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ENDPOINTS OF SUZUKI TYPE QUASI-CONTRACTIVE MULTIFUNCTIONS

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Some researchers have been provided many results about endpoints of some contractive multifunctions. In this paper, we give some endpoint results about Suzuki type quasi-contractive multifunctions which have one of the properties (BS) or (SBS).

Keywords: Endpoint, multifunction, Suzuki type quasi-contraction, property (BS).

1. Introduction

Let (X,d) be a metric space, 2^X the set of all nonempty subsets of X, $CB(X)$ the set of all nonempty closed bounded subsets of X and $x \in X$. As we know, the Hausdorff metric H on $CB(X)$ is defined by $H(A,B) = max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$. An element $x \in X$ is said to be a fixed point of the multifunction $T : X \to 2^X$ whenever $x \in Tx$. Also, an element $x \in X$ is said to be an endpoint of T whenever $Tx = \{x\}$. We say that T has the approximate fixed point property whenever $\inf_{x\in X} \sup_{y\in Tx} d(x,y) = 0$. In 2010, Amini-Harandi proved that some multifunctions which have unique endpoint if and only if have approximate endpoint property ([2]). Then, Moradi and Khojasteh generalized his main result for generalized weak contractive multifunction ([7]). The technique of α-ψ-contractive mappings introduced by Samet, Vetro and Vetro in 2012 ([9]). Later, some authors used it for some subjects in fixed point theory (see for example [3], [5] and [8]) or generalized it by using different methods for some contractive multifunctions (see for example [1], [4] and [6]). Denote by Ψ the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ ([9]). It is known that $\psi(t) < t$ for all $t > 0$ ([9]). We say that a multifunction $T : X \to CB(X)$ has the property (BS) whenever for each $x \in X$ there exists $y \in Tx$ such that $H(Tx,Ty) = \sup_{b\in Ty} d(y,b)$. In fact, there are many multifunctions which have the property (BS). For see this, let $X = [0,\infty)$, $d(x,y) = |x-y|$, $s,t > 0$, $T_1, T_2 : X \to CB(X)$ be defined by $T_1x = [0, sx]$ and $T_2x = [x, x+t]$. It is easy to check that the multifunctions T_1 and T_2 have the property (BS). Also, we say that the multifunction T has the property (SBS) whenever for each sequence $\{x_n\}$ with $d(x_n, Tx_n) \le d(x_n, x_{n+1}) + \psi(d(x_n, x_{n+1}))$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, Tx_{n_k}) \le d(x_{n_k}, x) + \psi(d(x_{n_k}, x))$ for all k. Let $\alpha : X \times X \to [0,\infty)$ be a mapping and $T : X \to CB(X)$ a multifunction. We say that T is α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \ge 1$ we have $\alpha(y,z) \ge 1$ for all $z \in Ty$ ([6]). Also, we say that X has the condition (C_α) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \ge 1$ for all k ([6]). Recall that T is continuous whenever $H(Tx_n, Tx) \to 0$ for all sequence $\{x_n\}$ in X with $x_n \to x$. In this papers, by using and combining the idea of the papers [2], [7], [9] and

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[6], we give some results about endpoints of Suzuki type quasi-contractive multifunctions which have one of the properties (BS) or (SBS).

2. **Main results**

Here, we provide our main results.

Theorem 2.1. *Let (X, d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0, \infty)$ a mapping and $T : X \to CB(X)$ an α-admissible such that T has the property (BS) and $\alpha(x, y)H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$, where*

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or ψ is right upper semi-continuous and X has the condition (C_α), then T has an endpoint.

Proof. Choose $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Since T has the property (BS), there exists $x_2 \in Tx_1$ such that $H(Tx_1, Tx_2) = \sup_{b \in Tx_2} d(x_2, b)$. Since T is α-admissible, $\alpha(x_1, x_2) \geq 1$. By continuing this process, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) \geq 1$ and $H(Tx_n, Tx_{n+1}) = \sup_{b \in Tx_{n+1}} d(x_{n+1}, b)$ for all n. Note that,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \sup_{b \in Tx_n} d(x_n, b) = H(Tx_{n-1}, Tx_n) \\
&\leq \alpha(x_{n-1}, x_n)H(Tx_{n-1}, Tx_n) \leq \psi(M(x_{n-1}, x_n)) \\
&= \psi(\max\{d(x_{n-1}, x_n)), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}\}) \\
&\leq \psi(\max\{d(x_{n-1}, x_n)), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\}) \\
&\leq \psi(\max\{d(x_{n-1}, x_n)), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\}) \\
&\leq \psi(\max\{d(x_{n-1}, x_n)), d(x_n, x_{n+1})\})
\end{aligned}$$

for all $n \geq 2$. If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then $d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n+1}))$. Hence, $d(x_n, x_{n+1}) = 0$ and so $d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)))$. On the other hand, we get $d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))$ whenever $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n))$. Since ψ is nondecreasing, we obtain $d(x_n, x_{n+1}) \leq \psi^{n-1}(d(x_1, x_2))$ for all $n \geq 2$. Since

$$d(x_n, x_m) \leq \Sigma_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \Sigma_{i=n}^{m-1} \psi^{i-1}(d(x_1, x_2)),$$

$\{x_n\}$ is a Cauchy sequence. Choose $x^\star \in X$ such that $x_n \to x^\star$. If T is continuous, then

$$H(\{x_n\}, Tx_n) \leq H(Tx_{n-1}, Tx_n) \leq H(Tx_{n-1}, Tx^*) + H(Tx_n, Tx^*) \to 0.$$

Hence,

$$\begin{aligned}
H(\{x^*\}, Tx^*) &\leq H(\{x^*\}, \{x_n\}) + H(\{x_n\}, Tx_n) + H(Tx_n, Tx^*) \\
&= d(x^*, x_n) + H(\{x_n\}, Tx_n) + H(Tx_n, Tx^*) \to 0
\end{aligned}$$

and so $\{x^*\} = Tx^*$. If ψ is right upper semi-continuous and X has the condition (C_α), then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k and we get

$$\begin{aligned}
H(\{x_{n_k}\}, Tx_{n_k}) &\leq H(Tx_{n_k-1}, Tx_{n_k}) \leq \alpha(x_{n_k-1}, x_{n_k})H(Tx_{n_k-1}, Tx_{n_k}) \\
&\leq \psi(M(x_{n_k-1}, x_{n_k})) \leq \psi(\max\{d(x_{n_k-1}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\})
\end{aligned}$$

for all k and so $H(\{x_{n_k}\}, Tx_{n_k}) \to 0$. Also, we have

$$\begin{aligned}
H(\{x^*\}, Tx^*) &\leq d(x^*, x_{n_k}) + H(\{x_{n_k}\}, Tx_{n_k}) + H(Tx_{n_k}, Tx^*) \\
&\leq d(x^*, x_{n_k}) + H(\{x_{n_k}\}, Tx_{n_k}) + \alpha(x_{n_k}, x^*)H(Tx_{n_k}, Tx^*) \\
&\leq d(x^*, x_{n_k}) + H(\{x_{n_k}\}, Tx_{n_k}) + \psi(M(x_{n_k}, x^*))
\end{aligned}$$

for all k. But,

$$d(x^*,Tx^*) \le M(x_{n_k},x^*) \le \max\{d(x_{n_k},x^*), d(x_{n_k},Tx_{n_k}), d(x^*,Tx^*), \frac{d(x_{n_k},Tx^*)+d(x^*,Tx_{n_k})}{2}\},$$

$d(x_{n_k},Tx_{n_k}) \le d(x_{n_k},x_{n_k+1})$ and $d(x^*,Tx_{n_k}) \le d(x^*,x_{n_k+1})$ for all k. Hence,

$$d(x^*,Tx^*) \le \lim_{k\to\infty} M(x_{n_k},x^*) \le \max\{0,0,d(x^*,Tx^*), \frac{d(x^*,Tx^*)+0}{2}\} = d(x^*,Tx^*).$$

Thus, $H(\{x^*\},Tx^*) \le \psi(d(x^*,Tx^*)) \le \psi(H(\{x^*\},Tx^*))$. This implies that $H(\{x^*\},Tx^*) = 0$ and so $\{x^*\} = Tx^*$. □

Example 2.1. Let $X = [0, \frac{9}{2}]$, $d(x,y) = |x-y|$, $\alpha : X \times X \to [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 1 & x,y \in [0,1] \text{ or } x \in (\frac{5}{2},\frac{9}{2}] \text{ and } y = 0 \\ 0 & \text{otherwise}, \end{cases}$$

$T : X \to CB(X)$ defined by $Tx = \{\frac{x}{2}\}$ whenever $x \in [0,1]$, $Tx = \{4x - \frac{3}{2}\}$ whenever $x \in (1,\frac{3}{2}]$ and $Tx = \{0\}$ whenever $x \in (\frac{3}{2},\frac{9}{2}]$ and $\psi(t) = \frac{t}{2}$ for all $t \ge 0$. By using Theorem 2.1, T has an endpoint. Note that for $x = 1$ and $y = \frac{3}{2}$ we have $H(Tx,Ty) = H([0,\frac{1}{2}],\frac{9}{2}) = 4 > 3 = M(x,y) > \psi(M(x,y))$. Thus, T is not a generalized weak contractive multifunction and so we can not use main Theorem of ([7]).

Corollary 2.2. *Let (X,d) be a complete metric space, $\psi \in \Psi$ and $T : X \to CB(X)$ a multifunction such that $d(x,Tx) \le d(x,y) + \psi(d(x,y))$ implies $H(Tx,Ty) \le \psi(M(x,y))$ for all $x,y \in X$ and T has the property (BS). If T is continuous or T has the property (SBS) and ψ is right upper semi-continuous, then T has an endpoint.*

Proof. Define the map $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & d(x,Tx) \le d(x,y) + \psi(d(x,y)) \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to check that T is α-admissible. Also for every $x_0 \in X$ and $x_1 \in Tx_0$, we have $d(x_0,Tx_0) \le d(x_0,x_1) \le d(x_0,x_1) + \psi(d(x_0,x_1))$. Hence, $\alpha(x_0,x_1) = 1$. Also, it is easy to check that $\alpha(x,y)H(Tx,Ty) \le \psi(M(x,y))$ for all $x,y \in X$. Note that, X has the condition (C_α) whenever T has the property (SBS). Now by using Theorem 2.1, T has an endpoint. □

Corollary 2.3. *Let (X,d) be a complete metric space, $r \in [0,1)$ and $T : X \to CB(X)$ a multifunction such that $\frac{1}{1+r}d(x,Tx) \le d(x,y)$ implies $H(Tx,Ty) \le rM(x,y)$ for all $x,y \in X$ and T has the property (BS). If T is continuous or T has the property (SBS), then T has an endpoint.*

Corollary 2.4. *Let (X,d) be a complete metric space, $\psi \in \Psi$ and $T : X \to CB(X)$ a multifunction such that $d(x,Tx) \le d(x,y) + \psi(d(x,y))$ implies $H(Tx,Ty) \le \psi(K(x,y))$ for all $x,y \in X$ and T has the property (BS), where $K(x,y) = \max\{d(x,y), d(x,Tx), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. If T is continuous or T has the property (SBS) and ψ is right upper semi-continuous, then T has an endpoint.*

Proof. It is sufficient to note that, $K(x,y) \le M(x,y)$ for all $x,y \in X$. □

Corollary 2.5. *Let (X,d) be a complete metric space, $r \in [0,1)$ and $T : X \to CB(X)$ a multifunction such that $\frac{1}{1+r}d(x,Tx) \le d(x,y)$ implies $H(Tx,Ty) \le rK(x,y)$ for all $x,y \in X$ and T has the property (BS). If T is continuous or T has the property (SBS), then T has an endpoint.*

3. Conclusions

In 2010, Amini-Harandi proved that some multifunctions which have unique endpoint if and only if have approximate endpoint property. In 2011, Moradi and Khojasteh generalized his main result for generalized weak contractive multifunction. In this paper, by using the recent technique of Samet, we provide some new results about endpoints of Suzuki type quasi-contractive multifunctions. Also by providing an example, we show that our results improve some old results in a sense.

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