GRAPHS OF ORDER n WITH FAULT-TOLERANT PARTITION DIMENSION $n-1$

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This paper gives the characterization of all the connected graphs G of order $n \geq 8$ having fault-tolerant partition dimension $n-1$.

Keywords: resolving partition, fault-tolerant resolving partition, fault-tolerant partition dimension, diameter.

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1. Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G with vertex set $V(G)$ and edge set $E(G)$ is the minimum number of edges in a $u-v$ path. For a vertex v in G, the *eccentricity* $ecc(v)$ is the maximum distance between v and any other vertex of G. The *diameter* of G, denoted by $\mathcal{D}$, is the maximum eccentricity of a vertex v in G. Two vertices u and v in G are called the *diametral vertices* if $d(u, v) = \mathcal{D}$. If two vertices u and v are adjacent (form an edge) in G, then we write as $u \sim v$ and if they are non-adjacent (do not form an edge), then we write as $u \nsim v$. We refer [1] for the general graph theoretic notations and terminology not described in this paper.

Given an ordered set W "related to $\{w_1, w_2, \ldots, w_k\} \subseteq V(G)$". For each $v \in V(G)$, the *representation* of v with respect to W is the k-vector $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, denoted by $r(v|W)$. The set W is called a *resolving set* for G if all the vertices of G have distinct representations with respect to W. The minimum cardinality of a resolving set for G is called the *metric dimension* of G, denoted by $dim(G)$.

The metric dimension was first studied by Slater [2] and independently by Harary and Melter [3]. Slater described the usefulness of this notion when working with U.S. Sonar and Coast Guard Loran (Long range aids to navigation) stations. It was noted in [4] and an explicit construction was given in [5] showing that finding the metric dimension of a graph is NP-hard. For more results about the notion of metric dimension and its applications, we refer to a nice survey by Saenpholphat and Zhang [6] (see also [7, 8, 9, 10, 11, 12]).

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Possibly to gain insight into the metric dimension, Chartrand *et al*. introduced the notion of a resolving partition and partition dimension [13, 14]. To define the partition dimension, the distance $d(v, S)$ between a vertex v of G and $S \subseteq V(G)$ is defined as $\min_{s \in S} d(v, s)$. Let Π be an ordered k-partition "related to $\{S_1, S_2, \ldots, S_k\}$" of $V(G)$ and v be a vertex of G, then the k-vector $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$ is called the *representation* $r(v|\Pi)$ of v with respect to the partitison Π. A partition Π is called a *resolving partition* if for distinct vertices u and v of G, $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of G is the cardinality of a minimum resolving partition, denoted by $pd(G)$.

Based on the Chartrand et al. method of vertex-partitioning, Javaid et al. [15] partitioned the vertex set of a connected graph G into classes in such a way that any two distinct vertices in G have different distances from at least two classes of the partition. They referred this partition as a fault-tolerant resolving partition of $V(G)$, defined as follows: Let Π be an ordered k-partition "related to $\{U_1, U_2, \ldots, U_k\}$" of $V(G)$, then Π is called a *fault-tolerant resolving partition* if for every pair of distinct vertices v, w in G, the representations $r(v|\Pi)$ and $r(w|\Pi)$ differ by at least two coordinates. The cardinality of a minimum fault-tolerant resolving partition is called the *fault-tolerant partition dimension* of G, denoted by $\mathcal{P}(G)$.

We say that a class S *distinguishes* the vertices x and y of G if $d(x, S) \neq d(y, S)$. A partition Π distinguishes x and y if a class of Π distinguishes x and y. From these definitions, it can be observed that the property of a given partition Π of a graph G to be a fault-tolerant resolving partition of G can be verified by investigating that every pair of vertices in the same class is separated by at least two classes of Π. That is, for two classes U_i and U_j $(i \neq j)$ of a partition Π, $d(x, U_i) \neq d(y, U_i)$ and $d(x, U_j) \neq d(y, U_j)$ for all $x, y \in U_k$, $k \neq i, j$.

A useful property for finding the fault-tolerant partition dimension of a connected graph G is Lemma 3.1 placed in Annex-I.

The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and an edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

This paper is aim to characterize all the connected graphs G of order $n \geq 8$ having fault-tolerant partition dimension $n - 1$. In the next section, we list all the connected graphs having fault-tolerant partition dimension one less than the order of the graph and prove that these are the only graphs having this property.

2. Classification of graphs of order n with fault-tolerant partition dimension $n - 1$

The graph $G - e$ is a subgraph of G and can be obtained by deleting an edge e from G. The following is the list of graphs of order n having fault-tolerant partition dimension $n - 1$. It is worth mentioning that, in the list of graphs below, the graphs K with single subscript represent the complete graphs; and the graphs K with two subscripts separated by comma represent the complete bipartite graphs.

$G_1 := K_{1,n-1}$; $G_2 := K_1 + (K_1 \cup K_{n-2})$; $G_3 := K_n - E(P_3)$; $G_4 := K_n - E(P_4)$; $G_5 := K_n - E(K_3)$; $G_6 := K_{1,n-1} + e$; $G_7 := K_n - E(2K_2)$; $G_8 := K_n - E(3K_2)$;
$G_9 :=$ $K_{n-1} - e$ and another vertex adjacent to end vertices of e;
$G_{10} :=$ K_{n-1} and a vertex adjacent to two vertices of K_{n-1};
$G_{11} :=$ $\overline{K_2} + K_{n-3}$ with one edge deleted between $\overline{K_2}$ and K_{n-3} and a vertex adjacent

to the vertices of $\overline{K_2}$;
G_{12}:= The same construction as G_{11} with K_2 instead of $\overline{K_2}$;
G_{13}:= $K_{n-1}-e$ and a vertex adjacent to two vertices of K_{n-1}, one of them being an end vertex of e; and the following four families of graphs:

$\mathcal{G}_1 := \{K_n - E(K_{1,p}+e),\ \text{where}\ 3 \leq p \leq n-2\}$,
$\mathcal{G}_2 := \{K_n - E(K_{1,p}$ and a path P_3 having one adge in common with $K_{1,p})$, where $3 \leq p \leq n-3\}$,
$\mathcal{G}_3 := \{K_{n-1}-e$ and a vertex adjacent to p vertices of K_{n-1}, where $2 \leq p \leq n-3\}$,
$\mathcal{G}_4 := \{K_n - E(K_{1,p}),\ \text{where}\ 2 \leq p \leq n-3\}$.
Figure 1, shown in Annex-II, illustrates one graph of each family mentioned above for $n=8$ and $p=3$.

We also list 7 graphs of order n with the fault-tolerant partition dimension $n-2$ which will appear in the proofs of our lemmas.
H_1:= $K_2+\overline{K_{n-3}}$ and a vertex adjacent to the vertices of $\overline{K_{n-3}}$;
H_2:= K_{n-2} and a path P_4 joining two vertices of K_{n-2};
H_3:= K_{n-2} and a cycle C_3 having a common vertex;
H_4:= K_{n-2} and a path P_3 having in common the central vertex of P_3;
H_5:= K_{n-2} and a path P_3 having an end vertex common with K_{n-2};
H_6:= $K_{1,n-1}$ and a vertex adjacent to a diametral vertex of the star $K_{1,n-1}$;
H_7:= K_{n-2} and a path P_4 having the central edge in common with K_{n-2}.

The relationship between the fault-tolerant partition dimension and the diameter of a connected graph was obtained by Javaid *et al.* in [8] (see Theorem 3.1 in Annex-I). Following is a consequence of Theorem 3.1, cited in Annex-I, will helps in proof of next lemmas.

Corollary 2.1. *Lat G be a connected graph of order n with $\mathcal{P}(G)=n-1$. Then diameter of G is at most three.*

The connected graphs having fault-tolerant partition dimension equal to the order of the graph have been characterized by Javaid *et al* (see Theorem 3.3 in Annex-I). Now, we show that the graphs listed above are the only graphs with fault-tolerant partition dimension $n-1$.

Let u be a diametral vertex in G with eccentricity 2. Denote

$$V_i(u) = \{v : v \in V(G), d(u,v)=i\}\ \text{for}\ i=1,2.$$

Then $u \sim u'$ for each $u' \in V_1(u)$ and for each $w \in V_2(u)$, $w \sim w'$ for at least one $w' \in V_1(u)$. Now, we prove several lemmas which will help to prove our main result Theorem 2.1.

Lemma 2.1. *Let G be a connected graph of order $n \geq 8$ with $\mathcal{P}(G)=n-1$ and diameter $\mathcal{D}=2$. If $min(|V_1(u)|,|V_2(u)|) \geq 3$, then G belongs to $\mathcal{G}_1$, or $\mathcal{G}_3$, or $\mathcal{G}_4$.*

Proof. With out loss of generality, we suppose that $3 \leq r = |V_1(u)| \leq |V_2(u)| = s = n-r-1$. Since $n \geq 8$ and $r \geq 3$, if there are three distinct vertices x,y,z in $V_1(u)$ (or in $V_2(u)$) such that $x \sim y$ and $x \not\sim z, y \not\sim z$ in G, then for two distinct vertices a,b in $V_2(u)$ (or in $V_1(u)$), $(u)(z)(a,x)(b,y)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, where π denotes a partition of $V(G)\setminus\{u,a,b,x,y,z\}$

having all the classes consisting a single vertex (which will be called a singleton sets partition). We deduce that $\mathcal{P}(G) \leq n-2$, a contradiction. It follows that $V_1(u)$ and $V_2(u)$ induces K_r and K_s, or $K_r - e$ and $K_s - e$, or $\overline{K_r}$ and $\overline{K_s}$, respectively. In the case when $V_1(u)$ induces $\overline{K_r}$ and $V_2(u)$ induces $\overline{K_s}$, we can chose distinct vertices $u_1, v_1 \in V_1(u)$ and $u_2, v_2 \in V_2(u)$ such that $(u)(u_1, u_2)(v_1, v_2)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, where π is a singleton sets partition of the remaining vertices, a contradiction. Now, we discuss the following two case: $V_1(u)$ induces K_r and $V_2(u)$ induces K_s or $K_s - e$ (case 1), $V_1(u)$ induces $K_r - e$ and $V_2(u)$ induces K_s or $K_s - e$ (case 2).

Case 1. If there are distinct vertices $x, y \in V_1(u)$ and $a, b \in V_2(u)$ such that $a \not\sim x$ and $b \not\sim y$ in G, then for a vertex $z \in V_1(u) \setminus \{x, y\}$, $(a)(b)(u, z)(x, y)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction. We deduce that $V_1(u) \cup V_2(u)$ induces $K_r + K_s$ (subcase 1.1) or $(K_r + K_s) - e$ (subcase 1.2) or $K_r + (K_s - e)$ (subcase 1.3) or $(K_r + (K_s - e)) - e$ (subcase 1.4).

Subcase 1.1. In this case, $G \in \mathcal{G}_4$.

Subcase 1.2. In this case, $G \in \mathcal{G}_3$.

Subcase 1.3. In this case, $G \in \mathcal{G}_1$.

Subcase 1.4. Let $a \not\sim b$ and $c \not\sim x$ in G for $x \in V_1(u)$ and $a, b, c \in V_2(u)$. Then we can chose a vertex $y \in V_1(u) \setminus \{x\}$ and a vertex $d \in V_2(u) \setminus \{a, b, c\}$ such that $(u)(a)(c)(b, y)(x, d)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, where π is a singleton sets partition of the remaining vertices, a contradiction.

Case 2. By the similar arguments as Case 1, $V_1(u) \cup V_2(u)$ induces $(K_r - e) + K_s$ (subcase 2.1) or $((K_r - e) + K_s) - e$ (subcase 2.2) or $(K_r - e) + (K_s - e)$ (subcase 2.3) or $((K_r - e) + (K_s - e)) - e$ (subcase 2.4).

Subcase 2.1. In this case, $G \in \mathcal{G}_3$.

Subcase 2.2. Let $a \not\sim b$ for $a, b \in V_1(u)$ and $c \not\sim x$ for $c \in V_1(u)$, $x \in V_2(u)$. Then

(i) For $c \neq a, b$, $(b)(u, c)(a, x)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

(ii) For $c = a$ or b, $(x)(u, d)(c, y)\pi$, where $d \in V_1(u) \setminus \{a, b, c\}$ and $y \in V_2(u) \setminus \{x\}$, is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

Subcase 2.3. Let $a \not\sim b$ and $x \not\sim y$ in G for $a, b \in V_1(u)$ and $x, y \in V_2(u)$. Then for $c \in V_1(u) \setminus \{a, b\}$, $(a)(x)(u, c)(b, y)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

Subcase 2.4. Let $a \not\sim b, x \not\sim y$ and $c \not\sim z$ in G for $a, b, c \in V_1(u)$ and $x, y, z \in V_2(u)$. Then

(i) For $c \neq a, b$ and $z \neq x, y$, $(a)(b)(y)(u, c)(z, x)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

(ii) For $c \neq a, b$ and $z = x$ or y, $(a)(b)(u, c)(z, w)\pi$, where $w \in V_2(u) \setminus \{x, y, z\}$, is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

(iii) For $c = a$ or b and $z \neq x, y$, $(y)(u, c)(z, x)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

(*iv*) For $c = a$ or b and $z = x$ or y, $(u, d)(c, z)\pi$, where $d \in V_1(u) \setminus \{a, b, c\}$, is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction.

Lemma 2.2. *Let G be a connected graph of order $n \geq 8$ with $\mathcal{P}(G) = n-1$ and diameter $\mathcal{D} = 2$. If $min(|V_1(u)|, |V_2(u)|) \leq 2$, then G belongs to $\mathcal{G} = \{G_1, G_2, \ldots, G_{13}\}$, or $\mathcal{G}_1$, or $\mathcal{G}_2$, or $\mathcal{G}_3$, or $\mathcal{G}_4$.*

Proof. We shall consider the following cases:

$$\text{Case 1. } |V_1(u)| = 2, |V_2(u)| = n-3,$$
$$\text{Case 2. } |V_1(u)| = n-3, |V_2(u)| = 2,$$
$$\text{Case 3. } |V_1(u)| = 1, |V_2(u)| = n-2,$$
$$\text{Case 4. } |V_1(u)| = n-2, |V_2(u)| = 1.$$

Case 1. Suppose that $V_1(u) = \{v, w\}$. If $V_2(u)$ contains three distinct vertices x, y, z such that $x \not\sim y$ and $x \sim z$ in G, then the pair $\{y, z\}$ distinguished by x. Since $n \geq 8$, we can find another vertex $u' \in V_2(u)$ such that $(u, u')(y, z)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, which contradicts the hypothesis. It follows that $V_2(u)$ induces $\overline{K_{n-3}}$ (subcase 1.1), or K_{n-3} (subcase 1.2).
Subcase 1.1. If one of the vertices of $V_1(u)$, say v, has the property that there exist $x, y \in V_2(u)$ such that $v \not\sim x$ and $v \sim y$ in G, then either $v \sim w$ or $v \not\sim w$ in G, $(x)(u, v)(y, w)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction. One deduce that v and w are adjacent to all the vertices in $V_2(u)$ or one of them is not adjacent to any vertex of $V_2(u)$. But in the last case, we get $\mathcal{D} = 3$ unless $v \sim w$, which contradicts the hypothesis. If $v \sim w$ in G and for example $v \not\sim v'$ for any vertex $v' \in V_2(u)$, then it follows that $w \sim w'$ for all $w' \in V_2(u)$. In this case $G \cong G_6$. If $v \sim z$ and $w \sim z$ in G for all $z \in V_2(u)$, then
(*i*) $G \cong K_{2,n-2}$ if $v \not\sim w$ in G, but $\mathcal{P}(G) = n-2$, by Theorem 3.2, a contradiction.
(*ii*) $G \cong K_2 + \overline{K_{n-2}}$ if $v \sim w$ in G, but $\mathcal{P}(G) \leq n-2$, Since there exists a vertex $x \in V_2(u)$ such that $(u, v)(w, x)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, where π is a singleton partition of $V(G) \setminus \{u, v, w, x\}$, a contradiction.
Subcase 1.2. If one of the vertices of $V_1(u)$, say w, has the property that there exist three vertices $x, y, z \in V_2(u)$ such that $w \not\sim x, w \not\sim y$ and $w \sim z$ in G, then either $v \sim w$ or $v \not\sim w$ in G, $(x)(y)(u, z)(v, w)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction. It follows that if $v \sim c$ or $w \sim c$ for at least one vertex $c \in V_2(u)$, then it is adjacent to at least $n-4$ vertices in $V_2(u)$. If $v \not\sim w$ one obtains that both v and w adjacent to at least $n-4$ vertices in $V_2(u)$ since otherwise $\mathcal{D} = 3$. Consider now the case when both v and w are adjacent to at least $n-4$ vertices in $V_2(u)$. If v and w are adjacent to all $n-3$ vertices of $V_2(u)$, then $G \cong G_9$ if $v \not\sim w$ and $G \cong G_{10}$ if $v \sim w$. If one of v and w is adjacent to $n-4$ vertices in $V_2(u)$ and other one is adjacent to all $n-3$ vertices of $V_2(u)$, then $G \cong G_{11}$ if $v \not\sim w$ in G and $G \cong G_{12}$ if $v \sim w$ in G.

It is not possible that both v and w are adjacent to exactly $n-4$ vertices of $V_2(u)$. Indeed, if there exist distinct vertices $x, y \in V_2(u)$ such that $v \not\sim x, w \not\sim y$

and both v and w are adjacent to $n-4$ vertices of $V_2(u)$, then either $v \sim w$ in G or $v \not\sim w$ in G, there exists $z \in V_2(u) \setminus \{x, y\}$ such that $(x)(y)(u,z)(v,w)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, where π is a singleton sets partition of the remaining vertices, a contradiction. Consider now the case when $v \sim w$ in G and $v \not\sim v'$ for each vertex $v' \in V_2(u)$. If $w \sim w'$ for all $w' \in V_2(u)$, then $G \cong H_3$. In this case, there exist distinct vertices $x, y \in V_2(u)$ such that $(u,y)(v,x)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction. If $w \not\sim t$ for one vertex $t \in V_2(u)$, then $d(u,t) = 3$ which contradicts the equality $\mathcal{D} = 2$.

Case 2. In this case, let $V_2(u) = \{s, t\}$. If $V_1(u)$ contains three distinct vertices v, w, x such that $v \sim w$ and $v \not\sim x$ in G, then we can find another vertex $y \in V_1(u) \setminus \{v, w, x\}$ such that $(v)(u,y)(w,x)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$ which implies that $\mathcal{P}(G) \le n-2$, a contradiction. It follows that $V_1(u)$ induces $\overline{K_{n-3}}$ (subcase 2.1) or K_{n-3} (subcase 2.2).

Subcase 2.1. If $s \not\sim t$ in G, since $\mathcal{D} = 2$ we obtain that $V_1(u) \cup V_2(u)$ induces a subgraph isomorphic to $K_{2,n-3}$. By considering two distinct vertices $x, y \in V_1(u)$, we deduce that $(t)(u,x)(s,y)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, contradiction. It follows that $s \sim t$ in G. Suppose that one of the vertices s, t, say t, has the property that $t \not\sim t_1$ for one vertex $t_1 \in V_1(u)$, but $t \sim t_2$ for at least one vertex t_2 of $V_1(u)$. Then $(t)(s,t_1)(u,t_2)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction. Hence $s \sim v$ and $t \sim v$ in G for all $v \in V_1(u)$ and $G \cong H_1$, but $\mathcal{P}(G) \le n-2$ in this case. Since we can find two distinct vertices $x, y \in V_1(u)$ such that $(u)(s,x)(y,t)\pi$, where π is a singleton sets partition of $V(G) \setminus \{s, t, u, x, y\}$, is a fault-tolerant resolving partition having $n-2$ classes, a contradiction.

Subcase 2.2. Both the vertices s and t must adjacent to at least one vertex of $V_1(u)$, otherwise $\mathcal{D} = 3$. If $s \sim y$ and $t \sim y$ in G for all $y \in V_1(u)$, then $G \cong G_3$ if $s \sim t$ in G and $G \cong G_5$ if $s \not\sim t$ in G.

Consider now the case when, for all $v \in V_1(u)$, $v \sim t$ in G. When $s \not\sim t$ and $s \not\sim v_1, s \not\sim v_2, \ldots, s \not\sim v_i$ in G for $v_1, v_2, \ldots, v_i \in V_1(u)$, where $i \in \{1, 2, \ldots, n-4\}$, then $G \in \mathcal{G}_1$. When $s \sim t$; if $s \not\sim s'$ in G for a single vertex $s' \in V_1(u)$, then $G \cong G_4$, if $s \not\sim v_1, s \not\sim v_2, \ldots s \not\sim v_i$ in G for $v_1, v_2, \ldots, v_i \in V_1(u)$, where $i \in \{2, 3, \ldots, n-4\}$, then $G \in \mathcal{G}_2$. For $i = n-4$, $G \cong G_{13}$. A similar situation occurs when $v \sim s$ for all $v \in V_1(u)$. The remaining subcase is that when both s and t are not adjacent to at least one vertex of $V_1(u)$.

If there exist four distinct vertices $a, b, c, d \in V_1(u)$ such that $a \not\sim s, d \not\sim t$ and $c \sim s, b \sim t$ in G, let $v \in V_1(u) \setminus \{a, b, c, d\}$. If $v \sim s$ or $v \sim t$ in G, then either $s \not\sim t$ or $s \sim t$, $(u)(s,c)(t,b)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, which implies that $c \sim s, b \sim t$ are only edges joining s and t to vertices in $V_1(u)$. In this case, $G \cong H_2$ if $s \sim t$, but $\mathcal{P}(G) \le n-2$ since $(u)(c,s)(b,t)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction. If $s \not\sim t$ in G, then $d(s,t) = 3$, a contradiction.

If a and d coincide or b and c coincide, then it remains to consider only two subcases: $s \not\sim x$ and $t \not\sim x$ in G for a single vertex $x \in V_1(u)$ (subcase 2.2.1), $s \sim x'$ and $t \sim x'$ for a single vertex $x' \in V_1(u)$ (subcase 2.2.2).

Subcase 2.2.1. Either $s \not\sim t$ or $s \sim t$ (in this case $G \cong K_n - E(C_4)$), there are two distinct vertices $v, w \in V_1(u)$ such that $(u)(x)(s,v)(t,w)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, a contradiction.

Subcase 2.2.2. If $s \sim t \in E(G)$, then $G \cong H_3$ and if $s \not\sim t$, then $G \cong H_4$. In both the cases, $(u,t)(s,x)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, a contradiction.

Case 3. Let $V_1(u) = \{x\}$, it follows that $x \sim x'$ for all $x' \in V_2(u)$. If $V_2(u)$ induces $\overline{K_{n-2}}$ or K_{n-2}, then G is isomorphic to G_1 or G_2, respectively. Otherwise, there exists a diametral vertex $y \in V_2(u)$ such that $2 \leq |V_1(y)| \leq n-3$, hence $|V_2(y)| \in \{2, n-3\}$ and we are again in the Case 1, or in the Case 2, relatively to y.

Case 4. Let $V_2(u) = \{v\}$. If degree of v is one, then v is a diametral vertex and $|V_1(v)| = 1$, hence Case 3 occurs again. Otherwise, let degree of v, $d(v)$, is grater than or equal to two. If there exist six distinct vertices $a,b,c,d,e,f \in V_1(u)$ (since $n \geq 8$) such that $a \not\sim d, b \not\sim e, c \not\sim f$ and $a \sim b, a \sim c, a \sim e, a \sim f, b \sim c, b \sim d, b \sim f, c \sim d, c \sim e, d \sim e, d \sim f, e \sim f$ in G, then $(u)(a,b)(c,v)\pi$ is a faut-tolerant resolving partition of $V(G)$ having $n-2$ classes, where π is a singleton sets partition of the remaining vertices, a contradiction. It follows that $V_1(u)$ induces $K_{n-2} - e$ (subcase 4.1), or $K_{n-2} - E(2P_2)$ (subcase 4.2), or $\overline{K_{n-2}}$ (subcase 4.3), or K_{n-2} (subcase 4.4).

Subcase 4.1. Since $d(v) \geq 2$, if $v \sim v_1, v \sim v_2, \ldots, v \sim v_i$ in G for $v_1, v_2, \ldots, v_i \in V_1(u)$, where $i \in \{2,3,\ldots,n-3\}$, then $G \in \mathcal{G}_3$. For $i=2$, if $v_1, v_2 \in V_1(u)$ are end vertices of e then $G \cong G_9$ and if v_1 is end vertex of e and v_2 is not, then $G \cong G_{13}$. If $v \sim v'$ for all $v' \in V_1(u)$, then $G \cong G_7$.

Subcase 4.2. Let $a,b,c,d \in V_1(u)$ such that $a \not\sim c$ and $b \not\sim d$ in G. If there exist two distinct vertices $x, y \in V_1(u) \setminus \{a,b,c,d\}$ such that $v \sim a, v \sim y$ but $v \not\sim x$ in G, then $(u)(v)(c)(d)(a,x)(b,y)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction. We deduce that $v \sim v'$ for all $v' \in V_1(u)$. In this case $G \cong G_8$.

Subcase 4.3. Since $\mathcal{D} = 2$ it follows that $v \sim v'$ for all $v' \in V_1(u)$ and $G \cong K_{2,n-2}$, but $\mathcal{P}(G) \leq n-2$ since for every two distinct vertices s,t of $V_1(u)$, $(t,u)(s,v)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction.

Subcase 4.4. If $v \not\sim v_1, v \not\sim v_2, \ldots, v \not\sim v_i$ in G for $v_1, v_2, \ldots, v_i \in V_1(u)$, where $i \in \{1,2,\ldots,n-4\}$, then $G \in \mathcal{G}_4$. The case for $i = n-3$ is not occur since $d(v) \geq 2$. If $v \sim v'$ for all $v' \in V_1(u)$, then $G \cong K_n - e$, but $\mathcal{P}(K_n - e) = n$, by Theorem 3.3.

Lemma 2.3. *Let G be a connected graph of order $n \geq 8$ with $\mathcal{P}(G) = n-1$ and diameter $\mathcal{D} = 3$. Then $G \cong G_2$.*

Proof. Let s be a diametral vertex having $ecc(s) = 3$. Denote

$$V_i(s) = \{s' : s' \in V(G), d(s', s) = i\} \text{ for } i = 1,2,3.$$

Let $t \in V_1(s)$, $u \in V_2(s)$ and $v \in V_3(s)$. If there are $w, x \in V(G) \setminus \{s,t,u,v\}$ belonging to different sets from $V_1(s), V_2(s), V_3(s)$, then $(s)(x)(t,w)(u,v)\pi$ is a fault-tolerant partition of $V(G)$ having $n-2$ classes, a contradiction. It follows that we can consider only three cases.

$$\text{Case 1. } |V_1(s)| = |V_2(s)| = 1, |V_3(s)| = n-3,$$
$$\text{Case 2. } |V_1(s)| = |V_3(s)| = 1, |V_2(s)| = n-3,$$
$$\text{Case 3. } |V_2(s)| = |V_3(s)| = 1, |V_1(s)| = n-3.$$

Case 1. Suppose that $V_1(s) = \{t\}$, $V_2(s) = \{u\}$, then $u \sim u'$ for all $u' \in V_3(s)$ otherwise, there exists a vertex $v \in V_3(s)$ such that $d(s,v) = 4$, contradiction. If there exist three distinct vertices $x, y, z \in V_3(s)$ such that $x \not\sim y$ and $x \sim z$ in

G, then $(z)(s)(t,x)(u,y)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction. Hence $V_3(s)$ induces K_{n-3} or $\overline{K_{n-3}}$. In the first case $G \cong H_5$ but $\mathcal{P}(G) \leq n-2$ since there is a vertex $v \in V_3(s)$ such that $(s,v)(t,u)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction. In the second case $G \cong H_6$ and we get a contradiction by the same argument as previous case.
Case 2. Let $V_1(s) = \{u\}$ and $V_3(s) = \{v\}$ then $u \sim u'$ for all $u' \in V_2(s)$. As above, $V_2(s)$ induces K_{n-3} (subcase 2.1) or $\overline{K_{n-3}}$ (subcase 2.2). $v \sim w$ for at least one vertex $w \in V_2(s)$. If there exist two vertices $x, y \in V_2(s) \setminus \{w\}$ such that $v \not\sim x$ and $v \sim y$ in G, then $(x)(w)(u,s)(v,y)\pi$ ia s fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, a contradiction. It follows that $v \sim v'$ for all $v' \in V_2(s)$ or $v \not\sim v'$ for any vertex in $v' \in V_2(s) \setminus \{w\}$.
Subcase 2.1. If $v \sim v'$ for all $v' \in V_2(s)$, then $G \cong G_2$. Otherwise, $G \cong H_7$ but $\mathcal{P}(G) \leq n-2$ since for every $t \in V_2(s) \setminus \{w\}$, $(w)(s,t)(u,v)\pi$ is a fault-tolerant resolving $(n-2)$-partition of $V(G)$, a contradiction.
Subcase 2.2. If $v \sim v'$ for all $v' \in V_2(s)$, then $G \cong K_{2,n-2} - e$. Otherwise, $G \cong H_6$. In both the cases, $\mathcal{P}(G) \leq n-2$ since for every $t \in V_2(s) \setminus \{w\}$, $(w)(u,s)(v,t)\pi$ is a fault-tolerant resolving partition of $V(G)$ having $n-2$ classes, a contradiction.
Case 3. Suppose that $V_2(s) = \{x\}$ and $V_3(s) = \{y\}$. In this case y is a diametral vertex and $|V_1(y)| = 1$. Hence y instead of s we have Cases 1 and 2, which completes the proof.

Now, our main result is the following:

Theorem 2.1. *Let G be a connected graph of order $n \geq 8$. Then $\mathcal{P}(G) = n-1$ if and only if G belongs to $\mathcal{G} = \{G_1, G_2, \ldots, G_{13}\}$, or $\mathcal{G}_1$, or $\mathcal{G}_2$, or $\mathcal{G}_3$, or $\mathcal{G}_4$.*

Proof. By using Lemma 3.1, it is a routine exercise to verify that all the graphs enumerated in the statement have the fault-tolerant partition dimension $n-1$.

Conversely, let G be a connected graph of order $n \geq 8$ having $\mathcal{P}(G) = n-1$. Then $\mathcal{D} \leq 3$, by Corollary 2.1. When $\mathcal{D} = 1$, then G is isomorphic to the complete graph K_n and $\mathcal{P}(G) = n$, by Theorem 3.3. When $\mathcal{D} = 2, 3$, then Lemmas 2.1, 2.2 and 2.3 conclude the proof.

3. **Conclusion**

In this paper, we considered the generalization of the fault-tolerant metric dimension “the fault-tolerant partition dimension”. Inspired by the works, done by Chartrand *et al.* in [13, 14] and by Javaid *et al.* in [8], on the characterization of all the connected graphs of order n having partition dimension (the generalization of the metric dimension) $n, n-1, n-2$, and the fault-tolerant partition dimension n, we classified all the connected graphs with fault-tolerant partition dimension one less than the order of the graphs.

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Annex-I

Lemma 3.1. [15] *Let Π be a fault-tolerant resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all $w \in V(G) \setminus \{u, v\}$, then u and v belong to distinct classes of Π.*

Theorem 3.1. [8] *Let G be a connected graph of order $n \geq 3$ and diameter $\mathcal{D}$. Then*

$$\eta(n, \mathcal{D}) \leq \mathcal{P}(G) \leq n - \mathcal{D} + 2,$$

where $\eta(n, \mathcal{D})$ is the least positive integer ν for which $n \leq (\mathcal{D}+1)^{\nu}$.

Theorem 3.2. [8] *If $K_{m,n}$ be the complete bipartite graph for $m, n \geq 1$, then*

$$\mathcal{P}(K_{m,n}) = \begin{cases} m+1 & if \quad m-n=0, \\ max(m,n)+1 & if \quad |m-n|=1, \\ max(m,n) & if \quad |m-n| \geq 2. \end{cases}$$

Theorem 3.3. [8] *Let G be a connected graph of order n. Then $\mathcal{P}(G) = n$ if and only if G is one of the graphs K_n and $K_n - e$.*

Annex-II

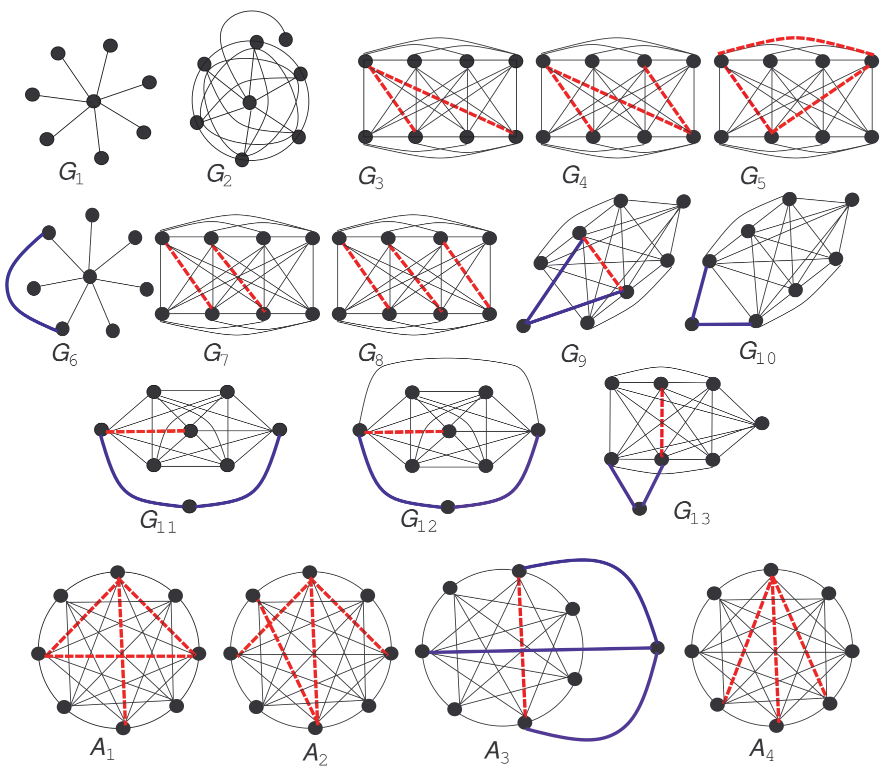

FIGURE 1. Illustration of the graphs for $n = 8$. $A_i \in \mathcal{G}_i$ for $i = 1, 2, 3, 4$ and $p = 3, n = 8$. Deleted edges colored by red (dotted edges) and new edges colored by blue (thick edges).

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