ONE-SIDED TAUBERIAN CONDITIONS FOR THE $(\overline{N}, p)$ SUMMABILITY OF INTEGRALS

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Let p be a function on $\mathbb{R}_+ := [0, \infty)$ which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol: $p \in L^1_{loc}(\mathbb{R}_+)$ such that $P(x) = \int_0^x p(t)dt \neq 0$ for each $x > 0$, $P(0) = 0$ and $P(x) \to \infty$ as $x \to \infty$. For a real-valued function $f \in L^1_{loc}(\mathbb{R}_+)$, we set $s(x) := \int_0^x f(t)dt$ and $\sigma_p^{(1)}(x) := \frac{1}{P(x)} \int_0^x s(t)p(t)dt$, $x > 0$, provided that $P(x) > 0$.

We say that $\int_0^\infty f(x)dx$ is summable by the weighted mean method determined by the function $P(x)$ if there exists some $s \in \mathbb{R}$ such that

$$\lim_{x \to \infty} \sigma_p^{(1)}(x) = s.$$

If the limit $\lim_{x \to \infty} s(x) = s$ exists, then so does $\lim_{x \to \infty} \sigma_p^{(1)}(x) = s$.

In this paper, we obtain some new Tauberian conditions in terms of the weighted general control modulo for the weighted mean method of integrals in order that the converse implication hold true. Our results generalize some classical type Tauberian theorems given for Cesàro summability method of integrals.

Keywords: Tauberian theorem, Tauberian condition, weighted mean method of integrals, general control modulo.

MSC2010: 40E05, 40A10, 40G05.

1. Introduction

A number of authors such as Hardy [9], Móricz and Rhoades [12], Tietz [16], Çanak and Totur [2, 5] obtained Tauberian theorems for the weighted mean method of summability of sequences. Hardy [9] proved a classical two-sided bounded Tauberian theorem and Móricz and Rhoades [12] obtained a one-sided bounded Tauberian theorem for $(\overline{N}, p)$ summability of sequences. Çanak and Totur [2, 5] introduced some certain general one-sided bounded Tauberian conditions for this method.

In recent years, there has been an interest on summability method of integrals. Çanak and Totur [4] proved classical type some Tauberian theorems for the Cesàro summability method of integrals in parallel with sequences. They also proved some new Tauberian theorems by using the general control modulo which was defined by Dik [7] for the numerical sequences and generalized Hardy-Littlewood type Tauberian theorem in [3]. Moreover, one-sided and two-sided Tauberian conditions for the weighted mean method of integrals are given by Móricz [13] and Fekete and Móricz [8]. Móricz and Stadtmüller [14] characterized

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the convergence of weighted means of a function. Totur and Okur [18] have presented alternative proofs of the Landau-type and Schmidt-type Tauberian theorems for $(\overline{N}, p)$ summability of integrals under some certain conditions imposed on the sequence p.

The purpose of this paper is two fold: First, we introduce the weighted Kronecker identity and weighted general control modulo of integer order. Next, we prove more general theorems for the weighted mean method of integrals than the classical ones mentioned above.

2. Preliminaries

Let p be a function on $\mathbb{R}_+ := [0, \infty)$ which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol: $p \in L^1_{loc}(\mathbb{R}_+)$ such that $P(x) = \int_0^x p(t)dt \neq 0$ for each $x > 0$, $P(0) = 0$ and $P(x) \to \infty$ as $x \to \infty$. For a real-valued function $f \in L^1_{loc}(\mathbb{R}_+)$, the weighted mean of $s(x)$ is defined by

$$\sigma_p^{(1)}(x) = \frac{1}{P(x)} \int_0^x s(t)p(t)dt, \tag{1}$$

where $s(x) = \int_0^x f(t)dt$. If the limit

$$\lim_{x \to \infty} \sigma_p^{(1)}(x) = s \tag{2}$$

exists, then the integral

$$\int_0^\infty f(x)dx \tag{3}$$

is said to be summable by the weighted mean method determined by the function $P(x)$, in short; the $(\overline{N}, p)$ summable to a finite number s and we write $s(x) \to s\,(\overline{N}, p)$. We note that the concept of $(\overline{N}, p)$ summability here is the integral analogue of the one given in [9, page 57].

It is known that the existence of the integral

$$\int_0^\infty f(x)dx = s \tag{4}$$

implies (2) (see [15, page 16]). However, the converse implication is not always true.

Adding some suitable condition on $s(x)$, which is called a Tauberian condition, one may obtain (4) from (2). Any theorem which states that convergence of (3) follows from the $(\overline{N}, p)$ summability of $s(x)$ and a Tauberian condition is said to be a Tauberian theorem. The purpose of this paper is to investigate the converse implication of (4) $\Rightarrow$ (2).

The main results of this paper involve the concept of regularly varying of index $\alpha > 0$ which was introduced by Karamata [11] as follows (see [1] for more details):

Definition 2.1. *A positive function P is called regularly varying of index $\alpha > 0$ if*

$$\lim_{x \to \infty} \frac{P(\lambda x)}{P(x)} = \lambda^\alpha, \quad \lambda > 0. \tag{5}$$

We remind the reader that if a positive function P is regularly varying of index $\alpha > 0$, then the following conditions are cleary satisfied (see [6]):

$$\limsup_{x \to \infty} \frac{P(\lambda x)}{P(x)} < 1, \text{ for } 0 < \lambda < 1, \tag{6}$$

and

$$\limsup_{x \to \infty} \frac{P(x)}{P(\lambda x)} < 1, \text{ for } \lambda > 1. \tag{7}$$

For integrals, an analogous definition of weighted general control modulo of oscillating behaviors which is presented by Totur and Çanak [17] for sequence of real numbers is defined as follows.

We define the weighted classical control modulo of $s(x)$ by $\omega_p^{(0)}(x) = \frac{P(x)}{p(x)} f(x)$ and the weighted general control modulo of integer order $m \geq 1$ of $s(x)$ by

$$\omega_p^{(m)}(x) = \omega_p^{(m-1)}(x) - \sigma_p^{(1)}(\omega_p^{(m-1)}(x)). \qquad (8)$$

For each integer $m \geq 0$, we define $\sigma_p^{(m)}(x)$ and $v_p^{(m)}(x)$ by

$$\sigma_p^{(m)}(x) = \begin{cases} \frac{1}{P(x)} \int_0^x \sigma_p^{(m-1)}(t) p(t) dt, & m \geq 1 \\ s(x), & m = 0, \end{cases}$$

and

$$v_p^{(m)}(x) = \begin{cases} \frac{1}{P(x)} \int_0^x v_p^{(m-1)}(t) p(t) dt, & m \geq 1 \\ v_p(x), & m = 0, \end{cases}$$

where $v_p(x) = \frac{1}{P(x)} \int_0^x f(t) P(t) dt$.

For a function f, we define

$$\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_m f(x) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m-1} \left(\frac{P(x)}{p(x)} \frac{d}{dx} f(x) \right)$$
$$= \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right) \left(\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m-1} f(x) \right),$$

where $\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_0 f(x) = f(x)$ and $\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_1 f(x) = \frac{P(x)}{p(x)} \frac{d}{dx} f(x)$.

3. Main Results

By the following theorem, we obtain that every $(\overline{N}, p)$ summable integral is convergent provided $\omega_p^{(m)}(x)$ is one-sided bounded for sufficiently large x. This theorem generalizes Theorem 2 in [3].

Theorem 3.1. *Let P be regularly varying of index $\alpha > 0$. If $\int_0^\infty f(t) dt$ is $(\overline{N}, p)$ summable to s and $\omega_p^{(m)}(x) \geq -C$ for some $C > 0$, sufficiently large x and some nonnegative integer m, then the integral $\int_0^\infty f(t) dt$ converges to s.*

Corollary 3.1. *Let P be regularly varying of index $\alpha > 0$. If $\int_0^\infty f(t) dt$ is $(\overline{N}, p)$ summable to s and $\omega_p^{(m)}(x) = O(1)$ for sufficiently large x and some nonnegative integer m, then the integral $\int_0^\infty f(t) dt$ converges to s.*

By choosing $p(x) = 1$ and $m = 0$ in Corollary 3.1, we obtain the following integral analogue of a classical one-sided Tauberian theorem of Landau [10] which states that if a reel sequence (s_n) is Cesàro summable and there exists a positive constant $H > 0$ such that $k(s_k - s_{k-1}) > -H$ for all $k = 1, 2, ...$, then (s_n) is convergent.

Corollary 3.2. *If $\int_0^\infty f(t) dt$ is Cesàro summable to s and $xf(x) \geq -C$ for some $C > 0$, sufficiently large x, then the integral $\int_0^\infty f(t) dt$ converges to s.*

We recall that the integral $\int_0^\infty f(t) dt$ is said to be Cesàro summable to s if the limit $\lim_{x \to \infty} \frac{1}{x} \int_0^x s(t) dt = s$.

If we take $(\overline{N}, p)$ summability of $\sigma_p^{(1)}(x)$ instead that of $s(x)$ as a hypothesis with same Tauberian condition in Theorem 3.1, then we get the convergence of $s(x)$ again. By this theorem, we generalize Theorem 3 in [3].

Theorem 3.2. *Let P be regularly varying of index $\alpha > 0$. If $\sigma_p^{(k)}(x)$ is $(\overline{N}, p)$ summable to s for any nonnegative integer k and $\omega_p^{(m)}(x) \geq -C$ for some $C > 0$, sufficiently large x and some nonnegative integer m, then the integral $\int_0^\infty f(t) dt$ converges to s.*

We may not obtain the convergence of $s(x)$ by weakening the conditions in Theorem 3.2 more. However we obtain $(\overline{N}, p)$ summability of $s(x)$ by one-sided boundedness condition of $(\overline{N}, p)$ mean of $\omega_p^{(m)}(x)$.

Theorem 3.3. *Let P be regularly varying of index $\alpha > 0$. If $\sigma_p^{(k)}(x)$ is $(\overline{N}, p)$ summable to s for any nonnegative k and $\sigma_p^{(1)}(\omega_p^{(m)}(x)) \geq -C$ for some $C > 0$, sufficiently large x and some nonnegative integer m, then the integral $\int_0^\infty f(t)dt$ is $(\overline{N}, p)$ summable to s.*

4. Auxiliary Results

In order to prove our main results, we need the following lemmas.

Lemma 4.1. *([18]) The following identities hold:*

(i) *For $\lambda > 1$,*

$$s(x) - \sigma_p^{(1)}(x) = \frac{P(\lambda x)}{P(\lambda x) - P(x)}(\sigma_p^{(1)}(\lambda x) - \sigma_p^{(1)}(x))$$

$$- \frac{1}{P(\lambda x) - P(x)} \int_x^{\lambda x} (s(t) - s(x))p(t)dt.$$

(ii) *For $0 < \lambda < 1$,*

$$s(x) - \sigma_p^{(1)}(x) = \frac{P(\lambda x)}{P(x) - P(\lambda x)}(\sigma_p^{(1)}(x) - \sigma_p^{(1)}(\lambda x))$$

$$+ \frac{1}{P(x) - P(\lambda x)} \int_{\lambda x}^{x} (s(x) - s(t))p(t)dt.$$

Lemma 4.2 is a classical-type Tauberian theorem for the weighted mean method of integrals.

Lemma 4.2. *Let P be regularly varying of index $\alpha > 0$. If $\int_0^\infty f(t)dt$ is $(\overline{N}, p)$ integrable to s and*

$$\frac{P(x)}{p(x)}f(x) \geq -C, \tag{9}$$

for some $C > 0$ and sufficiently large x, then the integral $\int_0^\infty f(t)dt$ converges to s.

Proof. The condition $\frac{P(x)}{p(x)}f(x) \geq -C$ implies $-\frac{d}{dx}s(x) \leq C\frac{p(x)}{P(x)}$ for some $C > 0$ and sufficiently large x. From Lemma 4.1 (i), we have

$$s(x) - \sigma_p^{(1)}(x) \leq \frac{P(\lambda x)}{P(\lambda x) - P(x)}(\sigma_p^{(1)}(\lambda x) - \sigma_p^{(1)}(x)) + C\log\frac{P(\lambda x)}{P(x)},$$

for $\lambda > 1$.

Since P is regularly varying of index α, it is plain that for all $\lambda > 1$ and sufficiently large x,

$$\frac{\lambda^\alpha}{2(\lambda^\alpha - 1)} \leq \frac{P(\lambda x)}{P(\lambda x) - P(x)} \leq \frac{3\lambda^\alpha}{2(\lambda^\alpha - 1)}. \tag{10}$$

As the limit of $\sigma_p^{(1)}(x)$ exists, we obtain

$$\limsup_{x \to \infty}\left(s(x) - \sigma_p^{(1)}(\lambda x)\right) \leq \limsup_{x \to \infty}\left(C\log\frac{P(\lambda x)}{P(x)}\right)$$

for some $C > 0$. Taking the limit of both sides as $\lambda \to 1^+$, we get

$$\limsup_{x \to \infty}\left(s(x) - \sigma_p^{(1)}(x)\right) \leq 0 \tag{11}$$

by the hypothesis that P is regularly varying of index α.

In a similar way from Lemma 4.1 (ii), we get

$$\liminf_{x \to \infty} \left(s(x) - \sigma_p^{(1)}(x) \right) \geq 0. \tag{12}$$

Therefore, the proof is completed by (11) and (12). □

The following lemma provides an identity which is called the weighted Kronecker identity.

Lemma 4.3. $s(x) - \sigma_p^{(1)}(x) = v_p(x)$ where $v_p(x) = \frac{1}{P(x)} \int_0^x f(t) P(t) dt.$

Proof. From (1), we have

$$
\begin{aligned}
\sigma_p^{(1)}(x) = \frac{1}{P(x)} \int_0^x s(t) p(t) dt &= \frac{1}{P(x)} \int_0^x \left(\int_0^t f(u) du \right) p(t) dt \\
&= \frac{1}{P(x)} \int_0^x f(u) \left(\int_u^x p(t) dt \right) du \\
&= \frac{1}{P(x)} \int_0^x (P(x) - P(u)) f(u) du \\
&= \int_0^x \left(1 - \frac{P(t)}{P(x)} \right) f(t) dt \\
&= \int_0^x f(t) dt - \frac{1}{P(x)} \int_0^x f(t) P(t) dt \\
&= s(x) - v_p(x),
\end{aligned}
$$

which completes the proof. □

Lemma 4.4. For each integer $m \geq 0$, $\dfrac{P(x)}{p(x)} \dfrac{d}{dx} \sigma_p^{(m)}(x) = v_p^{(m)}(x).$

Proof. If we take the derivative of $\sigma_p^{(m)}(x)$, we obtain

$$\frac{d}{dx} \sigma_p^{(m)}(x) = \frac{p(x)}{P(x)} \left(\sigma_p^{(m-1)}(x) - \sigma_p^{(m)}(x) \right).$$

By Lemma 4.3, we obtain $\dfrac{d}{dx} \sigma_p^{(m)}(x) = \dfrac{p(x)}{P(x)} v_p^m(x)$. This completes the proof. □

Lemma 4.5. For each $m \geq 0$ integer,

(i) $\dfrac{P(x)}{p(x)} \dfrac{d}{dx} v_p(x) = \dfrac{P(x)}{p(x)} f(x) - v_p(x).$

(ii) $v_p^{(m)}(x) - v_p^{(m+1)}(x) = \dfrac{P(x)}{p(x)} \dfrac{d}{dx} v_p^{(m+1)}(x).$

Proof. (i) If we take the derivative of the weighted Kronecker identity, then we obtain

$$f(x) - \left(-\frac{p(x)}{P^2(x)} \int_0^x s(t) p(t) dt + \frac{s(x) p(x)}{P(x)} \right) = \frac{d}{dx} v_p(x). \tag{13}$$

Multiplying both sides of (13) by $P(x)$ and using the weighted Kronecker identity, we get

$$P(x) \frac{d}{dx} v_p(x) = P(x) f(x) - p(x) v_p(x). \tag{14}$$

Then dividing both sides of (14) by $p(x)$, we have the proof of (i).

(ii) Applying $\sigma_p^{(m)}(x)$ to Lemma 4.3 and taking the derivative of both sides, we have

$$\frac{d}{dx} \sigma_p^{(m)}(x) - \frac{d}{dx} \sigma_p^{(m+1)}(x) = \frac{d}{dx} v_p^{(m+1)}(x). \tag{15}$$

Then multiplying both sides of (15) by $\frac{P(x)}{p(x)}$ and using Lemma 4.4, we have the proof of (ii). □

Lemma 4.6. *For each integer* $k \geq 1$, $\sigma_p^{(k)} \left(\frac{P(x)}{p(x)} \frac{d}{dx} v_p(x) \right) = \frac{P(x)}{p(x)} \frac{d}{dx} v_p^{(k)}(x)$.

Proof. If we take the weighted mean of order k of the both sides of the identity in Lemma 4.5 (i), we get

$$\sigma_p^{(k)} \left(\frac{P(x)}{p(x)} \frac{d}{dx} v_p(x) \right) = v_p^{(k-1)}(x) - v_p^{(k)}(x).$$

Also taking $m = k - 1$ in Lemma 4.5 (ii), we obtain $v_p^{(k-1)}(x) - v_p^{(k)}(x) = \frac{P(x)}{p(x)} \frac{d}{dx} v_p^{(k)}(x)$. This completes the proof of Lemma. □

In the following lemma, we now give a different representation of the weighted general control modulo of integer order of functions.

Lemma 4.7. *For each integer* $m \geq 1$, $\omega_p^{(m)}(x) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_m v_p^{(m-1)}(x)$.

Proof. We establish the proof by the method of induction. Taking $m = 1$ in (8) and using Lemma 4.5 (i), we get $\omega_p^{(1)}(x) = \frac{P(x)}{p(x)} \frac{d}{dx} v_p(x)$. Assume that the assertion is true for $m = k$. Therefore we obtain,

$$\omega_p^{(k)}(x) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_k v_p^{(k-1)}(x). \tag{16}$$

Taking $m = k + 1$ in (8) and using (16), we get

$$\omega_p^{(k+1)}(x) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_k \left(v_p^{(k-1)}(x) - v_p^{(k)}(x) \right).$$

From Lemma 4.5 (ii), we obtain

$$\omega_p^{(k+1)}(x) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_k \left(\frac{P(x)}{p(x)} \frac{d}{dx} v_p^{(k)}(x) \right) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{k+1} v_p^{(k)}(x).$$

Thus, we conclude that Lemma 4.7 is true for each integer $m \geq 1$. □

Lemma 4.8. *For each integer* $m \geq 0$ *and* $k \geq 1$, $\sigma_p^{(k)}(\omega_p^{(m)}(x)) = \omega_p^{(m)}(\sigma_p^{(k)}(x))$.

Proof. From Lemma 4.4, 4.6 and 4.7 we obtain,

$$\sigma_p^{(k)}(\omega_p^{(m)}(x)) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_m \left(\frac{P(x)}{p(x)} \frac{d}{dx} \sigma_p^{(m+k-1)}(x) \right).$$

Therefore we get,

$$\sigma_p^{(k)}(\omega_p^{(m)}(x)) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m+1} \sigma_p^{(m+k-1)}(x). \tag{17}$$

On the other hand by using Lemma 4.4 and 4.7, we obtain,

$$\omega_p^{(m)}(\sigma_p^{(k)}(x)) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_m \left(\frac{P(x)}{p(x)} \frac{d}{dx} \sigma_p^{(m-1)}(\sigma_p^{(k)}(x)) \right).$$

Hence we get,

$$\omega_p^{(m)}(\sigma_p^{(k)}(x)) = \left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m+1} \sigma_p^{(m+k-1)}(x). \tag{18}$$

Combining (17) and (18), we obtain $\sigma_p^{(k)}(\omega_p^{(m)}(x)) = \omega_p^{(m)}(\sigma_p^{(k)}(x))$. □

5. **Proofs**

Proof of Theorem 3.1

Suppose that $s(x)$ is $(\overline{N}, p)$ summable to s. Therefore $\sigma_p^{(1)}(x)$ is $(\overline{N}, p)$ summable to same value. By the difference of $s(x)$ and $\sigma_p^{(1)}(x)$, from Lemma 4.3, we get $v_p(x)$ is $(\overline{N}, p)$ summable to 0. Using the definition of the weighted general control modulo of integer order $m \geq 1$, we obtain

$$\sigma_p^{(1)}(w_p^{(m-1)}(x)) \to 0(\overline{N}, p). \tag{19}$$

If we use Lemma 4.7 and the hypothesis $\omega_p^{(m)}(x) \geq -C$ for some $C \geq 0$ and sufficiently large x, then we have

$$\omega_p^{(m)}(x) = \frac{P(x)}{p(x)} \frac{d}{dx} \left(\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m-1} v_p^{(m-1)}(x) \right) \geq -C. \tag{20}$$

Applying Lemma 4.1 (i) to $\sigma_p^{(1)}(\omega_p^{(m-1)}(x))$ we get

$$\begin{aligned}
\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) \quad &- \quad \sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \\
&= \quad \frac{P(\lambda x)}{P(\lambda x) - P(x)} (\sigma_p^{(2)}(\omega_p^{(m-1)}(\lambda x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x))) \\
&\quad - \quad \frac{1}{P(\lambda x) - P(x)} \int_x^{\lambda x} \left(\int_x^t \frac{d}{dz} \sigma_p^{(1)}(\omega_p^{(m-1)}(z)) dz \right) p(t) dt.
\end{aligned}$$

Using Lemma 4.6 and 4.7 and the condition (20), we get

$$\begin{aligned}
\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) \quad &- \quad \sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \\
&\leq \quad \frac{P(\lambda x)}{P(\lambda x) - P(x)} (\sigma_p^{(2)}(\omega_p^{(m-1)}(\lambda x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x))) \\
&\quad + C \log \frac{P(\lambda x)}{P(x)} .
\end{aligned}$$

Since P is regularly varying of index α, it is plain that for all $\lambda > 1$ and sufficiently large x,

$$\frac{\lambda^\alpha}{2(\lambda^\alpha - 1)} \leq \frac{P(\lambda x)}{P(\lambda x) - P(x)} \leq \frac{3\lambda^\alpha}{2(\lambda^\alpha - 1)}. \tag{21}$$

By (19) and (21), for all $\lambda > 1$,

$$\lim_{x \to \infty} \frac{P(\lambda x)}{P(\lambda x) - P(x)} (\sigma_p^{(2)}(\omega_p^{(m-1)}(\lambda x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x))) = 0. \tag{22}$$

Taking $\limsup$ of both sides as $x \to \infty$, we obtain

$$\begin{aligned}
&\limsup_{x \to \infty} \left(\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \right) \\
\leq \quad &\limsup_{x \to \infty} \frac{P(\lambda x)}{P(\lambda x) - P(x)} \left(\sigma_p^{(2)}(\omega_p^{(m-1)}(\lambda x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \right) \\
&+ \limsup_{x \to \infty} C \log \frac{P(\lambda x)}{P(x)}.
\end{aligned}$$

Taking (22) into account, we obtain

$$\limsup_{x \to \infty} \left(\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \right) \leq \limsup_{x \to \infty} C \log \frac{P(\lambda x)}{P(x)}.$$

Since P is regularly varying, taking the limit of both sides as $\lambda \to 1^+$, we get

$$\limsup_{x \to \infty} \left(\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \right) \leq 0. \tag{23}$$

In a similar way from Lemma 4.1 (ii), we get

$$\liminf_{x\to\infty}\left(\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) - \sigma_p^{(2)}(\omega_p^{(m-1)}(x))\right) \geq 0. \tag{24}$$

Combining (23) and (24), we obtain that

$$\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) = o(1).$$

Therefore, from (8) we get $\omega_p^{(m-1)}(x) \geq -C_1$ for some $C_1 > 0$. Using Lemma 4.7, we obtain

$$\omega_p^{(m-1)}(x) = \frac{P(x)}{p(x)}\frac{d}{dx}\left(\left(\frac{P(x)}{p(x)}\frac{d}{dx}\right)_{m-2} v_p^{(m-2)}(x)\right) \geq -C_1. \tag{25}$$

Using (8), we obtain

$$\sigma_p^{(1)}(\omega_p^{(m-2)}(x)) \to 0(\overline{N}, p). \tag{26}$$

Applying Lemma 4.1 (i) and (ii) to $\sigma_p^{(1)}(\omega_p^{(m-2)}(x))$ with similar steps, we get

$$\sigma_p^{(1)}(\omega_p^{(m-2)}(x)) = o(1).$$

Continuing in this way, we obtain that

$$\sigma_p^{(1)}(\omega_p^{(1)}(x)) = o(1).$$

Since $s(x)$ is $(\overline{N}, p)$ summable to s, we have $\sigma_p^{(1)}(v_p(x)) = o(1)$. Also from Lemma 4.5 (ii), 4.6 and 4.7, we get that $v_p(x) = o(1)$. Finally, from Lemma 4.3 we conclude that $s(x)$ converges to s. $\square$

Proof of Theorem 3.2

Assume that $\sigma_p^{(k)}(x)$ is $(\overline{N}, p)$ summable to s. Taking the weighted mean of the both sides of the identity in Lemma 4.3, then we obtain $v_p^{(k)}(x)$ is $(\overline{N}, p)$ summable to 0. Using this result in (8), we get that

$$\sigma_p^{(k+1)}(\omega_p^{(m-1)}(x)) \to 0(\overline{N}, p). \tag{27}$$

From Lemma 4.7 and the hypothesis $\omega_p^{(m)}(x) \geq -C$ for some $C \geq 0$ and sufficiently large x, then we have

$$\sigma_p^{(k)}(\omega_p^{(m)}(x)) = \frac{P(x)}{p(x)}\frac{d}{dx}\sigma_p^{(k)}\left(\left(\frac{P(x)}{p(x)}\frac{d}{dx}\right)_{m-1} v_p^{(m-1)}(x)\right) \geq -C. \tag{28}$$

Applying Lemma 4.2 to $\sigma_p^{(k+1)}(\omega_p^{(m-1)}(x))$ and using Lemma 4.6 and conditions (27), (28), we obtain

$$\sigma_p^{(k+1)}(\omega_p^{(m-1)}(x)) = o(1).$$

By (8), we obtain $\sigma_p^{(k)}(\omega_p^{(m-1)}(x)) \geq -C_1$ for some $C_1 \geq 0$. Using Lemma 4.7 and the hypothesis $\omega_p^{(m)}(x) \geq -C$ for some $C \geq 0$ and sufficiently large x again, we get that

$$\sigma_p^{(k)}(\omega_p^{(m-1)}(x)) = \frac{P(x)}{p(x)}\frac{d}{dx}\sigma_p^{(k)}\left(\left(\frac{P(x)}{p(x)}\frac{d}{dx}\right)_{m-2} v_p^{(m-2)}(x)\right) \geq -C. \tag{29}$$

From (8), we obtain

$$\sigma_p^{(k+1)}(\omega_p^{(m-2)}(x)) \to 0(\overline{N}, p). \tag{30}$$

Applying Lemma 4.2 to $\sigma_p^{(k)}(\omega_p^{(m-2)}(x))$ and using Lemma 4.6, we obtain

$$\sigma_p^{(k+1)}(\omega_p^{(m-2)}(x)) = o(1).$$

Continuing in the same vein, we obtain

$$\sigma_p^{(k+1)}(\omega_p^{(1)}(x)) = o(1).$$

Since $\sigma_p^{(k)}(x) \to s(\overline{N}, p)$, we get that $v_p^{(k+1)}(x) = o(1)$. Using Lemma 4.5 (ii), 4.6 and 4.7, we have $v_p^{(k)}(x) = o(1)$. Using the weighted mean of order k of the identity in Lemma 4.3, we get $\sigma_p^{(k)}(x) \to s(\overline{N}, p)$. If we do these steps $k - 1$ times, we obtain $\sigma_p^{(1)}(x) \to s(\overline{N}, p)$.

If we take the weighted mean of the both sides of the identity in Lemma 4.3, then we obtain $v_p^{(1)}(x)$ is $(\overline{N}, p)$ summable to 0 and we have

$$\sigma_p^{(2)}(\omega_p^{(m-1)}(x)) \to 0(\overline{N}, p). \tag{31}$$

By Lemma 4.7 and the hypothesis $\omega_p^{(m)}(x) \geq -C$ for some $C > 0$ and sufficiently large x, then we have

$$\sigma_p^{(1)}(\omega_p^{(m)}(x)) = \frac{P(x)}{p(x)} \frac{d}{dx} \sigma_p^{(1)} \left(\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m-1} v_p^{(m-1)}(x) \right) \geq -C. \tag{32}$$

Applying Lemma 4.2 to $\sigma_p^{(2)}(\omega_p^{(m-1)}(x))$ and using Lemma 4.6 and conditions (31) and (32), we obtain that

$$\sigma_p^{(2)}(\omega_p^{(m-1)}(x)) = o(1).$$

By (8), we get $\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) \geq -C_1$, for some $C_1 > 0$. By using Lemma 4.7 and the hypothesis $\omega_p^{(m)}(x) \geq -C$, for some $C > 0$ and enough large x again, we get that

$$\sigma_p^{(1)}(\omega_p^{(m-1)}(x)) = \frac{P(x)}{p(x)} \frac{d}{dx} \sigma_p^{(1)} \left(\left(\frac{P(x)}{p(x)} \frac{d}{dx} \right)_{m-2} v_p^{(m-2)}(x) \right) \geq -C_1. \tag{33}$$

From (8), we obtain

$$\sigma_p^{(2)}(\omega_p^{(m-2)}(x)) \to 0(\overline{N}, p). \tag{34}$$

Applying Lemma 4.2 to $\sigma_p^{(2)}(\omega_p^{(m-2)}(x))$ with using (33), (34) and Lemma 4.6, we obtain that

$$\sigma_p^{(2)}(\omega_p^{(m-2)}(x)) = o(1).$$

If we continue in the same vein, then we get that

$$\sigma_p^{(2)}(\omega_p^{(1)}(x)) = o(1).$$

From the $(\overline{N}, p)$ summability of $\sigma_p^{(1)}(x)$ to s and Lemma 4.3, we get $\sigma_p^{(2)}(v_p(x)) = o(1)$. Using Lemma 4.5 (ii), 4.6 and 4.7, we get that $v_p^{(1)}(x) = o(1)$. Finally, using the weighted mean of the identity in Lemma 4.3 we get $\sigma_p^{(1)}(x) \to s$. Therefore the conditions in Theorem 3.1 holds and proof is completed. $\square$

Proof of Theorem 3.3

From Lemma 4.8 and the hypothesis $\sigma_p^{(1)}(\omega_p^{(m)}(x)) \geq -C$ for some $C \geq 0$, we obtain

$$\omega_p^{(m)}(\sigma_p^{(k)}(x)) \geq -C, \tag{35}$$

for some $C \geq 0$. Since $\sigma_p^{(k)}(x)$ is $(\overline{N}, p)$ summable to s, from Theorem 3.1 and condition (35), we get

$$\sigma_p^{(k-1)}(x) \to s(\overline{N}, p). \tag{36}$$

Also from Lemma 4.8 and the same hypothesis, we get

$$\omega_p^{(m)}(\sigma_p^{(k-1)}(x)) \geq -C, \tag{37}$$

for some $C \geq 0$. Hence, from Theorem 3.1, conditions (36) and (37), we get

$$\sigma_p^{(k-2)}(x) \to s(\overline{N}, p). \tag{38}$$

If we continue in the same vein, then we get

$$\sigma_p^{(1)}(x) \to s(\overline{N}, p). \tag{39}$$

From Lemma 4.8 and the same hypothesis, we get

$$\omega_p^{(m)}(\sigma_p^{(1)}(x)) \geq -C, \tag{40}$$

for some $C \geq 0$. Finally, from Theorem 3.1, conditions (39) and (40), we get $s(x) \to s(\overline{N}, p)$. This completes the proof. $\square$

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