A GENERALIZATION OF DOWSON'S RESULT

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In this paper we obtain S-spectral operators which appear as restrictions and quotients of spectral operators. We generalize the result obtained by H.R. Dowson about the fact that the restriction and quotient of a spectral operator at a closed subspace of a complex Banach space are, in certain conditions, spectral operators. In this article we found a condition in which the restriction and quotient are even S-spectral operators thorough the separation and extraction of the restriction and quotient operators' spectral part. Finally, the results are extended from one operator to an operator system.

Keywords: scalar; spectral; S-scalar; S-spectral; spectral measure; S-spectral measure; restriction and quotient of an operator (operator system).

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1. Introduction

Throughout the paper, it is investigated the behavior of restriction and quotient of a spectral operator (respectively, spectral operator systems) with respect to an invariant closed subspace. These spectral decompositions are related to differential equations and to differential equation systems ([18]) and can have various applications in quantum mechanics, in bifurcation and fractal theories ([1]).

Let us consider X a complex Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X and let $\mathcal{P}_X$ be the set of all projectors on X. If $T \in \mathbf{B}(X)$ and Y is a closed subspace of X invariant to T, then $T|Y$ is the restriction operator of T to Y and $\dot{T}$ is the quotient operator induced by T on the quotient space $\dot{X} = X/Y$; for an operator system $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$, $a|Y = (a_1|Y, a_2|Y, \ldots, a_n|Y) \subset \mathbf{B}(Y)$ is the restriction operator system of a to Y and $\dot{a} = (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_n) \subset \mathbf{B}(\dot{X})$ is the quotient operator system induced by a on the quotient space $\dot{X}$. For $T \in \mathbf{B}(X)$, we denote by $\rho(T)$ the resolvent set of T (in X) and by $\sigma(T)$ the spectrum of T (in X); for $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$, we denote by $\sigma(a, X)$ the spectrum of a (in X)[2, 3].

A closed subspace $Y \subset X$ is called *spectral maximal space* of $T \in \mathbf{B}(X)$ if Y is invariant to T and for any other closed subspace $Z \subset X$, also invariant to T, such that $\sigma(T|Z) \subset \sigma(T|Y)$ we have $Z \subset Y$ ([7], [8]).

We recall that $T \in \mathbf{B}(X)$ has the *single-valued extension property* if for any analytic function $f : D \to X$ ($D \subset \mathbb{C}$ open set) with $(\lambda I - T)f(\lambda) = 0$, we have $f(\lambda) \equiv 0$ ([11, 12]).

If $T \in \mathbf{B}(X)$ has the single-valued extension property and $x \in X$, we consider the set $\rho_T(x)$ of all elements $\lambda_0 \in \mathbb{C}$ such that there is an X-valued analytic function $\lambda \to x_T(\lambda)$ defined in a neighborhood V of λ_0 which verifies $(\lambda I - T)x_T(\lambda) \equiv x$ on V. We take $\sigma_T(x) = \complement\rho_T(x) = \mathbb{C} \setminus \rho_T(x)$ and $X_T(F) = \{x \in X; \sigma_T(x) \subset F\}$, where $F \subset \mathbb{C}$ is closed. $\rho_T(x)$ is the *local resolvent set of x with respect to T* and $\sigma_T(x)$ is the *local spectrum of x with respect to T* ([11, 15]).

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In [9] and [10], H.R. Dowson has shown that if $T \in \mathbf{B}(X)$ is a spectral operator and Y is a closed subspace of X invariant to T, then the restriction operator $T|Y$ is spectral if and only if the quotient operator $\dot{T}$ is spectral if and only if Y is invariant to the spectral measure E of T. Furthermore, if T is spectral and its spectrum $\sigma(T)$ is totally disconnected (respectively, if T is spectral and the spectrum $\sigma(T|Y)$ of $T|Y$ is totally disconnected), then the restriction $T|Y$ and the quotient $\dot{T}$ are spectral (respectively, the restriction $T|Y$ is spectral).

In this paper, we prove that these assertions occur if the intersection of the spectra $\sigma(T|Y)$ and $\sigma(\dot{T})$ is totally disconnected, i.e. $\dim S = 0$, where $S = \sigma(T|Y) \cap \sigma(\dot{T})$. We will observe that $\dim S = 0$, where $S = \sigma(T|Y) \cap \sigma(\dot{T})$, instead of $\dim(\sigma(T|Y)) = 0$. We consider that the important part of the paper is obtaining the following result: if T is a spectral operator and Y an invariant subspace to T, then the restriction $T|Y$ and the quotient $\dot{T}$ are S-spectral operators, where S is a compact subset of $\sigma(T)$. In this way, we obtain S-spectral operators, which are spectral operators only on a subset of the spectrum $\sigma(T|Y)$ ([17, 23]).

2. **Extension of Dowson's result**

Definition 2.1. ([4]) Let X be a Banach space and let $\mathfrak{B}_S$ be the family of all Borelian sets B of the complex plane $\mathbb{C}$ with the property that $B \cap S = \emptyset$ or $B \supset S$, where S is a compact fixed set of $\mathbb{C}$.

A map $E_S : \mathfrak{B}_S \to \mathcal{P}_X$ is said to be *S-spectral measure* if

1. $E_S(\emptyset) = 0$, $E_S(\mathbb{C}) = I$
2. $E_S(B_1 \cap B_2) = E_S(B_1)E_S(B_2)$, $B_1, B_2 \in \mathfrak{B}_S$
3. $E_S\left(\bigcup_{m=1}^{\infty} B_m\right) x = \sum_{m=1}^{\infty} E_S(B_m)x$, $B_m \in \mathfrak{B}_S$, $B_p \cap B_m = \emptyset$, $p \neq m$, $x \in X$.
4. $\sup_{B \in \mathfrak{B}_S} \|E_S(B)\| < \infty$.

An operator $T \in \mathbf{B}(X)$ is said to be *S-spectral* if there is an S-spectral measure E_S such that

5. $TE_S(B) = E_S(B)T$, $B \in \mathfrak{B}_S$
6. $\sigma(T|E_S(B)X) \subset \overline{B}$, $B \in \mathfrak{B}_S$.

For $S = \emptyset$, we obtain a spectral measure and a spectral operator ([11, 20]).

Remark 2.1. An operator T is S-spectral if and only if it is written as a direct sum $T = T_1 \oplus T_2$, where T_1 is spectral and $\sigma(T_2) \subset S$.

Proof. Indeed, if T is S-spectral and E_S is its S-spectral measure, then it is a simple exercise to show that the map $E : \mathfrak{B} \to \mathcal{P}_X$ (where $\mathfrak{B} = \mathfrak{B}_\emptyset$), defined by $E(B) = E_S(B \cap \complement S)$, $B \in \mathfrak{B}$, is a spectral measure for $T_1 = T|E_S(\complement S)X$, hence $T = T_1 \oplus T_2$, where $T_2 = T|E_S(S)X$ and $\sigma(T_2) = \sigma(T|E_S(S)X) \subset S$.

Conversely, if $T_1 \in \mathbf{B}(X_1)$ is spectral and $T_2 \in \mathbf{B}(X_2)$ is nonspectral, with $\sigma(T_2) \not\supset \sigma(T_1)$, by putting $S = \sigma(T_2)$, $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$, we obtain that the map $E_S : \mathfrak{B}_S \to \mathcal{P}_X$ defined by the equalities $E_S(B) = E(B) \oplus 0$, if $B \cap S = \emptyset$ and $E_S(B) = E(B) \oplus I_2$, if $B \supset S$, $B \in \mathfrak{B}_S$, is an S-spectral measure of T, where E is the spectral measure of T_1 and I_2 is the identity operator in X_2. □

Definition 2.2. ([16]) A subset of complex plane is said to have *dimension* 0 or to be *totally disconnected* if the connected component of each point is the set consisting of the point itself; in other words, if any subset of it is both open and closed in its relative topology (the trace of the topology of S)

Lemma 2.1. *Let X be a Banach space and let X_1, X_2 be two linear closed subspaces of X such that $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2$ are closed. If $Y_i \subset X_i$ $(i = 1, 2)$ are two linear closed subspaces, then $Y_1 + Y_2$ is also closed.*
Moreover, if $a = (a_1, a_2, \dots, a_n) \subset \boldsymbol{B}(X)$ is a decomposable (spectral) system and Z_1, Z_2 are two closed subspaces of X invariant to a such that $\sigma(a, Z_1) \cap \sigma(a, Z_2) = \emptyset$, then $Z_1 + Z_2$ is closed.

Proof. Indeed, if $y_n \in Y_1 + Y_2$, $y_n \to y \in X$, then $y_n = y_n^1 + y_n^2$, $y_n^i \in Y_i \subset X_i$ $(i = 1, 2)$. Since $X_1 + X_2$ is closed, by the closed graph theorem it follows that $y_n^i \to y_i \in Y_i$ $(i = 1, 2)$, hence $y = y_1 + y_2 \in Y_1 + Y_2$, i.e. $Y_1 + Y_2$ is closed.

We have $Z_1 \subset X_a(\sigma(a, Z_1))$, $Z_2 \subset X_a(\sigma(a, Z_2))$, $X_a(\sigma(a, Z_1)) \cap X_a(\sigma(a, Z_2)) \subset X_a(\sigma(a, Z_1) \cap \sigma(a, Z_2)) = X_a(\emptyset) = \{0\}$ and from Proposition 2.2.8, [14], it results that

$$X_a(\sigma(a, Z_1)) \oplus X_a(\sigma(a, Z_2)) = X_a(\sigma(a, Z_1) \cup \sigma(a, Z_2))$$

is closed; consequently, according to the previous result, we have that $Z_1 + Z_2$ is closed. □

Remark 2.2. a) Let $T \in \mathbf{B}(X)$ be decomposable and let $Y_1, Y_2 \subset X$ be two closed invariant subspaces to T such that $\sigma(T|Y_1) \cap \sigma(T|Y_2) = \emptyset$. If we denote by $\dot{X} = X/Y_1$ the quotient space and by $\varphi : X \to \dot{X}$ the canonical map, then $Y_1 + Y_2$ is closed, Y_2 can be identified with $\varphi(Y_2) = \dot{Y}_2$, since Y_2 and $\dot{Y}_2$ are (topologically) isomorphic, hence $T|Y_2$ and $\dot{T}|\dot{Y}_2$ are similar and $\sigma(T|Y_2) = \sigma(\dot{T}|\dot{Y}_2)$.
b) In a similar way, let $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ be decomposable and let $Z_1, Z_2 \subset X$ be two closed invariant subspaces to a such that $\sigma(a, Z_1) \cap \sigma(a, Z_2) = \emptyset$. If we make the notations $\dot{X} = X/Z_1$ and $\varphi : X \to \dot{X}$ the canonical map, it follows that $Z_1 + Z_2$ is closed, Z_2 can be identified with $\varphi(Z_2) = \dot{Z}_2$, since Z_2 and $\dot{Z}_2$ are (topologically) isomorphic, again $a|Z_2$ and $\dot{a}|\dot{Z}_2$ are similar, with $\sigma(a, Z_2) = \sigma(\dot{a}, \dot{Z}_2)$.

Lemma 2.2. *Let $T \in \boldsymbol{B}(X)$, let Y be a closed subspace invariant to T and let $\dot{T}$ be the operator induced by T in the quotient space $\dot{X} = X/Y$. If T and $\dot{T}$ have the single-valued extension property, then*

$$X_T(\sigma(T|Y) \setminus \sigma(\dot{T})) \subset Y.$$

In a similar way, if $a = (a_1, a_2, \dots, a_n) \subset \boldsymbol{B}(X)$, Y is a closed subspace invariant to a and $\dot{a} = (\dot{a}_1, \dot{a}_2, \dots, \dot{a}_n) \subset \boldsymbol{B}(\dot{X})$ is the system induced by a on $\dot{X} = X/Y$ with $S_a = S_{\dot{a}} = \emptyset$ (S_a, $S_{\dot{a}}$ are the analytic spectral residuums of a, respectively of $\dot{a}$, [22]*) then*

$$X_{[a]}(\sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})) \subset Y.$$

Proof. If $x \in X_T(\sigma(T|Y) \setminus \sigma(\dot{T}))$, we have $\sigma_T(x) \subset \sigma(T|Y) \setminus \sigma(\dot{T})$ and

$$\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x) \cap \sigma(\dot{T}) \subset (\sigma(T|Y) \setminus \sigma(\dot{T})) \cap \sigma(\dot{T}) = \emptyset,$$

hence $\dot{x} = \dot{0}$ and consequently $x \in Y$ (because $S_T = S_{\dot{T}} = \emptyset$ implies that $\gamma_T(x) = \sigma_T(x)$, $\gamma_{\dot{T}}(\dot{x}) = \sigma_{\dot{T}}(\dot{x})$ and $\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x)$ ([5], Proposition 2.1)).

For an operator system $a \subset \mathbf{B}(X)$, J. Eschmeier proved in [13] that $\sigma(a, x) = sp(a, x)$, for any $x \in X$.

Let $x \in X_{[a]}(\sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X}))$, hence $sp(a, x) = \sigma(a, x) \subset \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$. We make the notation $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{C}^n$ and from the equality

$$x \equiv (\zeta_1 - a_1)f_1(\zeta) + (\zeta_2 - a_2)f_2(\zeta) + ... + (\zeta_n - a_n)f_n(\zeta)$$

with $\zeta \in \omega \subset \mathbb{C}^n$, f_j analytic functions $(j = 1, 2, \dots, n)$, it follows that

$$\dot{x} \equiv (\zeta_1 - \dot{a}_1)\dot{f}_1(\zeta) + (\zeta_2 - \dot{a}_2)\dot{f}_2(\zeta) + ... + (\zeta_n - \dot{a}_n)\dot{f}_n(\zeta)$$

hence $\sigma(\dot{a},\dot{x}) \subset \sigma(a,x) \subset \sigma(a,Y) \setminus \sigma(\dot{a},\dot{X})$. Then

$$\sigma(\dot{a},\dot{x}) \subset (\sigma(a,Y) \setminus \sigma(\dot{a},\dot{X})) \cap \sigma(\dot{a},\dot{X}) = \emptyset$$

hence $\dot{x} = \dot{0}$, therefore $X_{[a]}(\sigma(a,Y) \setminus \sigma(\dot{a},\dot{X})) \subset Y$. □

Lemma 2.3. *Let $T \in \boldsymbol{B}(X)$ be a spectral operator and let $A \subset \mathbb{C}$ be a Borelian set. Then the restriction $T_1 = T|E(A)X$ is a spectral operator, with the spectral measure E_A given by the relation $E_A(B) = E(A \cap B)$, for any $B \subset \mathbb{C}$ Borelian, where E is the spectral measure of T.*

Proof. One can easily verify that E_A is a spectral measure for T_1; this fact follows by the equality

$$T_1 E_A(B) = T_1 E(A \cap B) = E(A \cap B) T_1 = E_A(B) T_1$$

and from the inclusion

$$\sigma(T_1 | E_A(B)E(A)X) \subset \sigma(T_1 | E(A \cap B)X) \subset \overline{B}.$$

□

Proposition 2.1. *Let $T \in \boldsymbol{B}(X)$ be a spectral (scalar) operator having the spectral measure E and let Y be a linear closed subspace invariant to T. If $\dot{T}$ is the operator induced by T on $\dot{X} = X/Y$, $\varphi : X \to \dot{X}$ is the canonical map, then $\dot{T} = \dot{T}_1 \oplus \dot{T}_2$, where $\dot{T}_1 = \dot{T}|\varphi(E(\sigma')X)$ is spectral (scalar), $\dot{T}_2 = \dot{T}|\varphi(E(\sigma)X)$, $\sigma = \sigma(T|Y)$, $\sigma' = \sigma(\dot{T}) \setminus \sigma(T|Y)$ and $\sigma(\dot{T}_2) \subset S = \sigma(T|Y) \cap \sigma(\dot{T})$.*

Proof. The operator $T|E(\sigma')X$ is spectral (scalar) (Lemma 2.3) and since $Y \subset X_T(\sigma) = E(\sigma)X$, we have $Y \cap E(\sigma')X = \{0\}$. $E(\sigma')X + Y$ being closed (Lemma 2.1), then $E(\sigma')X + Y = E(\sigma')X \oplus Y$, hence $\varphi(E(\sigma')X)$ can be identified with $E(\sigma')X$ and $\dot{T}_1$ with $T|E(\sigma')X$ (Remark 2.2), meaning $\dot{T}_1$ is spectral (scalar). One easily verify that $\varphi(X_T(\sigma)) = \dot{X}_{\dot{T}}(\sigma) = \dot{X}_{\dot{T}}(S)$ is a spectral maximal space of $\dot{T}$ (Theorem 2.13, [5]), consequently

$$\sigma(\dot{T}_2) = \sigma(\dot{T}|\varphi(X_T(\sigma))) = \sigma(\dot{T}|\dot{X}_{\dot{T}}(S)) \subset S.$$

□

Proposition 2.2. *Let $T \in \boldsymbol{B}(X)$ be spectral (scalar) operator and let Y be a closed subspace invariant to T with $X_T(\sigma) \subset Y$, where $\sigma = \sigma(T|Y) \setminus \sigma(\dot{T})$. Let also $S = \sigma(T|Y) \cap \sigma(\dot{T})$ and $T_Y = T|Y$. Then $T_Y|E(\sigma)Y$ and $T_Y|\overline{X_T(\sigma)}$ are spectral (scalar), $T_Y = (T_Y|E(\sigma)Y) \oplus (T_Y|E(S)Y)$ and $\sigma(T_Y|E(S)Y) \subset \widetilde{S} \cap \sigma(T|Y)$.*

Proof. σ being open in $\sigma(T)$, there is a growing sequence of closed sets $(\sigma_n)_{n\in\mathbb{N}}$ with $\sigma = \bigcup_{n\in\mathbb{N}} \sigma_n$; from the continuity of the measure $E(\cdot)x$, it results that $E(\sigma) = \lim_{n\to\infty} E(\sigma_n)$, hence $E(\sigma_n)X = X_T(\sigma_n) \subset X_T(\sigma)$ implies $E(\sigma)X \subset \overline{X_T(\sigma)} \subset Y$. The closed subspaces $E(\sigma)X$ and $\overline{X_T(\sigma)}$ are invariant to T and to spectral measure E, hence $T_Y|E(\sigma)Y$ and $T_Y|\overline{X_T(\sigma)}$ are spectral (scalar) ([9], [10]). From $Y \subset X_T(\sigma(T|Y)) = E(\sigma(T|Y))X$, it follows that $Y = E(\sigma(T|Y))Y = E(\sigma)Y + E(S)Y$, hence Y is invariant to both $E(\sigma)$ and $E(S)$. Consequently, $E(\sigma)|Y$ and $E(S)|Y$ are projectors in Y, $E(\sigma)Y$ and $E(S)Y$ are closed subspaces and $Y = E(\sigma)Y \oplus E(S)Y$. Then $T_Y = (T_Y|E(\sigma)Y) \oplus (T_Y|E(S)Y)$. We also obtain that

$$\sigma(T_Y|E(S)Y) \subset \sigma(\widetilde{T|E(S)}X) \cap \sigma(T|Y) \subset \widetilde{S} \cap \sigma(T|Y),$$

$\widetilde{S} = \mathbb{C} \setminus D^\infty$, where D^∞ is the unbounded component of $\mathbb{C} \setminus S$. □

Theorem 2.1. *Let $T \in \boldsymbol{B}(X)$ be a spectral operator having the spectral measure E, let Y be a closed subspace invariant to T such that $X_T(\sigma) \subset Y$, where $\sigma = \sigma(T|Y) \setminus \sigma(\dot{T})$ and $S = \widetilde{S}$, with $S = \sigma(T|Y) \cap \sigma(\dot{T})$. Then $T|Y$ and $\dot{T}$ are S-spectral operators.*

Proof. These assertions follow by Proposition 2.1, Proposition 2.2 and Remark 2.1. According to Proposition 2.2 it follows that

$$T|Y = T_Y = T_Y|E(\sigma)Y \oplus T_Y|E(S)Y$$

where $T_Y|E(\sigma)Y$ is spectral and $\sigma(T_Y|E(S)Y) \subset \widetilde{S} = S$.

On the other hand, from Proposition 2.1, it result that

$$\dot{T} = \dot{T}|\varphi(E(\sigma)X) \oplus \dot{T}|\varphi(E(\sigma(T|Y))X)$$

where $\dot{T}|\varphi(E(\sigma)X)$ is spectral, $\sigma(\dot{T}|\varphi(E(\sigma(T|Y))X)) \subset S$ and $\varphi : X \to \dot{X}$ is the canonical map.

Consequently, the operators $T|Y$ and $\dot{T}$ are S-spectral (Remark 2.1). $\square$

Corollary 2.1. *Let $T \in \boldsymbol{B}(X)$ be spectral (scalar) operator and let Y be a closed subspace invariant to T such that* $\dim(\sigma(T|Y) \cap \sigma(\dot{T})) = 0$. *Then $T|Y$ and $\dot{T}$ are spectral (scalar) operators.*

Proof. From $\dim(\sigma(T|Y) \cap \sigma(\dot{T})) = 0$, it follows that $S_{\dot{T}} = \emptyset$ (Proposition 2.7, [21]), $X_T(\sigma(T|Y) \setminus \sigma(\dot{T})) \subset Y$ (Lemma 2.2) and according to Proposition 2.2, we have $Y = Y_1 \oplus Y_2$, where $Y_1 = E(\sigma)X = E(\sigma)Y$ and $Y_2 = E(S)Y$ ($\sigma = \sigma(T|Y) \setminus \sigma(\dot{T})$, $S = \sigma(T|Y) \cap \sigma(\dot{T})$, E is the spectral measure of T).

Obviously, Y_1 is invariant to T and to spectral measure E.

But $\sigma(T|Y_2) \subset \sigma(T|E(S)X) \subset S$ (since $\complement S$ is connected and $S = \widetilde{S}$), hence Y_2 is also invariant to E. Then Y is invariant to E and from [9], [10], we have that $T|Y$ and $\dot{T}$ are spectral (scalar) operators. $\square$

3. Generalization of Dowson's result to operator systems

Definition 3.1. ([6]) Let X be a Banach space and let $\mathfrak{B}^n_S$ be the family of all Borelian sets B of $\mathbb{C}^n$ that have the property $B \cap S = \emptyset$ or $S \subset B$, where $S \subset \mathbb{C}^n$ is a compact fixed set.

A map $E_S : \mathfrak{B}^n_S \to \mathcal{P}_X$ is called a $(\mathbb{C}^n, X)$ *type S-spectral measure* if

(1) $E_S(\emptyset) = 0$, $E_S(\mathbb{C}^n) = I$

(2) $E_S(B_1 \cap B_2) = E_S(B_1)E_S(B_2)$, B_1, $B_2 \in \mathfrak{B}^n_S$

(3) $E_S\left(\bigcup_{m=1}^{\infty} B_m\right) x = \sum_{m=1}^{\infty} E_S(B_m)x$, $B_m \in \mathfrak{B}^n_S$, $B_p \cap B_m = \emptyset$ if $p \neq m$, $x \in X$.

A commuting system $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ is called S-*spectral system* if there is a $(\mathbb{C}^n, X)$ type S-spectral measure E_S such that

(4) $a_j E_S(B) = E_S(B) a_j$, $B \in \mathfrak{B}^n_S$, $1 \leq j \leq n$

(5) $\sigma(a, E_S(B)X) \subset \overline{B}$, $B \in \mathfrak{B}^n_S$.

For $S = \emptyset$, we have $\mathfrak{B}^n_\emptyset = \mathfrak{B}(\mathbb{C}^n)$, $\emptyset$-spectral measure is spectral measure and $\emptyset$-spectral system is spectral system ([14]).

Remark 3.1. ([26]) A commuting system $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ is S-spectral if and only if it is a direct sum $a = b \oplus c$, where $b \subset \mathbf{B}(X)$ is spectral system and $\sigma(c, X) \subset S$.

Proof. Indeed, if a is an S-spectral system with its S-spectral measure E_S, then one easily verify that the map $E : \mathfrak{B}(\mathbb{C}^n) \to \mathcal{P}_X$ defined by $E(B) = E_S(B \cap \complement S)$, $B \in \mathfrak{B}(\mathbb{C}^n)$, is a spectral measure for $b = a|E_S(\complement S)X$, while $c = a|E_S(S)X$, $\sigma(c, X) = \sigma(a, E_S(S)X) \subset S$.

Conversely, if $b = (b_1, b_2, \dots, b_n) \subset \mathbf{B}(X_1)$ is spectral and $c = (c_1, c_2, \dots, c_n) \subset \mathbf{B}(X_2)$ is nonspectral, with $\sigma(c, X_2) \not\supset \sigma(b, X_1)$, by putting $S = \sigma(c, X_2)$, $X = X_1 \oplus X_2$ and $a = b \oplus c$, it results that the map $E_S : \mathfrak{B}^n_S \to \mathcal{P}_X$ defined by the equalities $E_S(B) = E(B) \oplus 0$, if $B \cap S = \emptyset$ and $E_S(B) = E(B) \oplus I_2$, if $B \supset S$, $B \in \mathfrak{B}^n_S$, is an S-spectral measure of a, where E is the spectral measure of b and I_2 is the identity operator in X_2. $\square$

Lemma 3.1. ([26]) *If $a = (a_1, a_2, \dots, a_n) \subset \boldsymbol{B}(X)$ is a commuting operator system, Y is a linear closed subspace invariant to a and $\dot{a} = (\dot{a}_1, \dot{a}_2, \dots, \dot{a}_n) \subset \boldsymbol{B}(\dot{X})$ is the system induced by a in $\dot{X} = X/Y$, then, using the notations $\sigma(a, X) = \sigma_1$, $\sigma(a, Y) = \sigma_2$, $\sigma(\dot{a}, \dot{X}) = \sigma_3$, we have:*

1°. $\sigma_1 \subset \sigma_2 \cup \sigma_3$, $\sigma_2 \subset \sigma_1 \cup \sigma_3$, $\sigma_3 \subset \sigma_1 \cup \sigma_2$
2°. $\sigma_1 \setminus \sigma_2 = \sigma_3 \setminus \sigma_2$, $\sigma_1 \setminus \sigma_3 = \sigma_2 \setminus \sigma_3$, $\sigma_3 \setminus \sigma_1 = \sigma_2 \setminus \sigma_1$
3°. $\sigma_1 \cup \sigma_2 = \sigma_2 \cup \sigma_3 = \sigma_3 \cup \sigma_1 = \sigma_1 \cup \sigma_2 \cup \sigma_3$.

Proof. The inclusions 1° follow from [19], Lemma 1.2, and the equalities 2° and 3° are directly consequences of the inclusions 1°. The assertions and the primary verifications for the case of a single operator have been proved in [5] for the first time, independent of [19]. □

Remark 3.2. According to 1°, one may easily notice that if a point $z \in \mathbb{C}^n$ belongs to one of the spectrum then it also belongs to at least one more or to all the three ones, so it can not belong to only one spectrum.
In unidimensional case, $n = 1$, $a = T \in \mathbf{B}(X)$, the result is significant for both operators T and $\dot{T}$ (if it is properly formulated, we believe it is also true for systems $n > 1$; see Corollary 3.1).

Lemma 3.2. *Let $a = (a_1, a_2, \dots, a_n) \subset \boldsymbol{B}(X)$ be a spectral system having the spectral measure E and let $A \subset \mathbb{C}^n$ be a Borelian set. Then the restriction $b = a|E(A)X$ is a spectral system with the spectral measure E_A given by the relation $E_A(B) = E(A \cap B)$, for any $B \subset \mathbb{C}^n$ Borelian.*

Proof. One easily verify that E_A is a spectral measure for b; this fact follows by the equality

$$b_j E_A(B) = b_j E(A \cap B) = E(A \cap B) b_j = E_A(B) b_j, \ 1 \le j \le n$$

(where $b_j = a_j|Y$, $Y = E(A)X$) and from the relations

$$\sigma(b, E_A(B)Y) = \sigma(b, E(A \cap B)X) = \sigma(a, E(A \cap B)X) \subset \overline{B}.$$

□

Proposition 3.1. *Let $a = (a_1, a_2, \dots, a_n) \subset \boldsymbol{B}(X)$ be a spectral system having the spectral measure E, let Y be a linear closed subspace invariant to a, let $\dot{a} = (\dot{a}_1, \dot{a}_2, \dots, \dot{a}_n)$ be the system induced by a on $\dot{X} = X/Y$ and let $\varphi : X \to \dot{X}$ be the canonical map. Then $\dot{a} = \dot{b} \oplus \dot{c}$, where $\dot{b} = \dot{a}|\varphi(E(\sigma')X)$ is spectral system, $\dot{c} = \dot{a}|\varphi(E(\sigma)X)$, $\sigma = \sigma(a, Y)$, $\sigma' = \sigma(\dot{a}, \dot{X}) \setminus \sigma(a, Y)$ and $\sigma(\dot{c}, \varphi(E(\sigma)X)) \subset S = \sigma(a, Y) \cap \sigma(\dot{a}, \dot{X})$.*

Proof. The system $a|E(\sigma')X$ is spectral (Lemma 3.2) and since $Y \subset E(\sigma)X$ $= X_{[a]}(\sigma)$ (Theorem 2.2.1 and Proposition 3.1.3, [14]), we have $Y \cap E(\sigma')X = \{0\}$. Because $E(\sigma')X + Y$ is closed (Lemma 2.1), then $E(\sigma')X + Y = E(\sigma')X \oplus Y$ and so that $\varphi(E(\sigma')X)$ can be identified with $E(\sigma')X$, respectively $\dot{b}$ with $a|E(\sigma')X$ (Remark 2.2), meaning $\dot{b}$ is spectral system. It is easily to verify that $\varphi(X_{[a]}(\sigma)) = \dot{X}_{[\dot{a}]}(\sigma) = \dot{X}_{[\dot{a}]}(S)$ is a spectral maximal space of $\dot{a}$, consequently $\sigma(\dot{c}, \varphi(E(\sigma)X)) = \sigma(\dot{c}, \dot{X}_{[\dot{a}]}(S)) \subset S$. □

Proposition 3.2. *Let $a = (a_1, a_2, \dots, a_n) \subset \boldsymbol{B}(X)$ be a spectral system having the spectral measure E and let Y be a closed subspace invariant to a with $X_a(\sigma) \subset Y$, where $\sigma = \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$. Let also $S = \sigma(a, Y) \cap \sigma(\dot{a}, \dot{X})$ and $b = a|Y$. Then $b|E(\sigma)Y$ and $b|\overline{X_a(\sigma)}$ are spectral systems and $b = (b|E(\sigma)Y) \oplus (b|E(S)Y)$, with $\sigma(b, E(S)Y) \subset S \cap \sigma(b, Y)$.*

Proof. σ being open in $\sigma(b, Y)$ (and also in $\sigma(a, X)$, because $\sigma(a, X) \setminus \sigma(\dot{a}, \dot{X}) = \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$; see Lemma 3.1), there is a growing sequence of closed sets $(\sigma_n)_{n \in \mathbb{N}}$ with $\sigma = \bigcup\limits_{n \in \mathbb{N}} \sigma_n$;

from the continuity of the measure $E(\cdot)x$, it results that $E(\sigma) = \lim_{n\to\infty} E(\sigma_n)$, therefore $E(\sigma_n)X = X_a(\sigma_n) \subset X_a(\sigma)$ (Proposition 3.1.3, [14]) implies $E(\sigma)X \subset \overline{X_a(\sigma)} \subset Y$. The closed subspaces $E(\sigma)X$ and $\overline{X_a(\sigma)}$ are invariant to a and to spectral measure E, so $a|E(\sigma)Y$ and $a|\overline{X_a(\sigma)}$ are spectral. $E(\sigma)|Y$ and $E(S)|Y$ are projectors in Y, $E(\sigma)Y$ and $E(S)Y$ are closed subspaces and $Y = E(\sigma(a,Y))Y = E(\sigma)Y \oplus E(S)Y$. We also obtain that $b = (b|E(\sigma)Y) \oplus (b|E(S)Y)$, $\sigma(b, E(S)Y) \subset \sigma(b, E(S)X) \cap \sigma(a,Y) \subset \widetilde{S} \cap \sigma(a,Y)$ ([25]). □

Theorem 3.1. *Let $a = (a_1, a_2, \ldots, a_n) \subset \boldsymbol{B}(X)$ be a spectral system having the spectral measure E, let Y be a closed subspace un subspace invariant to a such that $X_a(\sigma) \subset Y$, where $\sigma = \sigma(a,Y) \setminus \sigma(\dot{a}, \dot{X})$, $S = \sigma(a,Y) \cap \sigma(\dot{a}, \dot{X})$. Then $a|Y$ şi $\dot{a}$ are S-spectral systems.*

Proof. The assertions follow from Proposition 3.1 and Proposition 3.2, and also from Remark 3.1. □

Corollary 3.1. *Let $a = (a_1, a_2, \ldots, a_n) \subset \boldsymbol{B}(X)$ be a spectral system and let Y be a closed subspace of X invariant to a such that $\dim(\sigma(a,Y) \cap \sigma(\dot{a}, \dot{X})) = 0$. Then the restriction system $a|Y = (a_1|Y, a_2|Y, \ldots, a_n|Y) \subset \boldsymbol{B}(Y)$ and the system $\dot{a} = (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_n)$ induced by a on $\dot{X} = X/Y$ are spectral.*

Proof. The proof of this corollary is similar to the proof from the case of a single operator (see Corollary 2.1). According to Lemma 2.2, Proposition 3.1, Proposition 3.2, Remark 3.1 and using similar arguments to those of Corollary 2.1, it results that $\dot{a}$ is a spectral system, therefore $a|Y$ is a spectral system. □

Proposition 3.3. *Let $T \in \boldsymbol{B}(X)$, let Y be a linear closed subspace invariant to T and let $\dot{T}$ be the quotient operator induced by T on the quotient space $\dot{X} = X/Y$. If D^∞ is the unbounded component of $\rho(T)$ and D_n ($n \in \mathbb{N}$) are the bounded components, then: $D^\infty \cap \sigma(\dot{T}) = \emptyset$ and $D_n \subset \sigma(\dot{T})$ if and only if $D_n \subset \sigma(T|Y)$ (i.e. if and only if there is $\lambda_0 \in D_n$ such that $R(\lambda_0, T)Y \nsubseteq Y$, where $R(\lambda, T)$ is the resolvent of T).*

Proof. I.E. Scroggs proved in [24] that $D^\infty \cap \sigma(T|Y) = \emptyset$ and $D_n \subset \sigma(T|Y)$ if and only if there is $\lambda_0 \in D_n$ such that $R(\lambda_0, T)Y \nsubseteq Y$. According to Lemma 3.1 and Remark 3.2, $D_n \subset \sigma(\dot{T})$ if and only if $D_n \subset \sigma(T|Y)$ and $\lambda \in D^\infty$ implies $\lambda \notin \sigma(\dot{T})$ (if $\lambda \in \sigma(\dot{T})$, $\lambda \notin \sigma(T)$, then $\lambda \in \sigma(T|Y)$, contradiction with $D^\infty \cap \sigma(T|Y) = \emptyset$). □

4. **Conclusions**

In this paper we achieve the extraction of the spectral part from the operator $T|Y$ also known as its "good part". In the last section we extend Downson's result from one operator to a system of operators. The theorems obtained in this paper are verified for spectral operators and spectral systems (and are extended to S-spectral operators and S-spectral systems).

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