1 = \frac{ba_0^2}{24b_{-1}d},\ v = d + c,$$

so

$$q(x,t) = g(s) = \frac{a_0}{b_{-1}e^{-s} + \frac{ba_0^2}{24b_{-1}d}e^s}$$

and

$$r(x,t) = h(s) = f(s),\ r(x,t) = h(s) = \frac{-1}{2}f(s) \pm \frac{\sqrt{3}}{2}f(s)I,$$

are the solution of eqs. (1) and (2) where $f(s) = \frac{((4v-4\alpha g(s))\beta^2)^{\frac{1}{3}}}{\beta}$ and $s = x-(d+c)t$.

6. CONCLUSIONS

This paper studied a coupled nonlinear wave equation that arises in the study of nonlinear wavre equations. The traveling wave hypothesis was used to extract the solitary wave solutions with restrictions on the parameters that are all enlisted. The mapping method, modified mapping method and the extended mapping method are then subsequently and succesfully applied to obtain cnoidal wave, snoidal wave and other doubly-periodic solutions to the coupled wave equations. Finally the G'/G and the exp-function method are used to obtain more solutions to the coupled wave equations.

These solutions are going to be extremely useful in further studies of this equation. In fact, these results will be used to extract the conserved quantities of this coupled system. Furthermore, the perturbation terms will be turned on and the soliton perturbation theory will be utilized to obtain the adiabatic parameter dynamics of the soliton parameters. These results will be reported in future publications.

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