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THE SHILKRET-LIKE INTEGRAL ON THE SYMMETRIC INTERVAL

Biljana Mihailović[1], Maddalena Manzi[2], Petar Đapić[3]

A Shilkret-like integral based on absolutely monotone and sign stable set functions is investigated. We discuss main properties of this new integral. We consider the sufficient condition for the Shilkret-like integral-based representation of a functional defined on a symmetric interval.

Keywords: symmetric pseudo-operations, absolutely monotone set function, Shilkret-like integral

MSC2010: 28E10, 26E50.

1. **Introduction**

In the last few decades the Choquet integral and the Sugeno integral have had several applications in many fields of mathematics, soft computing, pattern recognition and decision analysis [4, 5, 7, 17, 21]. It shows up in the recently published papers that some new types of integrals on symmetric scales are useful tools for applications in decision making problems [6, 8, 12, 21].

The Sugeno integral has been introduced by M. Sugeno in [20] and for an $\mathcal{A}$-measurable function $f : X \to [0,1]$ is defined by

$$S_m(f) \quad = \quad \sup_{t\in[0,1]} (t \wedge m(\{x|f(x) \geqslant t\})), \tag{1}$$

where $m : \mathcal{A} \to [0,1]$ is a fuzzy measure and $\mathcal{A}$ denotes a σ-algebra of subsets of the universal set X. The Sugeno integral is one of a non-linear functional on the class of measurable functions which is comonotone-maxitive, monotone and $\wedge$-homogeneous [1, 16].

As a special type of the Sugeno-like integrals [2], the Shilkret integral [2, 18] originally has been defined for maxitive ($\vee$-additive) measures, but it is also defined for any fuzzy measure. This integral can be obtained by replacing $\wedge$ (minimum) with $\cdot$ (product) in (2). The Shilkret integral of an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}^+$

[1]Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia, e-mail: lica@uns.ac.rs

[2]Department of Economics, Ca' Foscari University of Venice, San Giobbe, Cannaregio 873, 30121 Venice, Italy, e-mail: maddalena.manzi@econ.units.it

[3]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia, e-mail: petarn@dmi.uns.ac.rs

is given by:

$$Sh_m(f) \quad = \quad \sup_{t\in[0,\infty[} (t \cdot m(\{x|f(x) \geqslant t\})), \tag{2}$$

where $m : \mathcal{A} \to [0, \infty]$ is a fuzzy measure. The main properties of this integral are comonotone maxitivity and homogeneity. It is monotone and if the underlying fuzzy measure is maxitive, it is subadditive [18].

The Choquet integral, introduced in [3], of a non-negative, $\mathcal{A}$-measurable function f based on a fuzzy measure $m : \mathcal{A} \to [0, \infty[$ is defined by

$$C_m(f) = \int_0^\infty m(\{x|f(x) \geqslant t\})dt. \tag{3}$$

The main properties of the Choquet integral are monotonicity and comonotone additivity, see [4, 16]. There exist asymmetric and symmetric extensions of the Choquet integral to the class of all $\mathcal{A}$-measurable functions ([4, 19]).

A special type of the Choquet-like integral based on the couple of pseudo-operations ($\oplus$,$\odot$), presented in [10], is related to some non-decreasing function $g : [0, 1] \to [0, \infty]$, $g(0) = 0$, and it is defined for non-negative, $\mathcal{A}$-measurable function f and fuzzy measure m. This integral is also defined for a real-valued function f, if for g is taken its odd extension on the real line.

The asymmetry of integrals is a desirable property in applications in decision making problems when gains and losses should be treated separately, such as situations occurring in the problems studied in mathematical psychology and behavioral economics. In [21] A. Tversky and D. Kahneman showed that one of the basic phenomena of choice under risk and uncertainty is loss aversion. The cumulative prospect theory, proposed in [21], is one of the integrals models for a representation of utility functional.

In [14] an absolutely monotone and sign stable set function $m : \mathcal{A} \to [-1, 1]$, $m(\varnothing) = 0$ has been introduced. It has been shown that m can be represented as the symmetric maximum of two set functions. The class of such set functions denoted by $AMSS$ has been presented and some properties of a set function $m \in AMSS$ have been shown.

The aim of this paper is to discuss the Shilkret-like integral with respect to a set function $m \in AMSS$, $|m(X)| < 1$, introduced in [11] and its application in decisionmaking. This integral is related to the couple ($\varovee, \odot$) of pseudo-operations. As we shall see, this new integral is monotone, asymmetric, co-comonotone $\varovee$-additive and positively $\odot$-homogeneous.

In Section 2, preliminary notions are given, definitions of symmetric pseudo-operations and the class of absolutely monotone, sign stable set functions are presented. In Section 3, the Shilkret-like integral with respect to $m \in AMSS$, $|m(X)| < 1$ is introduced, and its basic properties are shown. Finally, we consider the necessary conditions for a functional $\mathbf{I}$ to be representable by the Shilkret-like integral with respect to $m \in AMSS$.

2. Preliminaries

Let us first recall definitions of two symmetric operations, the symmetric maximum, introduced in [7] and the pseudo-multiplication $\odot$ generated by an odd, symmetric, multiplicative generator g. Following [7, 8, 9] we have the next definition.

Definition 1. (i) *The symmetric maximum* $\varovee : [-a,a]^2 \to [-a,a]$, $a \in \overline{\mathbb{R}}^+$, *is given by*

$$x \varovee y \quad = \quad \operatorname{sign}(x+y)(|x| \vee |y|),$$

with the convention $\infty \varovee (-\infty) = 0$.

(ii) *Let* $g : [-1,1] \to [-\infty,\infty]$, $g(0) = 0$ *be an odd, strictly increasing, continuous function. The pseudo-multiplication* $\odot : [-1,1]^2 \to [-1,1]$ *is defined by*

$$x \odot y = g^{-1}(g(x)g(y)), \tag{4}$$

with the convention $\infty \cdot 0 = 0$ *or* $\infty \cdot 0 = \infty$.

The generator g is called a multiplicative generator for $\odot$.

The results presented in the following proposition have been shown in [7, 8, 9].

For more details we recommend [7, 9, 13].

Lemma 1. *Let* $g : [-1,1] \to [-\infty,\infty]$, $g(0) = 0$ *be an odd, strictly increasing, continuous function. The pseudo-multiplication* $\odot$ *generated by* g *is distributive with respect to the symmetric maximum* $\varovee : [-1,1]^2 \to [-1,1]$, *i.e., for all* $x, y, z \in [-1,1]$ *we have*

$$x \odot (y \varovee z) = (x \odot y) \varovee (x \odot z).$$

`Proof.` For any $x, t \in [-1,1]$ we have

$$x \odot t = \operatorname{sign}(x \cdot t) g^{-1}(g(|x|) \cdot g(|t|)). \tag{5}$$

Let $x, y, z \in [-1,1]$.

I case. $y = -z$. Then $g(y) = -g(z)$, hence, by recalling that the inverse of an odd function is also odd, we have

$$\begin{aligned} x \odot (y \varovee z) &= g^{-1}(g(x)g(0)) = 0 = g^{-1}(g(x)g(y)) \varovee (-g^{-1}(g(x)g(-z))) = \\ &= (x \odot y) \varovee (x \odot z). \end{aligned}$$

II case. $|y| \geqslant |z|$, $y \neq -z$. Since we have $y \varovee z = y$, then $x \odot (y \varovee z) = x \odot y$. On the other side if $|y| \geqslant |z|$, $y \neq -z$, then $g(|y|) \geqslant g(|z|)$ and by (5) we have $|x \odot y| = |g^{-1}(g(|x|)g(|y|))| \geqslant |g^{-1}(g(|x|)g(|z|))| = |x \odot z|$. Therefore

$$x \odot (y \varovee z) = x \odot y = (x \odot y) \varovee (x \odot z).$$

III case. $|y| < |z|$. Then $y \varovee z = z$ and similarly as above we obtain distributivity of $\odot$ with respect to $\varovee$. □

Example 1. *1.1) The pseudo-multiplication* $\odot$ *defined by:*

$$x \odot y = \operatorname{sign}(x \cdot y) \left(1 - e^{-\ln(1-|x|)\ln(1-|y|)}\right),$$

for all $x, y \in]-1,1[$, is generated by a multiplicative generator $g : [-1,1] \to [-\infty, \infty]$ given with:

$$g(x) = \operatorname{sign}(-x)\ln(1-|x|), \tag{6}$$

$g(1) = \infty$, $g(-1) = -\infty$. *The neutral element is* $e_\odot = 1 - \frac{1}{e}$, *and for* $x \in]-1,1[\setminus\{0\}$ *we have* $x^{-1} = \operatorname{sign}(x)\left(1 - e^{\frac{1}{\ln(1-|x|)}}\right)$.

1.2) Let g be defined by:

$$g(x) = \sqrt{3}\tan\frac{\pi x}{2}, \tag{7}$$

for $x \in]-1,1[$, $g(1) = \infty$, $g(-1) = -\infty$. *The pseudo-multiplication $\odot$, for all $x, y \in]-1,1[$ is given by*

$$x \odot y = \frac{2}{\pi}\arctan(\sqrt{3}\tan\frac{\pi x}{2} \cdot \tan\frac{\pi y}{2}).$$

The neutral element of $\odot$ is $e_\odot = \frac{1}{3}$. *The inverse element, for* $x \in]-1,1[\setminus\{0\}$ *is* $x^{-1} = \frac{2}{\pi}\arctan(\frac{1}{3}\cot\frac{\pi x}{2})$.

The positive part and the negative part of a set function m, $m : \mathcal{A} \to [-a, a]$, $a \in \overline{\mathbb{R}}^+$, are non-negative set functions $m^+, m^- : \mathcal{A} \to [0, a]$ defined by:

$$m^+(A) = m(A) \vee 0, \tag{8}$$

$$m^-(A) = (-m(A)) \vee 0. \tag{9}$$

Let $\mathcal{A}$ be a σ-algebra of subsets of a non-empty universal set X. According to [14, 16] we have the next definition.

Definition 2. *A set function* $m : \mathcal{A} \to [-a, a]$, $a \in \overline{\mathbb{R}}^+$, $m(\varnothing) = 0$ *is*

(i) *non-negative if* $m(A) \geqslant 0$ *for all* $A \in \mathcal{A}$,

(ii) *fuzzy measure if it is non-negative and monotone, i.e. if for all* $A, B \in \mathcal{A}$, $A \subset B$ *we have* $0 \leqslant m(A) \leqslant m(B)$,

(iii) *absolutely monotone if for all* $A, B \in \mathcal{A}$, $A \subset B$ *we have* $|m(A)| \leqslant |m(B)|$.

(iv) *sign stable if it fulfils:*

$$\sup_{E \subset A} m^+(E) < m^-(A), \text{ if } m(A) < 0,$$

$$\sup_{E \subset A} m^-(E) < m^+(A), \text{ if } m(A) > 0,$$

$$\text{for all } E \subset A \quad m^+(E) = m^-(E), \text{ if } m(A) = 0.$$

The representation theorem of a set function $m : \mathcal{A} \to [-a, a]$, $a \in \overline{\mathbb{R}}^+$ $m(\varnothing) = 0$, from the class $AMSS$ of all sign stable and absolutely monotone set functions was shown in [14]. Namely, if $m \in AMSS$ then the set functions m_1, $m_2 : \mathcal{A} \to [0, a]$, defined by

$$m_1(A) = \begin{cases} m^+(A), & m(A) \geqslant 0, \\ \sup\limits_{E \subset A} m^+(E), & m(A) < 0. \end{cases} \tag{10}$$

$$m_2(A) = \begin{cases} m^-(A), & m(A) \leqslant 0, \\ \sup\limits_{E \subset A} m^-(E), & m(A) > 0. \end{cases} \tag{11}$$

are fuzzy measures such that $m = m_1 \varovee (-m_2)$ and $|m| = m_1 \vee m_2$. Additionally, if there exist fuzzy measures $\tilde{m}_1$, $\tilde{m}_2 : \mathcal{A} \to [0, a]$, such that for each $A \subset B$, $A, B \in \mathcal{A}$, we have $\tilde{m}_1(A) = \tilde{m}_2(A)$ whenever $\tilde{m}_1(B) = \tilde{m}_2(B)$ and $m = \tilde{m}_1 \varovee (-\tilde{m}_2)$, then $m \in AMSS$ and $m_1 \leqslant \tilde{m}_1$ and $m_2 \leqslant \tilde{m}_2$. For more details we recommend [14].

Example 2. *Let $k = \text{card}(A)$ be a cardinal number of set $A \subseteq \{1, 2, 3, 4\}$. Let us define a set function m for all $A \subseteq \{1, 2, 3, 4\}$ by*

$$m(A) = \begin{cases} \frac{k}{5}, & \text{if } k \text{ is odd} \\ -\frac{k}{5}, & \text{if } k \text{ is even} \\ 0, & \text{else.} \end{cases}$$

Hence, we have

$$m_1(A) = \begin{cases} 0.6, & k = 4 \text{ or } k = 3 \\ 0.2 & k = 1 \text{ or } k = 2 \\ 0 & \text{else} \end{cases} \quad \textit{and} \quad m_2(A) = \begin{cases} 0.8, & k = 4 \\ 0.4, & k = 2 \text{ or } k = 3 \\ 0, & \text{else.} \end{cases}$$

The next lemma has been proven in [15].

Lemma 2. *Let $m : \mathcal{A} \to [-a, a]$, $a \in \overline{\mathbb{R}}^+$ be a set function. $m \in AMSS$ iff $-m \in AMSS$ and for m_1 and m_2 given by (10) and (11), respectively we have*

$$(-m)_1 = m_2 \qquad \textit{and} \qquad (-m)_2 = m_1.$$

3. The Shilkret-like integral based on $m \in AMSS$

Let us define the Shilkret-like integral for $f \in \mathcal{F}$ based on $m \in AMSS$. This integral is related to the couple $(\varovee, \odot)$ of pseudo-operations. Also, it is related to the Shilkret integral, since it has similar properties as the Shilkret integral, as we are going to prove in the sequel. We consider a measurable space $(X, \mathcal{A})$, where X is a universal set, and an $\mathcal{A}$-measurable function f, $f : X \to [-1, 1]$, which is bounded, i.e. such that $\sup_{x \in X} |f(x)| < 1$. $\mathcal{F}$ denotes the class of such functions and $\mathcal{F}^+$ the class of non-negative, bounded, $\mathcal{A}$-measurable functions $f : X \to [0, 1]$ from $\mathcal{F}$. For any $x \in X$ and $f, h \in \mathcal{F}$, let us define

$$(f \varovee h)(x) = f(x) \varovee h(x), \quad (f \odot h)(x) = f(x) \odot h(x),$$

and for each $a \in]-1, 1[$

$$(a \odot f)(x) = a \odot f(x).$$

Let us first define the asymmetric extension of Shilkret integral for an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}$, based on $m \in AMSS$, $m : \mathcal{A} \to [-\infty, \infty]$ in the following manner:

$$Sh_m(f) \;\; = \;\; Sh_{m_1}(f^+) \varovee (-Sh_{m_2}(f^-)),$$

where the fuzzy measures m_1 and m_2 are given by (10) and (11), respectively. Now, let us define the Shilkret-like integral for $f \in \mathcal{F}$ based on $m \in AMSS$, $m : \mathcal{A} \to [-1, 1]$.

Definition 3. *Let $f \in \mathcal{F}$ and $m \in AMSS$, such that $|m(X)| < 1$ and let $g : [-1,1] \to [-\infty, \infty]$, $g(0) = 0$ be an odd, strictly increasing, continuous function, which generates the pseudo-multiplication $\odot$. The Shilkret-like integral based on m is defined by*

$$\begin{aligned} Sh^g_m(f) &= g^{-1}\left(Sh_{g\circ m}(g \circ f)\right) \\ &= \sup_{t\in[0,1[} (t \odot m_1(\{f^+ \geqslant t\})) \oslash \Big(- \sup_{t\in[0,1[} (t \odot m_2(\{f^- \geqslant t\}))\Big), \end{aligned}$$

where the fuzzy measures m_1 and m_2 are given by (10) and (11), respectively.

In [15] it has been shown that the Shilkret-like integral can be obtained as a sequence of generated Choquet integrals with respect to a set function $m \in AMSS$, $|m(X)| < 1$.

Example 3. *Let $m \in AMSS$ be defined with m_1 and m_2 given by:*

$$m_1(A) = \begin{cases} a, & \text{if } A \neq \varnothing, \\ 0, & A = \varnothing, \end{cases} \quad \text{and} \quad m_2(A) = \begin{cases} b, & \text{if } A = X, \\ 0, & A \neq X, \end{cases}$$

where $a, b > 0$, $a \neq b$ and let $g : [-1,1] \to [-\infty, \infty]$ be the function defined in Example 1 by (6).
3.1) If $a = 1 - \frac{1}{e^p}$, $p > 0$ and $b = 1 - \frac{1}{e} = e_\odot$, then

$$Sh^g_m(f) = \begin{cases} 1 - (1 - \sup\limits_{x\in X} f^+(x))^p, & s > 0, \\ - \inf\limits_{x\in X} f^-(x), & s = 0, \end{cases}$$

where $s = \sup\limits_{x\in X} f^+(x)$.
3.2) If $a = 1 - \frac{1}{e} = e_\odot$ and $b = 1 - \frac{1}{e^r}$, $r > 0$, then

$$Sh^g_m(f) = \begin{cases} \sup\limits_{x\in X} f^+(x), & s > 0, \\ (1 - \inf\limits_{x\in X} f^-(x))^r - 1, & s = 0, \end{cases}$$

where $s = \sup\limits_{x\in X} f^+(x)$. For $X = \{1,2,3,4\}$ and $r = 2$, if we take $f, h : X \to [-1,1]$, $f = (-0.1, -0.3, 0.9, 0.8)$ and $h = (-0.1, -0.4, -0.7, -0.5)$, then we obtain $Sh^g_m(f) = 0.9$, $AI_m(-f) = 0.3$, $Sh^g_m(h) = -0.19$ and $Sh^g_m(-h) = 0.7$.

Having in mind possible applications in decision-making problems, constants p and r will be called the degree of gain and loss aversion, respectively.

In order to prove some properties of the Shilkret-like integral based on $m \in AMSS$, we are going to use a concept of comonotone functions proposed in [4]. We have the next definition of co-comonotone functions, see [14].

Definition 4. *Let f and h be functions from $\mathcal{F}$.*
(i) *f and h are comonotone if for all x, $x_1 \in X$ we have $f(x) < f(x_1) \Rightarrow h(x) \leqslant h(x_1)$.*
(ii) *f and h are cosigned functions if for all $x \in X$ we have $f(x) \cdot h(x) \geqslant 0$.*
Let us denote with $f \backsim_s h$ a couple of comonotone and cosigned functions f and h.

For an $\mathcal{A}$-measurable function $f \in \mathcal{F}$, we have $f = f^+ \otimes (-f^-)$, where $f^+, f^- \in \mathcal{F}^+$, are the positive part and the negative part of f defined with $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

For $f,\, h \in \mathcal{F}$ such that $f \backsim_s h$, we have the next lemma proven in [14].

Lemma 3. *For any $f,\, h \in \mathcal{F}$, such that $f \backsim_s h$ we have*
(i) $(f \otimes h)^+ = f^+ \otimes h^+$;
(ii) $(f \otimes h)^- = f^- \otimes h^-$;
(iii) $c \odot f \backsim_s d \odot h$ for every $c,\, d \in [0,1[$;
(iv) $f \backsim_s f \otimes h$.

Let $\mathcal{S}$ be a subclass of $\mathcal{F}$ of all simple functions. For $a \in [0,1[$ and $A \in \mathcal{A}$, let us define

$$(a \cdot \mathbf{1}_A)(x) = \begin{cases} a & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \qquad \text{and} \qquad (a \cdot (-\mathbf{1}_A))(x) = \begin{cases} -a & \text{if } x \in A, \\ \;\;0 & \text{if } x \notin A. \end{cases}$$

Note that $\mathbf{1} = e_\odot$, i.e. the notation $a \cdot \mathbf{1}_A$ instead of $a \odot e_{\odot A}$ ($e_\odot$ is a neutral element for $\odot$) will be used. In what follows, let us consider the Shilkret-like integral based on $m \in AMSS$ defined on $\mathcal{F}^+$. Note that if $f \in \mathcal{F}^+$ we have

$$Sh_m^g(f) \;\;=\;\; \sup_{t \in [0,1[} (t \odot m_1(\{f \geqslant t\})). \tag{12}$$

Theorem 1. *Let $Sh_m^g : \mathcal{F}^+ \to [0,1[$, be a Shilkret-like integral based on $m \in AMSS$, $|m(X)| < 1$. Then Sh_m^g is monotone, comonotone $\vee$-additive and $\odot$-homogeneous.*

`Proof.` Monotonicity follows from definition (12), the monotonicity of the fuzzy measure m_1 and the restriction of operation $\odot$ on $[0,1]^2$.

For comonotone functions f and h, $f, h \in \mathcal{F}^+$, for any $t \in [0,1[$, we have $m_1(\{f \vee h \geqslant t\}) = m_1(\{f \geqslant t\}) \vee m_1(\{h \geqslant t\})$, hence, by Lemma 1 and (12) we obtain

$$Sh_m^g(f \otimes h) = Sh_m^g(f) \otimes Sh_m^g(h),$$

i.e. Sh_m^g is comonotone $\vee$-additive (comonotone maxitive).

For any $f \in \mathcal{F}^+$, $a \in [0,1[$ and $t \in [0,1[$, we have

$$t \odot m_1(\{a \odot f \geqslant t\}) = a \odot \bar{t} \odot m_1(\{f \geqslant \bar{t}\}),$$

where $\bar{t} = g^{-1}\left(\frac{g(t)}{g(a)}\right)$. Hence, it follows that Sh_m is $\odot$-homogeneous. $\square$

For a functional $\mathbf{I} : \mathcal{F} \to]-1,1[$, the support of $\mathbf{I}$ is given by

$$\operatorname{supp}(\mathbf{I}) = \{f \in \mathcal{F} |\, \mathbf{I}(f) \neq 0\}\,.$$

In the next theorem the properties of Sh_m^g are given.

Theorem 2. *Let $Sh_m^g : \mathcal{F} \to]-1,1[$, be a Shilkret-like integral based on $m \in AMSS$, $|m(X)| < 1$. Then Sh_m^g is:*
(i) *monotone: for all $m \in AMSS$ such that $|m(X)| < 1$, for all $f,\, h \in \mathcal{F}$ we have*

$$f \leqslant h \Rightarrow Sh_m^g(f) \leqslant Sh_m^g(h);$$

(ii) *monotone with respect to* m*: for all* $f \in \mathcal{F}$*, for all* $m', m \in AMSS$*,* $|m(X)| < 1$*,* $|m'(X)| < 1$ *we have*

$$m' \leqslant m \Rightarrow Sh^g_{m'}(f) \leqslant Sh^g_m(f),$$

(iii) *positive weakly* $\odot$*-homogeneous: for any basic function from* $\mathcal{S}$ *we have*

$$Sh^g_m(a \cdot \mathbf{1}_A) = \begin{cases} a \odot Sh^g_m(\mathbf{1}_A) = a \odot m_1(A), & a \geqslant 0 \\ -a \odot Sh^g_m(-\mathbf{1}_A) = a \odot m_2(A), & a < 0. \end{cases}$$

(iv) *positive* $\odot$*-homogeneous: for all* $a \in [0, 1[$*,* $f \in \mathcal{F}$ *we have*

$$Sh^g_m(a \odot f) = a \odot Sh^g_m(f);$$

(v) *asymmetric: for all* $f \in \mathcal{F}$ *we have*

$$Sh^g_m(-f) = -Sh^g_{-m}(f);$$

(vi) *co-comonotone* $\otimes$*-additive on* $\mathrm{supp}(Sh^g_m) \cup \mathcal{F}^+ \cup \mathcal{F}^-$*: for all* $f \backsim_s h$*,* $f, h \in \mathrm{supp}(Sh^g_m) \cup \mathcal{F}^+ \cup \mathcal{F}^-$ *we have*

$$Sh^g_m(f \otimes h) = Sh^g_m(f) \otimes Sh^g_m(h).$$

`Proof.` (i) By Theorem 1, the Shilkret-like integral is a monotone functional on $\mathcal{F}^+$ and $\otimes$ is a monotone operation. For $f, h \in \mathcal{F}$ such that $f \leqslant h$ we have $f^+ \leqslant h^+$ and $f^- \geqslant h^-$, hence $Sh^g_m(f) \leqslant Sh^g_m(h)$.
(ii) Let $m', m \in AMSS$, $|m(X)| < 1$ and $|m'(X)| < 1$. By Theorem 1 [14] there exist m'_1 and m'_2 defined by (10) and (11) such that $m'(A) = m'_1(A) \otimes (-m'_2(A))$, for all $A \in \mathcal{A}$. Analogously, there exist m_1 and m_2 such that $m(A) = m_1(A) \otimes (-m_2(A))$, for all $A \in \mathcal{A}$. From (10) and (11) it follows that $m'_1 \leqslant m_1$ and $m'_2 \geqslant m_2$.

Since $\odot$ is a monotone operation on $[0, 1]^2$ by (12), we have that the Shilkret-like integral is a monotone functional with respect to a fuzzy measure $m \in AMSS$ and $\otimes$ is a monotone operation, hence, for all $f \in \mathcal{F}$ we have $Sh^g_{m'}(f) \leqslant Sh^g_m(f)$. The properties of the Shilkret-like integral (iii) and (iv) follow from Definition 3 and Theorem 1.
(v) Let $m \in AMSS$, $|m(X)| < 1$. By Lemma 2 we have that $-m \in AMSS$ and $|(-m)(X)| < 1$. Hence,

$$Sh^g_m(-f) \;\; = \;\; -Sh^g_{-m}(f).$$

(vi) Let us suppose $f \backsim_s h$, $f, h \in \mathrm{supp}(Sh^g_m) \cup \mathcal{F}^+ \cup \mathcal{F}^-$. By (v), Lemma 3 and comonotone $\oplus$-additivity of the Shilkret-like integral on $\mathcal{F}^+$, we have:

$$\begin{aligned} Sh^g_m(f \otimes h) &= Sh^g_{m_1}((f \otimes h)^+) \otimes (-Sh^g_{m_2}((f \otimes h)^-)) \\ &= Sh^g_{m_1}(f^+ \otimes h^+) \otimes (-Sh^g_{m_2}(f^- \otimes h^-)) \\ &= \big(Sh^g_{m_1}(f^+) \otimes Sh^g_{m_1}(h^+)\big) \otimes \big(-\big(Sh^g_{m_2}(f^-) \otimes Sh^g_{m_2}(h^-)\big)\big) \\ &= Sh^g_m(f) \otimes Sh^g_m(h). \end{aligned}$$

□

In the following, an asymmetric functional defined on $\mathcal{F}$ will be considered, i.e let us suppose that it is not symmetric in general, i.e.

$$-\mathbf{I}(\mathbf{1}_A) = \mathbf{I}(-\mathbf{1}_A), \text{ for } A \in \mathcal{A}$$

is not satisfied in general.

Definition 5. *Let* $\mathbf{I} : \mathcal{F} \to]-1,1[$ *be a functional. We say that* $\mathbf{I}$ *is:*
(i) positive weakly $\odot$*-homogeneous if for all basic functions* $a \cdot \mathbf{1}_A$*,* $A \in \mathcal{A}$*,* $a \in [0,1[$*, we have*

$$\mathbf{I}(a \cdot \mathbf{1}_A) = a \odot \mathbf{I}(\mathbf{1}_A), \quad \mathbf{I}(a \cdot (-\mathbf{1}_A)) = a \odot \mathbf{I}(-\mathbf{1}_A),$$

(ii) comonotone-cosigned $\otimes$*-additive if for all co-comonotone functions* $f, h \in \mathcal{F}$ *we have*

$$\mathbf{I}(f \otimes h) = \mathbf{I}(f) \otimes \mathbf{I}(h),$$

(iii) continuous from below if for each non-decreasing sequence $\{f_n\}_{n\in\mathbb{N}}$*,* $f_n \in \mathcal{F}$ *we have*

$$\mathbf{I}(\sup_n f_n) = \sup_n \mathbf{I}(f_n),$$

(iv) weakly sign-dependant continuous if for every $A \in \mathcal{A}$ *it fulfils the next conditions*
a) if $-\mathbf{I}(\mathbf{1}_A) > \mathbf{I}(-\mathbf{1}_A)$*, then*

$$\mathbf{I}(\mathbf{1}_A) = \sup\{\mathbf{I}(\mathbf{1}_E) | E \subset A, -\mathbf{I}(\mathbf{1}_E) < \mathbf{I}(-\mathbf{1}_E)\},$$

b) if $-\mathbf{I}(\mathbf{1}_A) < \mathbf{I}(-\mathbf{1}_A)$*, then*

$$\mathbf{I}(-\mathbf{1}_A) = \inf\{\mathbf{I}(-\mathbf{1}_E) | E \subset A, \mathbf{I}(-\mathbf{1}_E) < -\mathbf{I}(\mathbf{1}_E)\},$$

c) for all $E \subset A$*,* $E, A \in \mathcal{A}$*, the equality*

$$-\mathbf{I}(\mathbf{1}_A) = \mathbf{I}(-\mathbf{1}_A) \text{ implies } -\mathbf{I}(\mathbf{1}_E) = \mathbf{I}(-\mathbf{1}_E).$$

Note that if a functional $\mathbf{I}$ is continuous from below on $\mathcal{F}^+$, and continuous from above on $\mathcal{F}^-$, then it fulfills the conditions (iv) a), b) and c), i.e. it is weakly sign-dependant continuous.

In the next theorems the conditions for the representation of a functional $\mathbf{I} : \mathcal{F} \to]-1,1[$ by the Shilkret-like integral based on $m \in AMSS$ are given.

Theorem 3. *Let* $\mathbf{I} : \mathcal{F}^+ \to [0,1[$ *be a monotone, positive weakly* $\odot$*-homogenous, co-comonotone* $\otimes$*-additive and continuous from below functional. Then there exists a fuzzy measure* m *continuous from below, such that, for all* $f \in \mathcal{F}^+$*, we have*

$$\mathbf{I}(f) = Sh_m^g(f).$$

`Proof.` Since $\mathbf{I}$ is a monotone functional, a set function m defined for all $A \in \mathcal{A}$ by $m(A) = \mathbf{I}(\mathbf{1}_A)$ is a fuzzy measure. $\mathbf{I}$ is a functional continuous from below, hence m is continuous from below. Let $\{s_n\}_{n\in\mathbb{N}}$ be the non-decreasing sequence of non-negative, simple functions converging to $f \in \mathcal{F}^+$, given by $s_n = \bigvee_{i=1}^{n\cdot 2^n} \frac{i}{2^n} \cdot \mathbf{1}_{A_{n,i}}$, $n \in \mathbb{N}$ where $A_{n,i} = \{x \in X \mid f(x) \geqslant \frac{i}{2^n}\}$, for $i = 0, \ldots, n \cdot 2^n$. By the positive weakly $\odot$-homogeneousity and co-comonotone $\otimes$-additivity of the functional $\mathbf{I}$, when applying it on the above representation of $s_n \in \mathcal{S}^+$ we have $\mathbf{I}(s_n) = Sh_m^g(s_n)$ and hence, for $f \in \mathcal{F}^+$ by the continuity from below of $\mathbf{I}$ we obtain

$$\begin{aligned}
\mathbf{I}(f) &= \mathbf{I}(\sup_n s_n) = \sup_n \mathbf{I}(s_n) = \sup_n Sh_m^g(s_n) \\
&= \sup_{t\in[0,1[} (t \odot m(\{f \geqslant t\})) = Sh_m^g(f),
\end{aligned}$$

□

The representation of a functional $\mathbf{I}$, defined on a subclass $\mathcal{S}$ of functions with the finite range by the Shilkret-like integral, is established in the next theorem.

Theorem 4. *Let $\mathbf{I} : \mathcal{S} \to]-1, 1[$ be a monotone, positive weakly $\odot$-homogeneous, co-comonotone $\varovee$-additive and weakly sign-dependant continuous functional. Then, there exists $m \in AMSS$ such that for all $f \in \mathcal{S}$, we have*

$$\mathbf{I}(f) = Sh_m^g(f).$$

Proof. Let $\mathbf{I} : \mathcal{S} \to]-1, 1[$ be a monotone functional, which is positive weakly $\odot$-homogeneous, co-comonotone $\varovee$-additive and weakly sign-dependant continuous. Let $\tilde{m}_1$ and $\tilde{m}_2$ be fuzzy measures defined by $\tilde{m}_1(A) = \mathbf{I}(\mathbf{1}_A)$ and $\tilde{m}_2(A) = -\mathbf{I}(-\mathbf{1}_A)$, for $A \in \mathcal{A}$. By Theorem 1 in [14] we have that a set function defined by $m = \tilde{m}_1 \varovee (-\tilde{m}_2)$ belongs to $AMSS$. $\mathbf{I}$ is weakly sign-dependant continuous functional, hence for $m(A) < 0$, we have

$$\tilde{m}_1(A) = \mathbf{I}(\mathbf{1}_A) = \sup_{\substack{E \subset A, \\ \mathbf{I}(\mathbf{1}_E) > -\mathbf{I}(-\mathbf{1}_E)}} \mathbf{I}(\mathbf{1}_E) = \sup_{E \subset A} m^+(E) = m_1(A),$$

and for $m(A) \geqslant 0$, we have $\tilde{m}_1(A) = m_1(A)$. Therefore $\tilde{m}_1 = m_1$. Analogously, we obtain $\tilde{m}_2 = m_2$, where for $m = \tilde{m}_1 \varovee (-\tilde{m}_2)$ fuzzy measures m_1 and m_2 are defined by (10) and (11), respectively.

Let $s \in \mathcal{S}$. Its comonotone-cosigned $\varovee$-additive representation is given by: $s = s^+ \varovee (-s^-)$, where

$$s^+ = \bigvee_{i=1}^{n} a_i \mathbf{1}_{A_i}\,, \qquad -s^- = \bigvee_{i=1}^{n} b_i(-\mathbf{1}_{B_i})\,, \tag{13}$$

and $a_i = s^+_{\sigma(i)}$, $b_i = s^-_{\sigma(n+1-i)}$, $A_i = A_{\sigma(i)}$, $B_i = A_1 \setminus A_{\sigma(n+2-i)}$, and σ is a permutation, such that $-1 < s_{\sigma(1)} \leqslant \cdots \leqslant s_{\sigma(n)} < 1$, $A_{\sigma(i)} = \{x \in X \mid s(x) \geqslant s_{\sigma(i)}\}$, $A_{\sigma(n+1)} = \varnothing$. Since $\mathbf{I}$ is positive weakly $\odot$-homogeneous and co-comonotone $\varovee$-additive we obtain

$$\begin{aligned}
\mathbf{I}(s) &= \mathbf{I}(s^+ \varovee (-s^-)) = \mathbf{I}(s^+) \varovee \mathbf{I}(-s^-) \\
&= \bigvee_{i=1}^{n} a_i \odot \mathbf{I}(\mathbf{1}_{A_i}) \varovee \bigvee_{i=1}^{n} b_i \odot \mathbf{I}(-\mathbf{1}_{B_i}) \\
&= \bigvee_{i=1}^{n} a_i \odot m_1(A_i) \varovee \big(- \bigvee_{i=1}^{n} b_i \odot m_2(B_i)\big) \\
&= \sup_{t \in [0,1[} (t \odot m_1(\{s^+ \geqslant t\})) \bigvee \big(- \sup_{t \in [0,1[} (t \odot m_2(\{s^- \geqslant t\}))\big) \\
&= Sh_m^g(s)
\end{aligned}$$

□

4. **Conclusions**

The results of this paper are related to integrals based on absolutely monotone real set functions, considered in [15]. We have analyzed a new asymmetric integral, the Shilkret-like integral based on absolutely monotone and sign stable set functions and its main properties. These results can be applied in decision making problems when gains and losses should be treated separately. In [21] it was shown that one of the basic phenomena of choice under risk and uncertainty is loss aversion. Since it is monotone and asymmetric, the Shilkret-like integral defined on the class of prospects, could be utility functional and proposed results could be used for modeling the observed asymmetry between gains and losses. Finally, let us mention that in [15] alternative approach for constructing the Shilkret-like integral to the one described here has been presented; however, in this paper we do not consider it. We only recall that this method require preliminaries concerning the generated Choquet integrals based on absolutely monotone real set functions.

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