THE EXTRAGRADIENT METHOD WITH A TWO-STEP INERTIAL TECHNIQUE FOR QUASIMONOTONE VARIATIONAL INEQUALITIES

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In this paper, based on extragradient method and the two-step inertial technique, we introduce a new iterative scheme for finding an element of the set of solutions of a quasimonotone, Lipschitz continuous variational inequality problem in real Hilbert spaces. Under suitable conditions, we present a weak convergence theorem of the sequence generated by the proposed algorithm.

Keywords: Extragradient method; two-step inertial; variational inequality; quasimonotone mapping; Lipschitz continuity.

MSC2020: 7H09, 47H10, 47J20, 47J25.

1. Introduction

The paper deals with two new numerical approaches for finding a solution of the variational inequality problem (VIP) [13, 14] in a real Hilbert space H.

Let C be a nonempty, closed, and convex subset in H and $F\colon H \to H$ be an operator. Recall that VIP for the operator F on C is stated as follows:

$$\text{Find } x^* \in C \text{ such that } \langle Fx^*, y - x^*\rangle \geq 0 \text{ for all } y \in C. \tag{1}$$

The solution set is denoted by S.

The dual variational inequality problem of (1) is to find a point $x^* \in C$ such that

$$\langle Fy, y - x^*\rangle \geq 0 \ \ \forall y \in C. \tag{2}$$

We denote the solution set of the dual variational inequality problem (2) by S_D. It is obvious that S_D is a closed, convex set (possibly empty). In the case when F is continuous and C is convex, we get

$$S_D \subset S.$$

If F is a pseudomonotone and continuous mapping, then $S = S_D$ (see, Lemma 2.1 in [10]). The inclusion $S \subset S_D$ is false, if F is a quasimonotone and continuous mapping (see Example 4.2 in [47]).

Variational inequality theory is an important tool in economics, engineering mechanics, mathematical programming, transportation and others (see, [1, 4, 15, 23, 25]). One

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of the most interesting and important problems in the VIP theory is the study of efficient iterative algorithms for finding approximate solutions and the convergence analysis of algorithms. Several methods have been proposed to solve VIPs in finite and infinite dimensional spaces, see e.g. [7, 8, 9, 28, 45]. and the references therein. Among these iterative methods, the simplest one for VIP (1) is the following gradient projection method:

$$\begin{cases} v_0 \in C, \\ v_{n+1} = P_C(v_n - \tau F v_n), \end{cases}$$

where P_C denotes the metric projection of H onto the set C and τ is a positive real number. The main restriction of gradient projection methods is that the operators require to be Lipschitz continuous and strongly monotone (or inverse strongly monotone). The extragradient method which was introduced by Korpelevich [26] and Antipin [3] overcomes this disadvantage by performing an additional projection at each iteration in the following way:

$$\begin{cases} v_0 \in C, \\ u_n = P_C(v_n - \tau F v_n), \\ v_{n+1} = P_C(v_n - \tau F u_n), \end{cases} \tag{3}$$

where $F\colon C \to C$ is monotone and L-Lipschitz continuous, $\tau \in \left(0, \frac{1}{L}\right)$. Recently, the extragradient method has given conclusive results assuming monotone and the Lipschitz continuous mappings (see, e.g., [11, 27, 34, 39, 42]). It is well known that to implement the extragradient method, one needs to calculate two projections onto C in each iteration. Thus, if C is a general closed and convex set, then the computation of projections is rather expensive. Recently, we have some methods were introduced so that they can overcome this drawback as follows the subgradient extragradient method [7], Tseng's method [40], the projection and contraction method [18]. However, these methods require a mapping $F\colon H \to H$ instead of $F\colon C \to C$ in the extragradient method. Therefore, in the case the computation of projection onto feasible C is easy to calculate, when we can use the extragradient method instead of some recent methods. This makes us interested in the extragradient method in this work.

One of the new directions in this field is to combine well-known algorithms with the inertial technique for solving VIPs; the purpose of them is to improve the speed of convergence rates (see, e.g., [2, 6, 12, 16, 22, 24, 35, 36, 37, 38] and the references therein).

Let us now discuss an inertial type algorithm. We know the problem of finding a zero of a maximal monotone operator A on a real Hilbert space H can be expressed as follows:

$$\text{find } x \in H \text{ such that } 0 \in A(x). \tag{4}$$

One fundamental approach to solving this is the proximal method, which generates the next iteration x_{n+1} by solving the subproblem:

$$0 \in \lambda_n A(x) + (x - x_n),$$

where x_n is the current iteration and λ_n is a regularization parameter (see [5, 33]).

In 2001, Attouch and Alvarez [2] applied an inertial technique to the algorithm above to construct an inertial proximal method for solving the original problem (4). It works as follows: given $x_{n-1}, x_n \in H$ and two parameters $\theta_n \in [0, 1), \lambda_n > 0$, find $x_{n+1} \in H$ such that:

$$0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}),$$

which can be rewritten as:

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})),$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n. The inertia is induced by the term $\theta_n(x_n - x_{n-1})$ and can be viewed as a means to accelerate convergence (see e.g., [2, 31]).

In recent years, the class of quasimonotone (or non-monotone) mappings has been studied as a weaker alternative to the pseudomonotonicity assumption for solving the VIP [19, 20, 43, 44, 46, 47]. However, these methods primarily employ the one-step inertial technique [2, 31] in conjunction with extragradient method (3). Recently, in [32], the two-step inertial technique was investigated, demonstrating that this approach may enhance convergence more rapidly than the one-step inertial technique. In this paper, motivated and inspired by the above problems, we introduce a modified extragradient method by applying the technique of two-step inertial for solving a quasimonotone variational inequality in real Hilbert spaces.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with analyzing the convergence of the proposed algorithm. Finally, in Sect. 4, conclusion is provided.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed, convex subset of H. The weak convergence of $\{p_n\}_{n=1}^{\infty}$ to x is denoted by $p_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{p_n\}_{n=1}^{\infty}$ to x is written as $p_n \to x$ as $n \to \infty$. For each u, v, $w \in H$, and θ, $\beta \in \mathbb{R}$ we have

$$\|u+v\|^2 \le \|u\|^2 + 2\langle v, u+v\rangle,$$

and

$$\begin{aligned}\|(1+\theta)u - (\theta-\beta)v - \beta w\|^2 =&(1+\theta)\|u\|^2 - (\theta-\beta)\|v\|^2 - \beta\|w\|^2 \\ &+ (1+\theta)(\theta-\beta)\|u-v\|^2 \\ &+ \beta(1+\theta)\|u-w\|^2 - \beta(\theta-\beta)\|v-w\|^2. \end{aligned} \tag{5}$$

For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \text{ for all } y \in C,$$

where P_C is the *metric projection* of H onto C. We know that P_C is nonexpansive.

Lemma 2.1 ([17])**.** *Let C be a closed convex subset in a real Hilbert space H and $x \in H$. Then we have the following inequalities:*

(*i*) *Given $z \in C$, we have $z = P_C x \iff \langle x - z, z - y\rangle \ge 0$, for all $y \in C$;*
(*ii*) *$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$ for all $y \in H$;*
(*iii*) *$\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2$, for all $y \in C$.*

Definition 2.1 ([41])**.** *An operator $F\colon H \to H$ is said to be:*

(*i*) *L-Lipschitz continuous with $L > 0$ if*

$$\|Fx - Fy\| \le L\|x - y\|, \text{ for all } x, y \in H.$$

In particular, when $L = 1$ then the operator F is called nonexpansive.

(*ii*) *monotone if*

$$\langle Fx - Fy, x - y\rangle \ge 0, \text{ for all } x, y \in H.$$

(*iii*) *pseudo-monotone in the sense of Karamardian* [21] *if*

$$\langle Fx, y - x\rangle \ge 0 \Longrightarrow \langle Fy, y - x\rangle \ge 0, \text{ for all } x, y \in H.$$

(*iv*) *quasimonotone, if*

$$\langle Fx, y - x\rangle > 0 \Longrightarrow \langle Fy, y - x\rangle \ge 0, \text{ for all } x, y \in H.$$

(v) δ-strongly pseudo-monotone if there exists a constant $\delta > 0$ such that

$$\langle Fx, x-y\rangle \geq 0 \Longrightarrow \langle Fy, y-x\rangle \geq \delta\|x-y\|^2, \text{ for all } x, y \in H.$$

(vi) sequentially weakly continuous if, for each sequence $\{p_n\}$ in H, the fact that $\{p_n\}$ converges weakly to a point $x \in H$ implies that $\{Fp_n\}$ converges weakly to Fx.

It is easy to see that every monotone operator is pseudo-monotone but the converse is not true. The following lemma provides some sufficient conditions for nonemptiness of S_D.

Lemma 2.2 ([47])**.** *Suppose that at least one of the following conditions holds true:*

(1) *F is pseudomonotone on C and $S \neq \emptyset$,*
(2) *F is the gradient of G, where G is a differential quasiconvex function on an open set $K, C \subset K$ and attains its global minimum on C,*
(3) *F is quasi-monotone on C, $F \neq 0$ on C and C is bounded,*
(4) *F is quasi-monotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $v \in C$ with $\|v\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle Fv, y-v\rangle \leq 0$,*
(5) *F is quasimonotone on C and $S_N \neq \emptyset$,*
(6) *F is quasi-monotone on C, $intC$ is nonempty and there exists $v^* \in S$ such that $Fv^* \neq 0$.*

Then, S_D is nonempty.

Lemma 2.3. *[*[29]*] Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(a) *for each $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(b) *every sequential weak cluster point of $\{x_n\}$ belongs to C.*

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.4 ([30])**.** *Let $\{\lambda\}, \{p_n\}$ and $\{q_n\}$ be three sequences of nonnegative numbers satisfying*

$$\lambda_{n+1} \leq (1+q_n)\lambda_n + p_n, \text{for all } n \geq 1,$$

where $\sum_{n=1}^{\infty} p_n < +\infty$ and $\sum_{n=1}^{\infty} q_n < +\infty$. Then $\lim_{n\to+\infty} \lambda_n$ exists.

3. The Main Results

In this paper, we introduce a new modified extragradient method for solving the quasimonotone VIP in real Hilbert spaces. In order to prove the convergence of the proposed algorithm, we assume the following conditions:

Condition 3.1. *$S_D \neq \emptyset$.*

Condition 3.2. *The mapping $F \colon C \to C$ is L-Lipschitz continuous on H. However, the size of L is not necessary to be known.*

Condition 3.3. *The mapping F is sequentially weakly continuous on C, i.e., for each sequence $\{x_n\} \subset C : \{x_n\}$ converges weakly to x^* implies $\{Fx_n\}$ converges weakly to Fx^*.*

Condition 3.4. *The mapping F is quasimonotone on H.*

Now, we introduce our algorithm:

Algorithm 3.1. *Given $\tau_1 > 0$, $\theta \in [0,1]$, $\beta \in [-1,0]$, $\mu \in \left(0, \frac{1}{3}\right)$. Let $y_{-1}, y_0, y_1 \in H$ be arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ be two nonnegative real numbers sequences such that $\sum_{n=1}^{\infty} \alpha_n < +\infty$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$.*

Iterative Steps: *Given the current iterate y_n, calculate y_{n+1} as follows:*

Step 1. *Compute*

$$\begin{cases} t_n = y_n + \theta(y_n - y_{n-1}) + \beta(y_{n-1} - y_{n-2}), \\ u_n = P_C(t_n - \tau_n F t_n), \end{cases}$$

If $t_n = u_n$ *or* $Ft_n = 0$ *then stop and* t_n *is a solution of* (1). *Otherwise*

Step 2. *Compute*

$$y_{n+1} = P_C(t_n - \tau_n F u_n),$$

update

$$\tau_{n+1} = \begin{cases} \min\left\{\mu \dfrac{\|t_n - u_n\|}{\|Ft_n - Fu_n\|}, (1+\alpha_n)\tau_n + \beta_n\right\} & \text{if } Ft_n \neq Fu_n, \\ (1+\alpha_n)\tau_n + \beta_n & \text{otherwise.} \end{cases} \tag{6}$$

Set $n := n+1$ *and go to* ***Step 1***.

The following lemmas will guide the convergence analysis.

Lemma 3.1. *Let* $\{\tau_n\}$ *be a sequence generated by* (6). *Then*

$$\lim_{n\to\infty} \tau_n = \tau \text{ and } \tau \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}.$$

Moreover, we also obtain

$$\|Ft_n - Fu_n\| \leq \frac{\mu}{\tau_{n+1}} \|t_n - u_n\|.$$

Proof. By the definition of $\{\tau_n\}$ we get $\tau_n \leq (1+\alpha_n)\tau_n + \beta_n$, for all n. Using Lemma **2**.4 then $\lim_{n\to\infty} \tau_n$ exists. Assume $\lim_{n\to\infty} \tau_n = \tau$. Using the definition of $\{\tau_n\}$ again, it is easy to see that $\tau_n \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}$. Thus $\tau \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}$. Moreover, it is obvious that

$$\|Ft_n - Fu_n\| \leq \frac{\mu}{\tau_{n+1}} \|t_n - u_n\|,$$

in both of these cases, $Ft_n \neq Fu_n$ or $Ft_n = Fu_n$. The proof is completed. □

Lemma 3.2. *Assume that Conditions* **3**.*1*–**3**.*4 hold. Let* $\{t_n\}$ *be a sequence generated by Algorithm* **3**.*1. If there exists a subsequence* $\{t_{n_k}\}$ *convergent weakly to* $z \in H$ *and* $\lim_{k\to\infty} \|t_{n_k} - u_{n_k}\| = 0$*, then* $z \in S_D$ *or* $Fz = 0$.

Proof. First, we see that $\{t_{n_k}\} \rightharpoonup z$ and $\lim_{k\to\infty} \|t_{n_k} - u_{n_k}\| = 0$ imply that $u_{n_k} \rightharpoonup z$ and since $u_n \in C$ we get $z \in C$.

Now, we divide the proof into two cases.

Case 1: If $\limsup_{k\to\infty} \|Fu_{n_k}\| = 0$, then we have $\lim_{k\to\infty} \|Fu_{n_k}\| = \liminf_{k\to\infty} \|Fu_{n_k}\| = 0$. Since u_{n_k} converges weakly to $z \in C$ and F satisfies Condition **3**.3 we get

$$0 \leq \|Fz\| \leq \liminf_{k\to\infty} \|Fu_{n_k}\| = 0.$$

This implies that $Fz = 0$.

Case 2: If $\limsup_{k\to\infty} \|Fu_{n_k}\| > 0$. Without loss of generality, we take $\lim_{k\to\infty} \|Fu_{n_k}\| = M > 0$. It then follows that there exists $K \in \mathbb{N}$ such that $\|Fu_{n_k}\| > \frac{M}{2}$ for all $k \geq K$. Since $u_{n_k} = P_C(t_{n_k} - \tau_{n_k} F t_{n_k})$, we have

$$\langle t_{n_k} - \tau_{n_k} F t_{n_k} - u_{n_k}, x - u_{n_k} \rangle \leq 0, \text{for all } x \in C,$$

or, equivalently,

$$\frac{1}{\tau_{n_k}} \langle t_{n_k} - u_{n_k}, x - u_{n_k} \rangle \leq \langle F t_{n_k}, x - u_{n_k} \rangle, \text{ for all } x \in C.$$

Consequently, we have

$$\frac{1}{\tau_{n_k}}\langle t_{n_k}-u_{n_k}, x-u_{n_k}\rangle+\langle Ft_{n_k}, u_{n_k}-t_{n_k}\rangle \leq \langle Ft_{n_k}, x-t_{n_k}\rangle, \text{ for all } x\in C. \tag{7}$$

Since $\{t_{n_k}\}$ is weakly convergent, $\{t_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{Ft_{n_k}\}$ is bounded. As $\|t_{n_k}-u_{n_k}\|\to 0$, $\{u_{n_k}\}$ is also bounded and $\tau_{n_k}\geq \min\left\{\tau_1, \frac{\mu}{L}\right\}$. Passing (7) to the limit as $k\to\infty$, we get

$$\liminf_{k\to\infty}\langle Ft_{n_k}, x-t_{n_k}\rangle \geq 0 \text{ for all } x\in C. \tag{8}$$

Moreover, we have

$$\begin{aligned}\langle Fu_{n_k}, x-u_{n_k}\rangle &= \langle Fu_{n_k}-Ft_{n_k}, x-t_{n_k}\rangle+\langle Ft_{n_k}, x-t_{n_k}\rangle\\ &\quad+\langle Fu_{n_k}, t_{n_k}-u_{n_k}\rangle.\end{aligned} \tag{9}$$

Since $\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\|=0$ and F is L-Lipschitz continuous on H, we get

$$\lim_{k\to\infty}\|Ft_{n_k}-Fu_{n_k}\|=0$$

which, together with (8) and (9), implies that

$$\liminf_{k\to\infty}\langle Fu_{n_k}, x-u_{n_k}\rangle \geq 0. \tag{10}$$

If $\limsup_{k\to\infty}\langle Fu_{n_k}, x-u_{n_k}\rangle>0$, then there exists a subsequence $\{u_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle Fu_{n_{k_j}}, x-u_{n_{k_j}}\rangle>0$. Consequently, there exists $j_0\in\mathbb{N}$ such that

$$\langle Fu_{n_{k_j}}, x-u_{n_{k_j}}\rangle>0, \text{ for all } j\geq j_0.$$

Using the quasimonotonicity of F, one gets $\langle Fx, x-u_{n_{k_j}}\rangle\geq 0$, hence, tending $j\to\infty$, we conclude $z\in S_D$.

If $\limsup_{k\to\infty}\langle Fu_{n_k}, x-u_{n_k}\rangle=0$, inequality (10) implies that

$$\lim_{k\to\infty}\langle Fu_{n_k}, x-u_{n_k}\rangle=0.$$

Let $\epsilon_k := |\langle Fu_{n_k}, x-u_{n_k}\rangle|+\dfrac{1}{k+1}$. Then we obtain

$$\langle Fu_{n_k}, x-u_{n_k}\rangle+\epsilon_k>0, \text{ for all } k\geq 1. \tag{11}$$

Furthermore, for each $k\geq 1$, since $\{u_{n_k}\}\subset C$, we can suppose $Fu_{n_k}\neq 0$ (otherwise, u_{n_k} is a solution) and, setting

$$q_{n_k}=\frac{Fu_{n_k}}{\|Fu_{n_k}\|^2},$$

we have $\langle Fu_{n_k}, q_{n_k}\rangle=1$ for each $k\geq 1$. Now, we can deduce from (11) that, for each $k\geq 1$,

$$\langle Fu_{n_k}, x+\epsilon_k q_{n_k}-u_{n_k}\rangle>0.$$

Since F is quasimonotone on H, we get

$$\langle F(x+\epsilon_k q_{n_k}), x+\epsilon_k q_{n_k}-u_{n_k}\rangle\geq 0. \tag{12}$$

Now, for all $k \geq 1$, using (12) we get

$$\begin{aligned}\langle Fx, x+\epsilon_k q_{n_k} - u_{n_k}\rangle &= \langle Fx - F(x+\epsilon_k q_{n_k}), x+\epsilon_k q_{n_k} - u_{n_k}\rangle + \langle F(x+\epsilon_k q_{n_k}), x+\epsilon_k q_{n_k} - u_{n_k}\rangle \\ &\geq \langle Fx - F(x+\epsilon_k q_{n_k}), x+\epsilon_k q_{n_k} - u_{n_k}\rangle \\ &\geq -\|Fx - F(x+\epsilon_k q_{n_k})\|\|x+\epsilon_k q_{n_k} - u_{n_k}\| \\ &\geq -\epsilon_k L\|q_{n_k}\|\|x+\epsilon_k q_{n_k} - u_{n_k}\| \\ &= -\epsilon_k L\frac{1}{\|Fu_{n_k}\|}\|x+\epsilon_k q_{n_k} - u_{n_k}\| \\ &\geq -\epsilon_k L\frac{2}{M}\|x+\epsilon_k q_{n_k} - u_{n_k}\|. \end{aligned} \tag{13}$$

In (13), letting $k \to \infty$ and using the fact that $\lim_{k\to\infty}\epsilon_k = 0$ and the boundedness of $\{\|x+\epsilon_k q_{n_k} - u_{n_k}\|\}$, we get

$$\langle Fx, x - z\rangle \geq 0, \quad \text{for all } x \in C.$$

This implies that $z \in S_D$.

□

Next, we present the convergence of Algorithm **3**.1.

Theorem 3.1. *Assume that Conditions* **3**.*1*–**3**.*4 hold and $Fx \neq 0$, for all $x \in C$. Then the sequence $\{y_n\}$ generated by Algorithm* **3**.*1 converges weakly to an element $z \in S$ provided that the parameters θ and β satisfy:* $0 \leq \theta < \dfrac{\sqrt{33}-5}{4}$ *and* $\max\left\{\dfrac{2\theta^2+5\theta-1}{8\theta+8}, 7\theta-1\right\} < \beta \leq 0$.

Proof. The proof is divided into several steps as follows:

Step 1. We first prove that

$$\|y_{n+1} - x^*\|^2 \leq \|t_n - x^*\|^2 - \frac{1}{3}\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_{n+1} - t_n\|^2, \text{ for all } x^* \in S_D. \tag{14}$$

Indeed, taking $x^* \in S_D \subset S \subset C$, we have

$$\begin{aligned}\|y_{n+1} - x^*\|^2 &= \|P_C(t_n - \tau_n Fu_n) - P_C x^*\|^2 \\ &\leq \langle y_{n+1} - x^*, t_n - \tau_n Fu_n - x^*\rangle \\ &= \frac{1}{2}\|y_{n+1} - x^*\|^2 + \frac{1}{2}\|t_n - \tau_n Fu_n - x^*\|^2 - \frac{1}{2}\|y_{n+1} - t_n + \tau_n Fu_n\|^2 \\ &= \frac{1}{2}\|y_{n+1} - x^*\|^2 + \frac{1}{2}\|t_n - x^*\|^2 + \frac{1}{2}\tau_n^2\|Fu_n\|^2 - \langle t_n - x^*, \tau_n Fu_n\rangle \\ &\quad - \frac{1}{2}\|y_{n+1} - t_n\|^2 - \frac{1}{2}\tau_n^2\|Fu_n\|^2 - \langle y_{n+1} - t_n, \tau_n Fu_n\rangle \\ &= \frac{1}{2}\|y_{n+1} - x^*\|^2 + \frac{1}{2}\|t_n - x^*\|^2 - \frac{1}{2}\|y_{n+1} - t_n\|^2 - \langle y_{n+1} - x^*, \tau_n Fu_n\rangle. \end{aligned}$$

This implies that

$$\|y_{n+1} - x^*\|^2 \leq \|t_n - x^*\|^2 - \|y_{n+1} - t_n\|^2 - 2\langle y_{n+1} - x^*, \tau_n Fu_n\rangle. \tag{15}$$

Since $x^* \in S_D$, we have $\langle Fx, x - x^*\rangle \geq 0$ for all $x \in C$. Taking $x := u_n \in C$, we get

$$\langle Fu_n, x^* - u_n\rangle \leq 0.$$

Thus we have

$$\langle Fu_n, x^* - y_{n+1}\rangle = \langle Fu_n, x^* - u_n\rangle + \langle Fu_n, u_n - y_{n+1}\rangle \leq \langle Fu_n, u_n - y_{n+1}\rangle. \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned}\|y_{n+1}-x^*\|^2 &\le \|t_n-x^*\|^2-\|y_{n+1}-t_n\|^2+2\tau_n\langle Fu_n,u_n-y_{n+1}\rangle\\ &=\|t_n-x^*\|^2-\|y_{n+1}-u_n\|^2-\|u_n-t_n\|^2-2\langle y_{n+1}-u_n,u_n-t_n\rangle\\ &\quad+2\tau_n\langle Fu_n,u_n-y_{n+1}\rangle\\ &=\|t_n-x^*\|^2-\|y_{n+1}-u_n\|^2-\|u_n-t_n\|^2\\ &\quad+2\langle t_n-\tau_nFu_n-u_n,y_{n+1}-u_n\rangle. \end{aligned} \tag{17}$$

Since $u_n=P_C(t_n-\tau_nFt_n)$ and $y_{n+1}\in C$, we have

$$\begin{aligned}2\langle t_n-\tau_nFu_n-u_n,y_{n+1}-u_n\rangle &= 2\langle t_n-\tau_nFt_n-u_n,y_{n+1}-u_n\rangle+2\tau_n\langle Ft_n-Fu_n,y_{n+1}-u_n\rangle\\ &\le 2\tau_n\langle Ft_n-Fu_n,y_{n+1}-u_n\rangle\\ &\le 2\tau_n\|Ft_n-Fu_n\|\|y_{n+1}-u_n\|\\ &\le 2\mu\frac{\tau_n}{\tau_{n+1}}\|t_n-u_n\|\|y_{n+1}-u_n\|\\ &\le \mu\frac{\tau_n}{\tau_{n+1}}\|t_n-u_n\|^2+\mu\frac{\tau_n}{\tau_{n+1}}\|u_n-y_{n+1}\|^2. \end{aligned} \tag{18}$$

Substituting (18) into (17), we obtain

$$\begin{aligned}\|y_{n+1}-x^*\|^2 &\le \|t_n-x^*\|^2-\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|u_n-t_n\|^2-\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_{n+1}-u_n\|^2\\ &=\|t_n-x^*\|^2-\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|u_n-t_n\|^2+\|y_{n+1}-u_n\|^2)\\ &\le \|t_n-x^*\|^2-\frac{1}{2}\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_{n+1}-t_n\|^2. \end{aligned} \tag{19}$$

From $\mu\in\Big(0,\dfrac{1}{3}\Big)$ and $\lim_{n\to\infty}\Big(1-\mu\dfrac{\tau_n}{\tau_{n+1}}\Big)=1-\mu\ge\dfrac{2}{3}$, it follows that there exists $N_1\in\mathbb{N}$ such that

$$1-\mu\frac{\tau_n}{\tau_{n+1}}\ge\frac{2}{3},\ \text{ for all } n\ge N_1.$$

It implies, from (19), that

$$\|y_{n+1}-x^*\|^2\le\|t_n-x^*\|^2-\frac{1}{3}\|y_{n+1}-t_n\|^2,\ \text{ for all } n\ge N_1. \tag{20}$$

Step 2.

$$\lim_{n\to\infty}\|y_{n-1}-y_{n-2}\|=0.$$

First, using the definition of t_n and using (5) we get

$$\begin{aligned}\|t_n-x^*\|^2 &= \|y_n+\theta(y_n-y_{n-1})+\beta(y_{n-1}-y_{n-2})-x^*\|^2\\ &=\|(1+\theta)(y_n-x^*)-(\theta-\beta)(y_{n-1}-x^*)-\beta(y_{n-2}-x^*)\|^2\\ &=(1+\theta)\|y_n-x^*\|^2-(\theta-\beta)\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2\\ &\quad+(1+\theta)(\theta-\beta)\|y_n-y_{n-1}\|^2+\beta(1+\theta)\|y_n-y_{n-2}\|^2-\beta(\theta-\beta)\|y_{n-1}-y_{n-2}\|^2\\ &\le(1+\theta)\|y_n-x^*\|^2-(\theta-\beta)\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2\\ &\quad+(1+\theta)(\theta-\beta)\|y_n-y_{n-1}\|^2-\beta(\theta-\beta)\|y_{n-1}-y_{n-2}\|^2. \end{aligned} \tag{21}$$

and

$$\begin{aligned}\|y_{n+1}-t_n\|^2 &= \|y_{n+1}-y_n+\theta(y_n-y_{n-1})-\beta(y_{n-1}-y_{n-2})\|^2\\ &= \|y_{n+1}-y_n\|^2-2\theta\langle y_{n+1}-y_n, y_n-y_{n-1}\rangle+\theta^2\|y_n-y_{n-1}\|^2\\ &\quad -2\beta\langle y_{n+1}-y_n, y_{n-1}-y_{n-2}\rangle+2\theta\beta\langle y_n-y_{n-1}, y_{n-1}-y_{n-2}\rangle+\beta^2\|y_{n-1}-y_{n-2}\|^2.\end{aligned} \tag{22}$$

On the other hand, it is easy to see the following inequalities

$$-2\theta\langle a,b\rangle \geq -2\theta\|a\|\|b\| \geq -\theta\|a\|^2-\theta\|b\|^2 \ (\theta>0)$$

and

$$2\beta\langle a,b\rangle \geq 2\beta\|a\|\|b\| \geq \beta\|a\|^2+\beta\|b\|^2 \ (\beta<0).$$

Using the above inequalities we deduce

$$-2\theta\langle y_{n+1}-y_n, y_n-y_{n-1}\rangle \geq -2\theta\|y_{n+1}-y_n\|\|y_n-y_{n-1}\| \geq -\theta\|y_{n+1}-y_n\|^2-\theta\|y_n-y_{n-1}\|^2,$$

$$-2\beta\langle y_{n+1}-y_n, y_{n-1}-y_{n-2}\rangle \geq 2\beta\|y_{n+1}-y_n\|\|y_{n-1}-y_{n-2}\| \geq \beta\|y_{n+1}-y_n\|^2+\beta\|y_{n-1}-y_{n-2}\|^2,$$

and

$$2\theta\beta\langle y_n-y_{n-1}, y_{n-1}-y_{n-2}\rangle \geq 2\theta\beta\|y_n-y_{n-1}\|\|y_{n-1}-y_{n-2}\| \geq \theta\beta\|y_n-y_{n-1}\|^2+\theta\beta\|y_{n-1}-y_{n-2}\|^2.$$

Substituting the three inequalities into (22), we obtain

$$\begin{aligned}\|y_{n+1}-t_n\|^2 &\geq (1-\theta+\beta)\|y_{n+1}-y_n\|^2+(\theta^2-\theta+\theta\beta)\|y_n-y_{n-1}\|^2\\ &\quad +(\beta^2+\theta\beta+\beta)\|y_{n-1}-y_{n-2}\|^2.\end{aligned} \tag{23}$$

Again, substituting (21) and (23) into (14) we get

$$\begin{aligned}\|y_{n+1}-x^*\|^2 &\leq (1+\theta)\|y_n-x^*\|^2-(\theta-\beta)\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2\\ &\quad +(1+\theta)(\theta-\beta)\|y_n-y_{n-1}\|^2-\beta(\theta-\beta)\|y_{n-1}-y_{n-2}\|^2\\ &\quad -\frac{1}{3}(1-\theta+\beta)\|y_{n+1}-y_n\|^2-\frac{1}{3}(\theta^2-\theta+\theta\beta)\|y_n-y_{n-1}\|^2\\ &\quad -\frac{1}{3}(\beta^2+\theta\beta+\beta)\|y_{n-1}-y_{n-2}\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&\|y_{n+1}-x^*\|^2-\theta\|y_n-x^*\|^2-\beta\|y_{n-1}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_{n+1}-y_n\|^2 \leq \|y_n-x^*\|^2\\ &-\theta\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_{n+1}-y_n\|^2\\ &-\left(\frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)\right)\|y_n-y_{n-1}\|^2\\ &-\left(\frac{1}{3}(\beta^2+\theta\beta+\beta)+\beta(\theta-\beta)\right)\|y_{n-1}-y_{n-2}\|^2.\end{aligned}$$

It is, equivalently,

$$\begin{aligned}
&\|y_{n+1}-x^*\|^2-\theta\|y_n-x^*\|^2-\beta\|y_{n-1}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_{n+1}-y_n\|^2\\
&\quad+\left(\frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)\right)\|y_n-y_{n-1}\|^2\\
&\le \|y_n-x^*\|^2-\theta\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_n-y_{n-1}\|^2\\
&\quad+\left(\frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)\right)\|y_{n-1}-y_{n-2}\|^2\\
&\quad-\Bigg(\frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)\\
&\quad+\frac{1}{3}(\beta^2+\theta\beta+\beta)+\beta(\theta-\beta)\Bigg)\|y_{n-1}-y_{n-2}\|^2. \qquad (24)
\end{aligned}$$

Let

$$\Gamma_n := \|y_n-x^*\|^2-\theta\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_n-y_{n-1}\|^2+C^1\|y_{n-1}-y_{n-2}\|^2,$$

where

$$C^1 := \frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta).$$

Let

$$C^2 := \frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)+\frac{1}{3}(\beta^2+\theta\beta+\beta)+\beta(\theta-\beta).$$

Using (24), we get

$$\Gamma_{n+1}-\Gamma_n \le -C_2\|y_{n+1}-y_n\|^2, \quad \text{for all } n.$$

Next, we show that

$$\lim_{n\to\infty}\Gamma_n \text{ exists}$$

and

$$\lim_{n\to\infty}\|y_{n-1}-y_{n-2}\| = 0. \qquad (25)$$

Note that we need the assumption $0 \le \theta < \frac{\sqrt{33}-5}{4}$ so that $2\theta^2+5\theta-1<0$. Now, we use $\max\left\{\frac{2\theta^2+5\theta-1}{8\theta+8}, 7\theta-1\right\} < \beta \le 0$ to prove that $C_1>0$ and $C_2>0$. Indeed, we have

$$\begin{aligned}
C_1 &= \frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)\\
&= \frac{1}{3}-\frac{5}{3}\theta+\frac{4}{3}\beta-\frac{2}{3}\theta^2+\frac{4}{3}\theta\beta.
\end{aligned}$$

It is easy to see that $C^1>0$ is equivalent to $\beta > \frac{2\theta^2+5\theta-1}{4\theta+4}$. By the assumption $\beta > \frac{2\theta^2+5\theta-1}{8\theta+8}$, we deduce $\beta > \frac{2\theta^2+5\theta-1}{4\theta+4}$. Hence, $C^1>0$.

Next, we have

$$\begin{aligned}C^2 :=&\frac{1}{3}(1-\theta+\beta)+\frac{1}{3}(\theta^2-\theta+\theta\beta)-(1+\theta)(\theta-\beta)+\frac{1}{3}(\beta^2+\theta\beta+\beta)+\beta(\theta-\beta)\\ =&\frac{1}{3}-\frac{5}{3}\theta+\frac{5}{3}\beta-\frac{2}{3}\theta^2+\frac{8}{3}\theta\beta-\beta^2\\ \geq&\frac{1}{3}-\frac{5}{3}\theta+\frac{5}{3}\beta-\frac{2}{3}\theta^2+\frac{8}{3}\theta\beta+\beta(\text{ by } -\beta^2\geq\beta)\\ =&\frac{1}{3}-\frac{5}{3}\theta+\frac{8}{3}\beta-\frac{2}{3}\theta^2+\frac{8}{3}\theta\beta.\end{aligned}$$

So $C^2>0$, by our assumption $\beta > \dfrac{2\theta^2+5\theta-1}{8\theta+8}$.

Now, we show that $\Gamma_n \geq 0$. Indeed, we have

$$\begin{aligned}\Gamma_n =&\|y_n-x^*\|^2-\theta\|y_{n-1}-x^*\|^2-\beta\|y_{n-2}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_n-y_{n-1}\|^2+C^1\|y_{n-1}-y_{n-2}\|^2\\ &\geq \|y_n-x^*\|^2-\theta\|y_{n-1}-x^*\|^2+\frac{1}{3}(1-\theta+\beta)\|y_n-y_{n-1}\|^2\\ &\geq \|y_n-x^*\|^2-2\theta\|y_n-x^*\|^2-2\theta\|y_n-y_{n-1}\|^2+\frac{1}{3}(1-\theta+\beta)\|y_n-y_{n-1}\|^2\\ &= (1-2\theta)\|y_n-x^*\|^2+\frac{1}{3}(1-7\theta+\beta)\|y_n-y_{n-1}\|^2. \qquad (26)\end{aligned}$$

By assumption $\beta > \max\left\{\dfrac{2\theta^2+5\theta-1}{8\theta+8}, 7\theta-1\right\}$, we obtain $\beta > 7\theta-1$. Combining this with (26), we get $\Gamma_n \geq 0$, for all n.

$$\Gamma_{n+1}-\Gamma_n < -C^2\|y_{n-1}-y_{n-2}\|^2 \leq 0, \quad \text{for all } n, \qquad (27)$$

Therefore, the sequence $\{\Gamma_n\}$ is below bounded and nonincreasing, hence $\lim_{n\to\infty}\Gamma_n$ exists. Using this and (27) we get

$$\lim_{n\to\infty}\|y_{n-1}-y_{n-2}\| = 0. \qquad (28)$$

Step 3.

$$\lim_{n\to\infty}\|y_n-x^*\|^2 \text{ exists for all } x^*\in S_D.$$

Indeed, combining (27), (28), $\lim_{n\to\infty}\Gamma_n$ exists. Therefore, the sequence $\{y_n\}$ is bounded, and from (28) it is easy to see that

$$\lim_{n\to\infty}\|y_{n-2}-y_n\| = 0.$$

Let

$$a_n := \|y_n-x^*\|^2-\theta\|y_{n-1}-x^*\|-\beta\|y_{n-2}-x^*\|,$$

$$b_n := \|y_{n-1}-y_n\|^2+2\langle y_{n-1}-y_n, y_n-x^*\rangle = \|y_{n-1}-x^*\|^2-\|y_n-x^*\|^2,$$

and

$$c_n := \|y_{n-2}-y_n\|^2+2\langle y_{n-2}-y_n, y_n-x^*\rangle = \|y_{n-2}-x^*\|^2-\|y_n-x^*\|^2.$$

We have

$$(1-\theta-\beta)\|y_n-x^*\|^2 = a_n+\theta b_n+\beta c_n. \qquad (29)$$

Moreover, since $\lim_{n\to\infty}\Gamma_n$ exists and $\lim_{n\to\infty}\|y_{n-1}-y_{n-2}\| = 0$ we deduce

$$\lim_{n\to\infty} a_n \text{ exists.} \qquad (30)$$

Since $\{y_n\}$ is bounded, $\lim_{n\to\infty}\|y_{n-1}-y_{n-2}\| = 0$ and $\lim_{n\to\infty}\|y_n-y_{n-2}\| = 0$ we obtain

$$\lim_{n\to\infty} b_n = 0 \text{ and } \lim_{n\to\infty} c_n = 0. \qquad (31)$$

Combining (32), (30) and (31), we get

$$\lim_{n\to\infty} \|y_n - x^*\|^2 \text{ exists for all } x^* \in S_D.$$

Step 4.

$$\lim_{n\to\infty} \|u_n - t_n\| = 0.$$

Indeed, we have

$$\|t_n - y_n\| \leq \theta\|y_n - y_{n-1}\| + \beta\|y_{n-1} - y_{n-2}\| \to 0 \text{ as } n \to \infty. \tag{32}$$

Using (32) and $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$, we get

$$\|y_{n+1} - t_n\| \leq \|y_{n+1} - y_n\| + \|y_n - t_n\| \to 0 \text{ as } n \to \infty. \tag{33}$$

On the other hand by (19), we have

$$\begin{aligned}\frac{1}{2}\Big(1 - \mu\frac{\tau_n}{\tau_{n+1}}\Big)\|u_n - t_n\|^2 \leq & \|t_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 \\ =& (\|t_n - x^*\| - \|y_{n+1} - x^*\|)(\|t_n - x^*\| + \|y_{n+1} - x^*\|) \\ \leq & M\|y_{n+1} - t_n\|, \quad \text{for some } M > 0. \end{aligned} \tag{34}$$

Combining (33) and (34) we deduce

$$\lim_{n\to\infty} \|u_n - t_n\| = 0. \tag{35}$$

Step 5. The sequence $\{y_n\}$ converges weakly to an element in S_D Now, since the sequence $\{y_n\}$ is bounded, we choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup z^*$. By (32), we have $t_{n_k} \rightharpoonup z^*$. From (35) and Lemma **3**.2, we get $z^* \in \mathrm{S_D} \subset \mathrm{S}$. Therefore, we proved that, for all $x^* \in \mathrm{S_D} \subset \mathrm{S}$, $\lim_{n\to\infty} \|y_n - x^*\|$ exists and each sequential weak cluster point of the sequence $\{y_n\}$ is in $\mathrm{S_D} \subset \mathrm{S}$. By Lemma **2**.3, the sequence $\{y_n\}$ converges weakly to an element of $\mathrm{S_D} \subset \mathrm{S}$. This completes the proof.

□

4. Conclusions

In this paper, we present a new version of the extragradient algorithm for solving the variational inequality problem in Hilbert spaces. We introduce new strategies for the inertial parameter and step size. Weak convergence is established under the assumptions of quasimonotonicity and Lipschitz continuity of the given mapping.

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