SPLIT FIXED POINT PROBLEMS FOR QUASI-NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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In this paper, we introduce an algorithm that converges to a solution of the split fixed point problem under some conditions. We apply our main results for solv-ing the split best proximity point problem. The main results of Suantai and Tiammee [J. Nonlinear Convex Anal. 22(2021) 2661-2670] related to the study of convergence of best proximity points for best proximally nonexpansive non-self mappings can be directly concluded from the convergence results of fixed points for quasi-nonexpansive self mappings. Therefore, these findings are not real generalizations. Furthermore, we apply our results to the common best proximity point problem in real Hilbert spaces. Finally, we give numerical results to demonstrate its convergence.

Keywords: best proximity point, fixed point, Hilbert space, quasi-nonexpansive mappings.
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1. Introduction

Many optimization problems can be reformulated as best approximation problems. Due to this optimization theory plays a key role in several areas such as variational inequalities problems, fixed point problems, split feasiblity problems so on. On the other hand, study of nonself mappings is also fascinating because in this case best approximation exists by Ky Fan [7] technique. In this case we find a point x is an approximate solution such that the error $||x-Tx||$ is minimum, that is, the point x is close proximity to Tx. However, when U is mapped into another subset C of X by T, the problem extends to determining a point that estimates the distance between these two subsets. These are referred to as best proximity points. If $T:U\to C$ is nonself mapping on a norm space, then a point $x\in U$ is known as the best proximity point of a nonself mapping T, satisfying the condition

$$||x-Tx|| = ||U-C|| = \inf\{||x-y|| : x\in U, y\in C\}$$

where U and C are non-empty subsets of X such that $U\cap C=\emptyset$. Best approximation is an invariant approximation in the case of self mappings.
The split feasibility problem (SFP), which is mathematically, formulated as:

$$\text{find a point } u\in U \text{ such that } Au\in C, \tag{1}$$

where U and C are non-empty convex and closed subsets of the Hilbert spaces H_1 and H_2 and $A: H_1\to H_2$ is a bounded linear operator. This problem was first proposed by Censor and Elfving [3] in Euclidean spaces.

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In the recent years, the best proximity point problem for nonself nonlinear mappings is an interesting topic in the optimization theory, see [1, 2, 4, 8, 11, 12, 14]. In nonlinear analysis, the convergence of iterative procedures has long been a fascinating problem. In 2017, G.K. Jacob et al. [9] gave a new iteration method and proved the iterative process converges strongly to the best proximity point of any non-expansive mapping.

A. Moudafi [15] investigated the split fixed point (SFP) problem for two operators:

$$\text{find a point } u \in U' \text{ such that } Au \in C', \tag{2}$$

where $U' = F(S)$ and $C' = F(T)$ are non-empty convex and closed subsets of the Hilbert spaces H_1 and H_2 repectively, $S : H_1 \to H_1$, $T : H_2 \to H_2$ are mappings, $A : H_1 \to H_2$ is a bounded linear operator and $F(S), F(T)$ denote the set of fixed point of the mappings S and T respectively. The solution set of the SFP problem is denoted by

$$\mathscr{S} = \left\{u \in U' : Au \in C'\right\}.$$

In 2020, Dadashi et al. [6] gave a forward-backward splitting algorithm for fixed point problem and proved some results. In 2022, Y. Yao et al. [19] gave iterative algorithms for split equilibrium problems of monotone operators, fixed point problems of pseudo-contractions and proved some results. Recently, Y. Yao et al. [20] gave approximation algorithm for solving a split problem of fixed point, variational inclusion problems and proved strong convergence theorems.

In 2021, Suantai and Tiammee [15] investigated the split best proximity point (SBPP) problem for two operators:

$$\text{find a point } u \in Best_U S \text{ such that } Au \in Best_C T, \tag{3}$$

where U, V, C and D are non-empty convex and closed subsets of the Hilbert spaces H_1 and H_2 repectively, $S : U \to V$, $T : C \to D$ are mappings, $A : H_1 \to H_2$ is a bounded linear operator and $Best_U S, Best_C T$ denote the set of best proximity point of the mappings S, T respectively. The solution set of the SBPP problem is denoted by

$$\mathscr{S}^* = \{u \in Best_U S : Au \in Best_C T\}.$$

In this paper, we introduce an algorithm which converges to solution of split fixed point problem under some conditions. We apply our main results for solving the split best proximity point problem. Also the main results of the paper [Suantai and Tiammee, split best proximity point problems for best proximally nonexpansive mappings in a real Hilbert space, J. Nonlinear Convex Anal. 22(2021), 2661-2670] which are related to study of convergence of best proximity points for best proximally nonexpansive non-self mappings can be concluded directly, from the convergence results of fixed points for quasi-nonexpansive self mappings and so they are not real generalizations. We also apply our results to the common best proximity point problem in real Hilbert spaces. Finally, we give numerical results to demonstrate its convergence.

2. Preliminaries

In this section, we give basic definitions, results and notations to be used in the sequel.

For two non-empty subsets A_1 and A_2 of a norm space X and non-self map $S' : A_1 \to A_2$ define

$$A_{1_0} = \{x \in A_1 : \text{there exists some } y \in A_2 \text{ such that } ||x-y|| = ||A_1 - A_2||\};$$
$$A_{2_0} = \{y \in A_2 : \text{there exists some } x \in A_1 \text{ such that } ||x-y|| = ||A_1 - A_2||\};$$
$$||z - A_1|| = \inf\{||x-z|| : x \in A_1 \text{ and } z \in X\},$$
$$\mathscr{P}_{A_1}(z) = \{x \in A_1 : ||x-z|| = ||z-A_1|| \text{ and } z \in X\},$$
$$Best_{A_1}S' = \left\{x \in A_1 : ||x - S'x|| = ||A_1 - A_2||\right\}.$$

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and the norm $||.||$. A mapping $T : H \to H$ is said to be

(I) nonexpansive [17] if
$$||Tx - Ty|| \leq ||x-y|| \text{ for all } x,y \in H;$$

(II) quasi nonexpansive [5] if $F(T) \neq \emptyset$ and
$$||Tx - q|| \leq ||x-q|| \text{ for all } x \in H, q \in F(T);$$

We denote by $F(T) = \{x \in H : Tx = x\}$.

Example 2.1. *Let $\mathbb{R}$ denotes the real numbers with the usual norm and $A_1 = [0,1)$. Assume that $T : A_1 \to A_1$ is defined by $T(x) = x^2$, for all $x \in A_1$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in [0,1]$ and $z = 0$, then*
$$||Tx - z|| = ||Tx - 0|| = |x^2| \leq |x| = ||x - 0||.$$
If we take $x = \frac{1}{2}$ and $y = 1$ then we get
$$||Tx - Ty|| = |Tx - Ty| = \frac{3}{4} > \frac{1}{2} = ||x-y||.$$
This shows that T is not nonexpansive mapping.

Remark 2.1. *If $T : H \to H$ is quasi nonexpansive then the set of fixed point a mapping T is closed and convex on which T is continuous (see [5]).*

Definition 2.1. *Let (A_1, A_2) be a pair of non-empty subsets of a norm space X. Then the pair (A_1, A_2) is said to have P-property [13] if for any $x, x' \in A_1$ and $y, y' \in A_2$,*
$$\begin{aligned} ||x-y|| &= \ ||A_1 - A_2||, \\ ||x'-y'|| &= \ ||A_1 - A_2||, \\ \textit{implies} \quad ||x-x'|| &= ||y-y'||. \end{aligned}$$

Remark 2.2. *It was proved that every non-empty, closed and convex pair in Hilbert spaces has the P-property (see [13]).*

Definition 2.2. *(see [15, 16]) Let A_1 and A_2 be two non-empty subsets of a real Hilbert space H and C be a subset of A_1. A mapping $T : A_1 \to A_2$ is said to be C-nonexpansive if*
$$||Tx - Ty|| \leq ||x-y|| \textit{ for all } x \in A_1 \textit{ and } y \in C.$$
If $C = Best_{A_1}T$, we say that T is a best proximally nonexpansive mapping.

Remark 2.3. *It is note that if T is nonself nonexpansive, then it is C-nonexpansive for every subset C of A_1, and if $C = F(T) \neq \emptyset$, then every C-nonexpansive is quasi-nonexpansive (see [16]).*

Definition 2.3. *Let A_1 be a subset of a Hilbert space H, satisfy Opial's condition* [21] *if $\{x_n\}$ converges weakly to a point p implies $\limsup_{n\to\infty}||x_n - p|| < \limsup_{n\to\infty}||x_n - l||$ for all $l \in A_1$ with $p \neq l$.*

Lemma 2.1. [16] *Let A_1 and A_2 be two non-empty subsets of a Hilbert space X such that A_1 is closed and convex. Suppose that $T : A_1 \to A_2$ is a best proximally nonexpansive mapping such that $T(A_{1_0}) \subseteq A_{2_0}$. Then $\mathscr{P}_{A_1}T|_{A_{1_0}}$ is a quasi-nonexpansive mapping.*

Lemma 2.2. [18] *Let C be a non-empty closed convex subset of a real Hilbert space H. Then, for any $x, y \in H$, the following assertions hold:*

(i) $\langle x - \mathscr{P}_C x, z - \mathscr{P}_C x\rangle \geq 0$ *for all* $z \in C$;
(ii) $||\mathscr{P}_C x - \mathscr{P}_C y||^2 \leq \langle \mathscr{P}_C x - \mathscr{P}_C y, x - y\rangle$ *for all* $x, y \in H$;
(iii) $||\mathscr{P}_C x - z||^2 \leq ||x - z||^2 - ||\mathscr{P}_C x - x||^2$ *for all* $z \in C$.

Lemma 2.3. [10] *Let T be a quasi nonexpansive mapping, and set $T_{\alpha'} = (1-\alpha')I + \alpha' T$ for $\alpha' \in (0,1]$. Then, the following assertions hold for all $(x,q) \in H \times F(T)$:*

(i) $\langle x - Tx, x - q\rangle \geq \frac{1}{2}||x - Tx||^2$ *and* $\langle x - Tx, q - Tx\rangle \leq \frac{1}{2}||x - Tx||^2$;
(ii) $||T_{\alpha'}x - q||^2 \leq ||x - q||^2 - \alpha'(1-\alpha')||Tx - x||^2$;
(iii) $\langle x - T_{\alpha'}x, x - q\rangle \geq \frac{\alpha'}{2}||x - Tx||^2$.

3. Main results

In this section, we introduce an algorithm that converges to a solution of the split fixed point problem under some conditions. Let $x_0 \in H_1$ be arbitrary. Define

$$\begin{cases} x_{n+1} = & (1-\eta_n)y_n + \eta_n S y_n \\ y_n = & (1-\gamma_n-\delta_n)z_n + (\gamma_n+\delta_n)Sz_n \\ z_n = & x_n + \alpha\beta A^*(T-I)Ax_n \end{cases} \tag{SS}$$

where $\eta_n, \gamma_n, \delta_n, (\gamma_n + \delta_n) \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\alpha \in (0, \frac{1}{\lambda\beta})$ with λ being the spectral radius of A^*A.

Remark 3.1. *If we take $(\gamma_n + \delta_n) = 0$ then algorithm (SS) reduces to the algorithm presented by Moudafi* [10] *as follows:*

$$\begin{cases} x_{n+1} = & (1-\eta_n)z_n + \eta_n S z_n \\ z_n = & x_n + \alpha\beta A^*(T-I)Ax_n \end{cases} \tag{M}$$

where $\eta_n \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\alpha \in (0, \frac{1}{\lambda\beta})$ with λ being the spectral radius of A^*A.

Theorem 3.1. *Let H_1, H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $S : H_1 \to H_1$, $T : H_2 \to H_2$ be continuous quasi-nonexpansive mappings. Then $F(S) = U'$ and $F(T) = C'$ are closed and convex subsets of H_1 and H_2, respectively. Furthermore, if U' is compact, $\mathscr{S} \neq \emptyset$, and sequence $\{x_n\}$ generated by algorithm (SS), then*

(i) *$\{x_n\}$ is Fejer monotone with respect to $\mathscr{S}$, that is, for every $z \in \mathscr{S}$,*

$$||x_{n+1} - z|| \leq ||x_n - z||, n \in \mathbb{N}.$$

(ii) *$\{x_n\}$ converges strongly to a point $x^* \in \mathscr{S}$.*

Proof. Let $z \in \mathscr{S}$, by Lemma 2.3, we get

$$||x_{n+1} - z||^2 \leq ||y_n - z||^2 - \eta_n(1-\eta_n)||Sy_n - y_n||^2.$$

Similarly

$$||y_n - z||^2 \leq ||z_n - z||^2 - (\gamma_n + \delta_n)(1 - \gamma_n - \delta_n)||Sz_n - z_n||^2. \tag{4}$$

On the other hand, we have

$$\begin{aligned} &||z_n - z||^2 \\ =&||x_n + \alpha\beta A^*(T-I)Ax_n - z||^2 \\ =&||x_n - z||^2 + \alpha^2\beta^2||A^*(T-I)Ax_n||^2 + 2\alpha\beta\langle x_n - z, A^*(T-I)Ax_n\rangle \\ =&||x_n - z||^2 + \alpha^2\beta^2\langle (T-I)Ax_n, AA^*(T-I)Ax_n\rangle + 2\alpha\beta\langle x_n - z, A^*(T-I)Ax_n\rangle. \end{aligned} \tag{5}$$

Since λ is the spectral radius of A^*A,

$$\begin{aligned} \alpha^2\beta^2\langle (T-I)Ax_n, AA^*(T-I)Ax_n\rangle &\leq \lambda\alpha^2\beta^2\langle (T-I)Ax_n, (T-I)Ax_n\rangle \\ &= \lambda\alpha^2\beta^2||(T-I)Ax_n||^2. \end{aligned} \tag{6}$$

By Lemma 2.3, we obtain

$$\begin{aligned} &2\alpha\beta\langle x_n - z, A^*(T-I)Ax_n\rangle \\ =&2\alpha\beta\langle A(x_n - z), (T-I)Ax_n\rangle \\ =&2\alpha\beta\langle A(x_n - z) + (T-I)Ax_n - (T-I)Ax_n, (T-I)Ax_n\rangle \\ =&2\alpha\beta(\langle A(x_n - z) + (T-I)Ax_n, (T-I)Ax_n\rangle - ||(T-I)Ax_n||^2) \\ =&2\alpha\beta(\langle TAx_n - Az, TAx_n - Ax_n\rangle - ||(T-I)Ax_n||^2) \\ \leq&2\alpha\beta\left(\frac{||(T-I)Ax_n||^2}{2} - ||(T-I)Ax_n||^2\right) \\ =&-\alpha\beta||(T-I)Ax_n||^2. \end{aligned} \tag{7}$$

Using (6) and (7) in (5), we obtain

$$||z_n - z||^2 \leq ||x_n - z||^2 - \alpha\beta(1-\lambda\alpha\beta)||(T-I)Ax_n||^2. \tag{8}$$

By equations (4) and (8), we get

$$\begin{aligned} ||x_{n+1} - z||^2 \leq&||x_n - z||^2 - \alpha\beta(1-\lambda\alpha\beta)||(T-I)Ax_n||^2 - \\ &(\gamma_n + \delta_n)(1-\gamma_n-\delta_n)||Sz_n - z_n||^2 - \eta_n(1-\eta_n)||Sy_n - y_n||^2 \\ \leq&||x_n - z||^2 - \alpha\beta(1-\lambda\alpha\beta)||(T-I)Ax_n||^2 - \\ &(\gamma_n + \delta_n)(1-\gamma_n-\delta_n)||Sz_n - z_n||^2. \end{aligned} \tag{9}$$

By (9), we get

$$||x_{n+1} - z||^2 \leq ||x_n - z||^2 - \alpha\beta(1-\lambda\alpha\beta)||(T-I)Ax_n||^2. \tag{10}$$

It follows that

$$||x_{n+1} - z|| \leq ||x_n - z||.$$

Hence $\{x_n\}$ is Fejer monotone with respect to $\mathscr{S}$ and $\{||x_n - z||\}$ is monotonically decreasing. So, assume that $\lim_{n\to\infty} ||x_n - z|| = l$ for some $l \geq 0$. It follows from (10) that

$$\lim_{n\to\infty} ||(T-I)Ax_n|| = 0. \tag{11}$$

Since the sequence $\{x_n\}$ is Fejer monotone, hence the sequence $\{x_n\}$ is bounded. As U' is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in U'$. Then it follows from (11) that $TAx^* = Ax^*$. This shows $Ax^* \in C'$.

On the other hand, by setting $z_n = x_n + \alpha\beta A^*(T-I)Ax_n$, we have

$$\begin{aligned}
||z_{n_k} - x^*|| =& ||x_{n_k} + \alpha\beta A^*(T-I)Ax_{n_k} - x^*|| \\
\leq& ||x_{n_k} - x^*|| + \alpha\beta ||A^*|| ||(T-I)Ax_{n_k}|| \\
\to& 0 \text{ as } k \to \infty.
\end{aligned}$$

From (9), (11) and the convergence of the sequence $\{||x_n - z||\}$, we obtain

$$\lim_{n\to\infty} ||Sz_{n_k} - z_{n_k}|| = 0. \tag{12}$$

it follows from that $Sx^* = x^*$. Hence $x^* \in \mathscr{S}$. Assume that another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges to $x_1 \in \mathscr{S}$. Assume that x^* and x_1 are distinct, then by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} ||x_n - x^*|| &= \lim_{k\to\infty} ||x_{n_k} - x^*|| < \lim_{k\to\infty} ||x_{n_k} - x_1|| = \lim_{n\to\infty} ||x_n - x_1|| \\
&= \lim_{j\to\infty} ||x_{n_j} - x_1|| < \lim_{j\to\infty} ||x_{n_j} - x^*|| = \lim_{n\to\infty} ||x_n - x^*||,
\end{aligned}$$

which is contradiction so $x^* = x_1$. Consequently, sequence $\{x_n\}$ generated by algorithm (SS) converges strongly to a point $x^* \in \mathscr{S}$. □

Corollary 3.1. *(see [10]) Let H_1, H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $S : H_1 \to H_1$, $T : H_2 \to H_2$ be continuous quasi-nonexpansive mappings. Then $F(S) = U'$ and $F(T) = C'$ are closed and convex subsets of H_1 and H_2, respectively. Furthermore, if U' is compact, $\mathscr{S} \neq \emptyset$, and sequence $\{x_n\}$ generated by algorithm (M), then*

(i) *$\{x_n\}$ is Fejer monotone with respect to $\mathscr{S}$, that is, for every $z \in \mathscr{S}$,*

$$||x_{n+1} - z|| \leq ||x_n - z||, n \in \mathbb{N}.$$

(ii) *$\{x_n\}$ converges strongly to a point $x^* \in \mathscr{S}$.*

4. **Consequences**

Now we apply our main result to obtain an algorithm for solving the (SBPP) problem. Let $x_0 \in U_0$ arbitrary. Define

$$\begin{cases}
x_{n+1} = & (1-\eta_n)y_n + \eta_n \mathscr{P}_U S y_n \\
y_n = & (1-\gamma_n-\delta_n)z_n + (\gamma_n+\delta_n)\mathscr{P}_U S z_n \\
z_n = & \mathscr{P}_U[x_n + \alpha\beta A^*(\mathscr{P}_C T - I)Ax_n]
\end{cases} \tag{SS'}$$

where $\eta_n, \gamma_n, \delta_n, (\gamma_n + \delta_n) \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\alpha \in (0, \frac{1}{\lambda\beta})$ with λ being the spectral radius of A^*A.

Remark 4.1. *If we take* $(\gamma_n + \delta_n) = 0$ *then algorithm (SS') reduces to the algorithm presented by Suantai* [15] *as follows:*

$$\begin{cases} x_{n+1} = & (1-\eta_n)z_n + \eta_n \mathscr{P}_U S z_n \\ z_n = & \mathscr{P}_U[x_n + \alpha\beta A^*(\mathscr{P}_C T - I)Ax_n] \end{cases} \tag{SU}$$

where $\eta_n \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ *and* $\alpha \in (0, \frac{1}{\lambda\beta})$ *with* λ *being the spectral radius of* A^*A.

Theorem 4.1. *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $U,V \subset H_1$, $C,D \subset H_2$ *be non-empty closed convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A(U) \subseteq C$ *and* $S : U \to V$, $T : C \to D$ *be best proximally nonexpansive mappings with non-empty* $Best_U(S)$ *and* $Best_C(T)$*. Suppose that* $S(U_0) \subseteq V_0$ *and* $T(C_0) \subseteq D_0$ *and* $\mathscr{S}^* \neq \emptyset$*. If* $Best_U(S)$ *is compact and* $\{x_n\}$ *is a sequence generated by algorithm (SS'), then*

(i) $\{x_n\}$ *is Fejer monotone with respect to* $\mathscr{S}^*$ *, that is, for every* $z \in \mathscr{S}^*$,

$$||x_{n+1} - z|| \leq ||x_n - z||, n \in \mathbb{N}.$$

(ii) $\{x_n\}$ *converges strongly to a split best proximity point* $x^* \in \mathscr{S}^*$.

Proof. Since $S : U \to V$ be best proximally nonexpansive mapping and $S(U_0) \subseteq V_0$ by Lemma 2.1 $\mathscr{P}_U S$ is quasi-nonexpansive mapping on U_0. Next, we prove that set of fixed point of a mapping $\mathscr{P}_U S$ is equal to best proximity point of S. Let $x^* \in F(\mathscr{P}_U S)$. Since $\mathscr{P}_U S$ is a quasi-nonexpansive mapping on U_0, we have $\mathscr{P}_U S(x^*) \in U_0$. So there exists $Sx^* \in V$ such that $||\mathscr{P}_U S(x^*) - Sx^*|| = ||U - V||$. If $S\hat{x} \in V$ is another point such that $||\mathscr{P}_U S(x^*) - S\hat{x}|| = ||U - V||$. Also by Remark 2.2 the pair (U,V) has P-property we have $||\mathscr{P}_U S(x^*) - \mathscr{P}_U S(x^*)|| = ||Sx^* - S\hat{x}||$. This shows $Sx^* = S\hat{x}$. Thus

$$||x^* - Sx^*|| = ||\mathscr{P}_U S(x^*) - Sx^*|| = ||U - V||,$$

where $x^* \in F(\mathscr{P}_U S)$. This shows that $F(\mathscr{P}_U S) \subseteq Best_U(S)$. Similar arguments shows that $Best_U(S) \subseteq F(\mathscr{P}_U S)$. Hence $F(\mathscr{P}_U S) = Best_U(S)$. Similarly we can prove that $\mathscr{P}_C T$ is quasi- nonexpansive mapping and set of fixed point of a mapping $\mathscr{P}_C T$ is equal to best proximity point of T. By Theorem 3.1, $\{x_n\}$ is Fejer monotone with respect to $\mathscr{W}$ and converges strongly to a point $x^* \in \mathscr{W}$ where

$$\mathscr{W} = \{x^* \in F(\mathscr{P}_U S) : Ax^* \in F(\mathscr{P}_C T)\}.$$

Since $F(\mathscr{P}_U S) = Best_U S$ and $F(\mathscr{P}_C T) = Best_C T$. This shows $\{x_n\}$ converges strongly to a point $x^* \in \mathscr{S}^*$. ☐

If we take U is compact in Theorem 4.1 then we have following result:

Theorem 4.2. *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $U,V \subset H_1$, $C,D \subset H_2$ *be non-empty closed convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A(U) \subseteq C$ *and* $S : U \to V$, $T : C \to D$ *be best proximally nonexpansive mappings with non-empty* $Best_U(S)$ *and* $Best_C(T)$*. Suppose that* $S(U_0) \subseteq V_0$ *and* $T(C_0) \subseteq D_0$ *and* $\mathscr{S}^* \neq \emptyset$*. If* U *is compact and* $\{x_n\}$ *is a sequence generated by algorithm (SS'), then*

(i) $\{x_n\}$ *is Fejer monotone with respect to* $\mathscr{S}^*$ *, that is, for every* $z \in \mathscr{S}^*$,

$$||x_{n+1} - z|| \leq ||x_n - z||, n \in \mathbb{N}.$$

(ii) $\{x_n\}$ *converges strongly to a split best proximity point* $x^* \in \mathscr{S}^*$.

Corollary 4.1. [15] *Let H_1 and H_2 be two real Hilbert spaces and $U,V \subset H_1$, $C,D \subset H_2$ be non-empty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A(U) \subseteq C$ and $S : U \to V$, $T : C \to D$ be best proximally nonexpansive mappings with non-empty $Best_U(S)$ and $Best_C(T)$. Suppose that $S(U_0) \subseteq V_0$ and $T(C_0) \subseteq D_0$ and $\mathscr{S}^* \neq \emptyset$. If U is compact and $\{x_n\}$ is a sequence generated by algorithm (SU), then*

(i) *$\{x_n\}$ is Fejer monotone with respect to $\mathscr{S}^*$, that is, for every $z \in \mathscr{S}^*$,*

$$||x_{n+1} - z|| \leq ||x_n - z||, n \in \mathbb{N}.$$

(ii) *$\{x_n\}$ converges strongly to a split best proximity point $x^* \in \mathscr{S}^*$.*

Remark 4.2. *Theorem 3.2 of Suantai ([15], p. 2665) is a straightforward consequence of Theorem 2.1 of Moudafi ([10], p. 4086).*

Theorem 4.3. *Let $U,V \subset H$ be two non-empty closed convex subsets of a real Hilbert space H with U is compact and $S : U \to V$, $T : U \to V$ be two best proximally nonexpansive mappings with non-empty $Best_U(S) \cap Best_U(T)$. Suppose that $S(U_0) \subseteq V_0$ and $T(U_0) \subseteq V_0$. Then the sequence $\{x_n\}$ generated by algorithm (SS'), converges strongly to a common best proximity point $x^* \in Best_U(S) \cap Best_U(T)$.*

Theorem 4.4. *(see [15]) Let $U,V \subset H$ be two non-empty closed convex subsets of a real Hilbert space H with U is compact and $S : U \to V$, $T : U \to V$ be two best proximally nonexpansive mappings with non-empty $Best_U(S) \cap Best_U(T)$. Suppose that $S(U_0) \subseteq V_0$ and $T(U_0) \subseteq V_0$. Then the sequence $\{x_n\}$ generated by algorithm (SU), converges strongly to a common best proximity point $x^* \in Best_U(S) \cap Best_U(T)$.*

Proof. This is a consequence of Theorem 4.3 by taking $H_1 = H_2 = H$, $A = I$, $U = C$ and $V = D$. □

5. Examples

In this section, we give some examples to validate our results.

Example 5.1. *Let $H_1 = H_2 = \mathbb{R}^2$ with Euclidean distance. Let $S : H_1 \to H_2$ be defined by $S(x,y) = (5 - x, y)$, for all $(x,y) \in H_1$. Let $T : H_2 \to H_2$ be defined by $T(x,y) = (10 - x, \frac{y}{3})$ and $A : H_1 \to H_2$ be defined by $A(x,y) = (2x, 2y)$ for all $(x,y) \in H_1$. It is clear that both S and T are quasi-nonexpansive mappings and S has a fixed point $(\dfrac{5}{2}, 0)$ and $A(\dfrac{5}{2}, 0) = (5,0)$ is fixed point of T. All the conditions of Theorem 3.1 are satisfied, so sequence generated by algorithm (SS) converges to $(\dfrac{5}{2}, 0)$.*

Example 5.2. *Let $H_1 = H_2 = \mathbb{R}^2$ with Euclidean distance. Let $U = [0,2] \times [0,2]$ and $V = [3,5] \times [0,2]$. Let $C = [0,4] \times [0,4]$ and $D = [6,10] \times [0,4]$. Let $S : U \to V$ be defined by $S(x,y) = \big(5 - x, \frac{y}{2}\big)$, for all $(x,y) \in U$. Let $T : C \to D$ be defined by $T(x,y) = (10 - x, y)$ for all $(x,y) \in C$. Let $A : H_1 \to H_2$ be defined by $A(x,y) = (2x, 2y)$ for all $(x,y) \in H_1$. It is clear that $U_0 = \{2\} \times [0,2], V_0 = \{3\} \times [0,2], C_0 = \{4\} \times [0,4], D_0 = \{6\} \times [0,4], d(U,V) = 1, d(C,D) = 2, Best_U S = \{(2,0)\}, Best_C T = \{4\} \times [0,4], \mathscr{S} = (2,0)$ and S, T are nonexpansive mappings such that $S(U_0) \subseteq V_0, T(C_0) \subseteq D_0$. Clearly S has best proximity point (2,0) and $A(2,0) = (4,0)$ is a best proximity point of T. All the conditions of Theorem 4.1 are satisfied, so the sequence generated by algorithm (SS') converges to (2,0) see (Table 1). Take $\gamma_n' = \frac{4n+1}{12n}$ and $\delta_n' = \frac{1}{n+25}$.*

n	$\hat{x}_n$	$A\hat{x}_n$
0	$(2,2)$	$(4,4)$
1	$(2,1.006173e+00)$	$(4,2.0123)$
2	$(2,5.473394e-01)$	$(4,1.0947)$
3	$(2,3.056327e-01)$	$(4,0.6113)$
4	$(2,1.729776e-01)$	$(4,0.3460)$
$\vdots$	$\vdots$	$\vdots$
37	$(2,1.177434e-08)$	$(4,0.0000)$
38	$(2,6.994626e-09)$	$(4,0.0000)$
39	$(2,4.156611e-09)$	$(4,0.0000)$

Table 1

6. Conclusion

In this paper, we introduce an algorithm which converges to solution of split fixed point problem under some conditions. We apply our main results for solving the split best proximity point problem. As a consequence, we obtain some fixed point and best proximity point results. Finally, we give numerical results to demonstrate there convergence.

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