α-REDUNDANCY FOR INFINITE FRAMES

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This paper is concerned with the lower and upper redundancy of infinite frames in a separable Hilbert space. For a given infinite frame, we introduce a new quantitative notion of redundancy (α-redundancy), which is between its lower redundancy and its upper redundancy and it is completely dependent on the number of repetitive nonzero frame vectors.

Keywords: Frame, lower redundancy, upper redundancy, redundancy function, α-redundancy function, α-redundancy.

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1. **Introduction**

The concept of frames in a Hilbert space was originally introduced by Duffin and Schaeffer in the context of the non-harmonic Fourier series [7]. From the last decade, various generalizations of the frames have been proposed such as frame of subspaces, pseudo-frames, oblique frames, continuous frames, fusion frames, g-frames, and so on. The concept of equal norm Parseval frames on finite-dimensional Hilbert spaces was first introduced by Casazza and Leonhard in [5] and it been developed very fast over the last ten years, especially in the context of wavelets and Gabor systems.

Given a separable Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \tag{1}$$

where A, B are respectively the lower and upper frame bounds. The second inequality of the frame condition (1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. $\{f_k\}_{k=1}^{\infty}$ is called a tight frame, if $A = B$. A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a frame sequence in $\mathcal{H}$, if it is a frame for $\overline{span}\,\{f_k\}_{k=1}^{\infty}$.

The bounded linear operator T defined by

$$T : \ell_2(\mathbb{N}) \to \mathcal{H}, \qquad T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

is called the pre-frame operator or synthesis operator of $\{f_k\}_{k=1}^{\infty}$. Also the bounded linear operator S defined by

$$S : \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$$

is called the frame operator of $\{f_k\}_{k=1}^{\infty}$. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ae_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $A \in B(\mathcal{H})$ is an invertible operator. Every

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Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$. Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{H}$ if

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

The frame $\left\{\tilde{f}_k\right\}_{k=1}^{\infty}$ defined by $\tilde{f}_k = S^{-1} f_k$ is a dual frame of the frame $\{f_k\}_{k=1}^{\infty}$ that is called canonical dual frame of $\{f_k\}_{k=1}^{\infty}$.

A tight frame with frame bound 1 is called a Parseval frame. Parseval frames are useful in applications, as they provide the decompostion

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

The sequence $\left\{S^{\frac{-1}{2}} f_k\right\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$, if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame operator S[6]. A frame $\{f_k\}_{k=1}^{\infty}$ is a unit norm frame if $\|f_k\| = 1$ for all k. For more information concerning frames refer to[1, 4, 6, 8, 9, 10].

Frames are redundant sets of vectors in a Hilbert space, which yield one natural representation of each vector in the space, but may have infinitely many different representations for any given vector. It is this redundancy that makes frames useful in applications. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations.

The number of frame vectors per dimension is defined as the redundancy of a frame in the finite-dimensional setting which is not an unsatisfactory definition. A more precise quantitative notion of redundancy for finite frames (lower and upper redundancies) has been introduced in [2]. This quantitative notion of redundancy is generalized to infinite frames in [3].

In this paper, we introduce a new quantitative notion of redundancy (α-redundancy) for infinite frames, which is completely dependent on the number of the repetitive nonzero frame vectors and we discuss some of its properties.

2. α-redundancy

Bodmann, Casazza, and Kutyniok introduced a quantitative notion of redundancy for finite frames, and Cahill, Casazza, and Heinecke generalized it to infinite frames.

Definition 2.1. [3] *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for Hilbert space $\mathcal{H}$. The redundancy function of $\mathcal{F}$ is defined on the unit sphere $\mathbb{S} := \{x \in \mathcal{H}; \|x\| = 1\}$ in $\mathcal{H}$ by*

$$\mathcal{R}_{\mathcal{F}} : \mathbb{S} \to \mathbb{R}^+, \quad \mathcal{R}_{\mathcal{F}}(x) := \sum_{i=1}^{\infty} \|P_{<f_i>}(x)\|^2,$$

where $P_{<f_i>}$ is the orthogonal projection onto $< f_i > := span\{f_i\}$.
The upper and lower redundancy of $\mathcal{F}$ are defined by

$$\mathcal{R}_{\mathcal{F}}^+ := \sup_{x \in \mathbb{S}} \mathcal{R}_{\mathcal{F}}(x) \quad and \quad \mathcal{R}_{\mathcal{F}}^- := \inf_{x \in \mathbb{S}} \mathcal{R}_{\mathcal{F}}(x),$$

respectively.
Moreover, $\mathcal{F}$ has a uniform redundancy, if $\mathcal{R}_{\mathcal{F}}^- = \mathcal{R}_{\mathcal{F}}^+$.

The properties of lower and upper redundancy for infinite frames can be found in [3, Theorem 3.1].

Since zero vectors have no effect on redundancy, throughout this paper, we assume that $f_i \neq 0$, for all $i \in \mathbb{N}$. Thus

$$\mathcal{R}_{\mathcal{F}}(x) := \sum_{i=1}^{\infty} \frac{|\langle x, f_i\rangle|^2}{\|f_i\|^2}.$$

The next example is a motivation to define a new quantitative notion of redundancy for infinite frames. Before it, we need the next lemma.

Lemma 2.1. *If* $\mathcal{R}_{\mathcal{F}}(x) = \sum_{i=1}^{\infty} c_i |\langle x, e_i\rangle|^2, \forall x \in \mathbb{S}$, *for some orthonormal basis* $\{e_i\}_{i=1}^{\infty}$ *and sequence* $\{c_i\}_{i=1}^{\infty}$ *of positive numbers such that* $\{c_i\}_{i=1}^{\infty}$ *has only a finite number of values, then* $\mathcal{R}_{\mathcal{F}}^- = c$ *and* $\mathcal{R}_{\mathcal{F}}^+ = C$, *where* $c = \min_{i\in\mathbb{N}} c_i$ *and* $C = \max_{i\in\mathbb{N}} c_i$.

Proof. For all $x \in \mathbb{S}$ we have $c \leq \mathcal{R}_{\mathcal{F}}(x) \leq C$. Thus $c \leq \mathcal{R}_{\mathcal{F}}^- \leq \mathcal{R}_{\mathcal{F}}^+ \leq C$. On the other hand there is $j, k \in \mathbb{N}$ such that $c = c_j$ and $C = c_k$. Thus $\mathcal{R}_{\mathcal{F}}^- \leq \mathcal{R}_{\mathcal{F}}(e_j) = c$ and $\mathcal{R}_{\mathcal{F}}^+ \geq \mathcal{R}_{\mathcal{F}}(e_k) = C$. □

Example 2.1. *Let* $\{e_i\}_{i=1}^{\infty}$ *be an orthonormal basis for* $\mathcal{H}$ *and let* $\mathcal{F}_k := \{e_1, e_1, e_2, e_2, \ldots, e_k, e_k, e_{k+1}, e_{k+2}, \ldots\}$, *for any* $k \in \mathbb{N}$. *Then for all* $x \in \mathbb{S}$, *we have*

$$\mathcal{R}_{\mathcal{F}_k}(x) = 2|\langle x, e_1\rangle|^2 + 2|\langle x, e_2\rangle|^2 + \cdots + 2|\langle x, e_k\rangle|^2 + |\langle x, e_{k+1}\rangle|^2 + \ldots,$$

which implies $\mathcal{R}_{\mathcal{F}_k}^- = 1$ *and* $\mathcal{R}_{\mathcal{F}_k}^+ = 2$, *for all* $k \in \mathbb{N}$ *by Lemma 2.1.*
Thus $\mathcal{R}_{\mathcal{F}_k}^-$ *and* $\mathcal{R}_{\mathcal{F}_k}^+$ *are independent of* k *and hence the repetitive elements do not have complete effect on the lower and upper redundancy of* $\mathcal{F}_k$.

For a given infinite frame, we introduce a new quantitative notion of redundancy (α-redundancy), which is between its lower redundancy and its upper redundancy and it is completely dependent with the number of the repetitive nonzero frame vectors.

Definition 2.2. *A redundancy coefficient function is a strictly increasing continuous function* α *of* $[0,\infty)$ *onto* $[0,1)$ *such that* $\alpha(0) = 0$ *and* $\lim_{t\to\infty} \alpha(t) = 1$.
We assume that α *be strictly increasing continuous function of* $[0,\infty]$ *onto* $[0,1]$ *by* $\alpha(\infty) := \lim_{t\to\infty} \alpha(t) = 1$.

Example 2.2. *The functions* $\alpha(t) := \frac{t}{1+t}$ *and* $\beta(t) := 1 - e^{-t}$ *are redundancy coefficient functions.*

It is easy to show that if α and β are redundancy coefficient functions, then so are $\alpha\beta$, $\min(\alpha, \beta)$ and $\max(\alpha, \beta)$.

Definition 2.3. *Let* $\mathcal{F} = \{f_i\}_{i=1}^{\infty}$ *be a frame for* $\mathcal{H}$ *with lower and upper redundancy* $\mathcal{R}_{\mathcal{F}}^-$ *and* $\mathcal{R}_{\mathcal{F}}^+ < \infty$, *respectively and let* α *be a redundancy coefficient function as above. The* α*-redundancy function associated to* $\mathcal{F}$ *is the function*

$$\mathcal{R}_{\mathcal{F}}^{\alpha} : [0,\infty] \to \mathbb{R}, \quad \mathcal{R}_{\mathcal{F}}^{\alpha}(t) := \mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\alpha(t)$$

and the α*-redundancy of* $\mathcal{F}$ *is* $\mathcal{R}_{\mathcal{F}}^{\alpha} := \mathcal{R}_{\mathcal{F}}^{\alpha}(n_{\mathcal{F}})$, *where*
$n_{\mathcal{F}} := card\{i;\quad f_i$ *is repetitive nozero vector*$\}$ *with* $n_{\mathcal{F}} := \infty$, *if*
$\{i;\quad f_i$ *is repetitive nozero vector*$\}$ *is infinite set.*

Example 2.3. *Let* α *be a redundancy coefficient function and let* $\mathcal{F}_k$ *be the frame in Example 2.1, for any* $k \in \mathbb{N}$. *Then* $\mathcal{R}_{\mathcal{F}_k}^{\alpha} = 1 + \alpha(2k)$. *If* $\mathcal{F}_\infty := \{e_1, e_1, e_2, e_2, \ldots\}$, *then we see that* $\mathcal{R}_{\mathcal{F}_\infty}^{\alpha} = \lim_{k\to\infty} \mathcal{R}_{\mathcal{F}_k}^{\alpha}$.

Example 2.4. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and $0<\delta<1$. Let $\mathcal{F}_\delta=\{f_i\}_{i=1}^{\infty}$ be the unit norm frame for $\mathcal{H}$ defined by $f_1=f_2=e_1$ and*

$$f_{2i-1}:=\sqrt{\frac{1-\delta}{2}}e_i+\sqrt{\frac{1+\delta}{2}}e_{i+1},\quad f_{2i}:=\sqrt{\frac{1-\delta}{2}}e_i-\sqrt{\frac{1+\delta}{2}}e_{i+1}$$

for any integer number $i\geq 2$. Using parallelogram law for complex numbers we have

$$\mathcal{R}_{\mathcal{F}_\delta}(x)=2\left|\langle x,e_1\rangle\right|^2+(1-\delta)\left|\langle x,e_2\rangle\right|^2+2\left|\langle x,e_3\rangle\right|^2+2\left|\langle x,e_4\rangle\right|^2+\dots,$$

for all $x\in\mathbb{S}$, which implies $\mathcal{R}^-_{\mathcal{F}_\delta}=1-\delta$ and $\mathcal{R}^+_{\mathcal{F}_\delta}=2$ by Lemma 2.1 and hence $\mathcal{R}^\alpha_{\mathcal{F}_\delta}=1-\delta+(1+\delta)\alpha(2)$, for any redundancy coefficient function α. If $\mathcal{F}_0:=\left\{e_1,e_1,\sqrt{\frac{1}{2}}e_2+\sqrt{\frac{1}{2}}e_3,\sqrt{\frac{1}{2}}e_2-\sqrt{\frac{1}{2}}e_3,\sqrt{\frac{1}{2}}e_3+\sqrt{\frac{1}{2}}e_4,\sqrt{\frac{1}{2}}e_3-\sqrt{\frac{1}{2}}e_4,\dots\right\}$, then we see that $\mathcal{R}^\alpha_{\mathcal{F}_0}=1+\alpha(2)=\lim_{\delta\to 0}\mathcal{R}^\alpha_{\mathcal{F}_\delta}$.

Example 2.5. *If $\mathcal{F}=\{f_i\}_{i=1}^{\infty}$ is a C-equal norm A- tight frame, then for all $x\in\mathbb{S}$,*

$$\begin{aligned}\mathcal{R}_{\mathcal{F}}(x)&=\sum_{i=1}^{\infty}\frac{\left|\langle x,f_i\rangle\right|^2}{\|f_i\|^2}\\&=\frac{A}{C^2}\end{aligned}$$

and hence $\mathcal{R}^-_{\mathcal{F}}=\mathcal{R}^+_{\mathcal{F}}=\frac{A}{C^2}$. Thus for any redundancy coefficient function α, the α-redundancy function associated to $\mathcal{F}$ is the fixed function

$$\mathcal{R}^\alpha_{\mathcal{F}}:[0,\infty]\to\mathbb{R},\quad \mathcal{R}^\alpha_{\mathcal{F}}(t)=\frac{A}{C^2}$$

and the α-redundancy of $\mathcal{F}$ is $\mathcal{R}^\alpha_{\mathcal{F}}=\frac{A}{C^2}$.

Properties of α-redundancy are given in the following proposition.

Proposition 2.1. *Let $\mathcal{F},\mathcal{F}_1,\mathcal{F}_2$ be frames for $\mathcal{H}$ and let $\mathcal{E}$ be an orthonormal basis for $\mathcal{H}$. Then for any redundancy coefficient function α we have*

a) $\mathcal{R}^\alpha_{\mathcal{F}}(0)=\mathcal{R}^-_{\mathcal{F}}$, $\mathcal{R}^\alpha_{\mathcal{F}}(\infty)=\mathcal{R}^+_{\mathcal{F}}$ and $range\left(\mathcal{R}^\alpha_{\mathcal{F}}\right)=\left[\mathcal{R}^-_{\mathcal{F}},\mathcal{R}^+_{\mathcal{F}}\right]$,
b) $\mathcal{F}$ has uniform redundancy if and only if the α-redundancy function associated to $\mathcal{F}$ is a fixed function,
c) $range\left(\mathcal{R}^\alpha_{\mathcal{F}_1\cup\mathcal{F}_2}\right)\subseteq range\left(\mathcal{R}^\alpha_{\mathcal{F}_1}\right)+range\left(\mathcal{R}^\alpha_{\mathcal{F}_2}\right)$. In particular, if $\mathcal{F}_1$ and $\mathcal{F}_2$ have uniform redundancy, then $range\left(\mathcal{R}^\alpha_{\mathcal{F}_1\cup\mathcal{F}_2}\right)=range\left(\mathcal{R}^\alpha_{\mathcal{F}_1}\right)+range\left(\mathcal{R}^\alpha_{\mathcal{F}_2}\right)$,
d) $range\left(\mathcal{R}^\alpha_{\mathcal{F}\cup\mathcal{E}}\right)=1+range\left(\mathcal{R}^\alpha_{\mathcal{F}}\right)$,
e) For any unitary operator $U\in\mathcal{B}(\mathcal{H})$ we have $\mathcal{R}^\alpha_{U(\mathcal{F})}(t)=\mathcal{R}^\alpha_{\mathcal{F}}(t)$, for all $t\in[0,\infty]$. In particular, we have $\mathcal{R}^\alpha_{U(\mathcal{F})}=\mathcal{R}^\alpha_{\mathcal{F}}$,
f) For any permutation π on $\mathbb{N}$, we have $\mathcal{R}^\alpha_{\left\{f_{\pi(i)}\right\}_{i=1}^{\infty}}(t)=\mathcal{R}^\alpha_{\{f_i\}_{i=1}^{\infty}}(t)$, for all $t\in[0,\infty]$. In particular $\mathcal{R}^\alpha_{\left\{f_{\pi(i)}\right\}_{i=1}^{\infty}}=\mathcal{R}^\alpha_{\{f_i\}_{i=1}^{\infty}}$,
g) For any sequence $\{c_i\}_{i=1}^{\infty}$ of nonzero complex numbers, we have $\mathcal{R}^\alpha_{\{c_if_i\}_{i=1}^{\infty}}(t)=\mathcal{R}^\alpha_{\{f_i\}_{i=1}^{\infty}}(t)$, for all $t\in[0,\infty]$. In particular $\mathcal{R}^\alpha_{\{c_if_i\}_{i=1}^{\infty}}=\mathcal{R}^\alpha_{\{f_i\}_{i=1}^{\infty}}$.

Proof. Parts a and b follow from definition of α-redundancy function associated to given frames.
Proof c) Let $\lambda\in range\left(\mathcal{R}^\alpha_{\mathcal{F}_1\cup\mathcal{F}_2}\right)$. Then $\lambda=\mathcal{R}^\alpha_{\mathcal{F}_1\cup\mathcal{F}_2}(t)$, for some $t\in[0,\infty]$. Thus

$$\mathcal{R}^-_{\mathcal{F}_1}+\mathcal{R}^-_{\mathcal{F}_2}\leq\mathcal{R}^-_{\mathcal{F}_1\cup\mathcal{F}_2}\leq\mathcal{R}^\alpha_{\mathcal{F}_1\cup\mathcal{F}_2}(t)=\lambda\leq\mathcal{R}^+_{\mathcal{F}_1\cup\mathcal{F}_2}\leq\mathcal{R}^+_{\mathcal{F}_1}+\mathcal{R}^+_{\mathcal{F}_2},$$

which implies $\lambda \in \left[\mathcal{R}_{\mathcal{F}_1}^-, \mathcal{R}_{\mathcal{F}_1}^+\right] + \left[\mathcal{R}_{\mathcal{F}_2}^-, \mathcal{R}_{\mathcal{F}_2}^+\right] = range\left(\mathcal{R}_{\mathcal{F}_1}^\alpha\right) + range\left(\mathcal{R}_{\mathcal{F}_2}^\alpha\right)$. In particular, if $\mathcal{F}_1$ and $\mathcal{F}_2$ have uniform redundancy, then

$$\begin{aligned} range\left(\mathcal{R}_{\mathcal{F}_1\cup\mathcal{F}_2}^\alpha\right) &= \left\{\mathcal{R}_{\mathcal{F}_1\cup\mathcal{F}_2}^-\right\} \\ &= \left\{\mathcal{R}_{\mathcal{F}_1}^- + \mathcal{R}_{\mathcal{F}_2}^-\right\} \\ &= \left\{\mathcal{R}_{\mathcal{F}_1}^-\right\} + \left\{\mathcal{R}_{\mathcal{F}_2}^-\right\} \\ &= range\left(\mathcal{R}_{\mathcal{F}_1}^\alpha\right) + range\left(\mathcal{R}_{\mathcal{F}_2}^\alpha\right). \end{aligned}$$

Proof d) $range\left(\mathcal{R}_{\mathcal{F}\cup\mathcal{E}}^\alpha\right) = \left[\mathcal{R}_{\mathcal{F}\cup\mathcal{E}}^-, \mathcal{R}_{\mathcal{F}\cup\mathcal{E}}^+\right] = \left[1+\mathcal{R}_{\mathcal{F}}^-, 1+\mathcal{R}_{\mathcal{F}}^+\right] = 1 + \left[\mathcal{R}_{\mathcal{F}}^-, \mathcal{R}_{\mathcal{F}}^+\right] = 1 + range\left(\mathcal{R}_{\mathcal{F}}^\alpha\right)$.

Parts e, f and g follow from definition of α-redundancy function associated to given frames and invariance of lower and upper redundancy under application of unitary operators on frame vectors, scaling of the frame vectors and permutations of frame vectors[3, Theorem 3.1]. □

The relation of $\mathcal{R}_{\mathcal{F}}^{\alpha\beta}$, $\mathcal{R}_{\mathcal{F}}^{\max(\alpha,\beta)}$ and $\mathcal{R}_{\mathcal{F}}^{\min(\alpha,\beta)}$ with $\mathcal{R}_{\mathcal{F}}^\alpha$ and $\mathcal{R}_{\mathcal{F}}^\beta$ is given in the following proposition.

Proposition 2.2. *Let $\mathcal{F}$ be frame for $\mathcal{H}$ with $\mathcal{R}_{\mathcal{F}}^+ < \infty$. Then for any pair of redundancy coefficient functions α and β, we have*
a) $\mathcal{R}_{\mathcal{F}}^{\alpha\beta}(t) \le \sqrt{\mathcal{R}_{\mathcal{F}}^\alpha(t)\,\mathcal{R}_{\mathcal{F}}^\beta(t)}$, for all $t \in [0,\infty]$. In particular $\mathcal{R}_{\mathcal{F}}^{\alpha\beta} \le \sqrt{\mathcal{R}_{\mathcal{F}}^\alpha\mathcal{R}_{\mathcal{F}}^\beta}$,
b) $\mathcal{R}_{\mathcal{F}}^{\max(\alpha,\beta)}(t) = \max\left(\mathcal{R}_{\mathcal{F}}^\alpha(t), \mathcal{R}_{\mathcal{F}}^\beta(t)\right)$, for all $t \in [0,\infty]$. In particular $\mathcal{R}_{\mathcal{F}}^{\max(\alpha,\beta)} = \max\left(\mathcal{R}_{\mathcal{F}}^\alpha, \mathcal{R}_{\mathcal{F}}^\beta\right)$,
c) $\mathcal{R}_{\mathcal{F}}^{\min(\alpha,\beta)}(t) = \min\left(\mathcal{R}_{\mathcal{F}}^\alpha(t), \mathcal{R}_{\mathcal{F}}^\beta(t)\right)$, for all $t \in [0,\infty]$. In particular, we have $\mathcal{R}_{\mathcal{F}}^{\min(\alpha,\beta)} = \min\left(\mathcal{R}_{\mathcal{F}}^\alpha, \mathcal{R}_{\mathcal{F}}^\beta\right)$.

Proof.
Proof a) For any $t \in [0.\infty]$, we have

$$\begin{aligned} \left(\mathcal{R}_{\mathcal{F}}^{\alpha\beta}(t)\right)^2 &= \left(\mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\alpha(t)\,\beta(t)\right)\left(\mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\alpha(t)\,\beta(t)\right) \\ &\le \left(\mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\alpha(t)\right)\left(\mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\beta(t)\right) \\ &= \mathcal{R}_{\mathcal{F}}^\alpha(t)\,\mathcal{R}_{\mathcal{F}}^\beta(t). \end{aligned}$$

Thus
$\mathcal{R}_{\mathcal{F}}^{\alpha\beta}(t) \le \sqrt{\mathcal{R}_{\mathcal{F}}^\alpha(t)\,\mathcal{R}_{\mathcal{F}}^\beta(t)}$. In particular, $\mathcal{R}_{\mathcal{F}}^{\alpha\beta}(n_{\mathcal{F}}) \le \sqrt{\mathcal{R}_{\mathcal{F}}^\alpha(n_{\mathcal{F}})\,\mathcal{R}_{\mathcal{F}}^\beta(n_{\mathcal{F}})}$, which implies $\mathcal{R}_{\mathcal{F}}^{\alpha\beta} \le \sqrt{\mathcal{R}_{\mathcal{F}}^\alpha\mathcal{R}_{\mathcal{F}}^\beta}$.
Proof b) For any $t \in [0.\infty]$, we have

$$\begin{aligned} \mathcal{R}_{\mathcal{F}}^{\max(\alpha,\beta)}(t) &= \mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\max(\alpha,\beta)(t) \\ &= \max\left(\mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\alpha(t), \mathcal{R}_{\mathcal{F}}^- + \left(\mathcal{R}_{\mathcal{F}}^+ - \mathcal{R}_{\mathcal{F}}^-\right)\beta(t)\right) \\ &= \max\left(\mathcal{R}_{\mathcal{F}}^\alpha(t), \mathcal{R}_{\mathcal{F}}^\beta(t)\right). \end{aligned}$$

In particular, $\mathcal{R}_{\mathcal{F}}^{\max(\alpha,\beta)}(n_{\mathcal{F}}) = \max\left(\mathcal{R}_{\mathcal{F}}^\alpha(n_{\mathcal{F}}), \mathcal{R}_{\mathcal{F}}^\beta(n_{\mathcal{F}})\right)$, which implies
$\mathcal{R}_{\mathcal{F}}^{\max(\alpha,\beta)} = \max\left(\mathcal{R}_{\mathcal{F}}^\alpha, \mathcal{R}_{\mathcal{F}}^\beta\right)$. The proof of part c is similar to part b. □

For any $x \in \mathbb{R}$ the least integer greater than or equal to x is denoted by $\lceil x \rceil$.
The relationship between $\mathcal{R}_{\mathcal{F}}^\alpha$ and linearly independent subsets of $\mathcal{F}$, is given in the next proposition.

Proposition 2.3. *Let $\mathcal{F}$ be frame for $\mathcal{H}$ with $\mathcal{R}_{\mathcal{F}}^{+} < \infty$ and let α be a redundancy coefficient function. Then there exists $t_0 \geq 0$ such that $\mathcal{F}$ can be partitioned into $\lceil \mathcal{R}_{\mathcal{F}}^{\alpha}(t_0) \rceil$ linearly independent sets. In particular t_0 can be chosen as*

$$t_0 := \begin{cases} n_{\mathcal{F}} & \lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil - 1 < \mathcal{R}_{\mathcal{F}}^{-}, \\ \alpha^{-1}\left(\frac{\lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil - \mathcal{R}_{\mathcal{F}}^{-} - 1}{\mathcal{R}_{\mathcal{F}}^{+} - \mathcal{R}_{\mathcal{F}}^{-}} \right) & o.w. \end{cases}$$

Proof. If $\lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil - 1 < \mathcal{R}_{\mathcal{F}}^{-}$, then $\lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil - 1 < \mathcal{R}_{\mathcal{F}}^{-} \leq \mathcal{R}_{\mathcal{F}}^{\alpha}(n_{\mathcal{F}}) \leq \mathcal{R}_{\mathcal{F}}^{+} \leq \lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil$ and hence $\lceil \mathcal{R}_{\mathcal{F}}^{\alpha}(t_0) \rceil = \lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil$ for $t_0 = n_{\mathcal{F}}$. Otherwise there exists $t_0 \geq 0$ such that $\alpha(t_0) = \frac{\lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil - \mathcal{R}_{\mathcal{F}}^{-} - 1}{\mathcal{R}_{\mathcal{F}}^{+} - \mathcal{R}_{\mathcal{F}}^{-}}$. Thus $\lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil - 1 = \mathcal{R}_{\mathcal{F}}^{\alpha}(t_0) \leq \mathcal{R}_{\mathcal{F}}^{+} \leq \lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil$ and hence $\lceil \mathcal{R}_{\mathcal{F}}^{\alpha}(t_0) \rceil = \lceil \mathcal{R}_{\mathcal{F}}^{+} \rceil$. Now the result is obtained by [3, Theorem 3.1]. □

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