A NUMERICAL METHOD OF SOLVING CAUCHY PROBLEM FOR DIFFERENTIAL EQUATIONS BASED ON A LINEAR APPROXIMATION

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An alternate method of numerical integration of first order ordinary differential equations (i.e. solving Cauchy problem) is proposed, based on the linearization of the expression which doesn't contain the derivative of the unknown function. It is proved that the approximation is of order 2 and, by given examples, the accuracy of method is illustrated. The method is extended for 2'nd order differential equations and one shows that it is also of order 2 of accuracy, but for some classes of such equations, the methods becomes of order 3. The advantage is that the proposed method applies directly to 2'nd order equations, without the need of solving systems of first order.

Keywords: Taylor series; linear approximation; order of approximation; Cauchy problem.

1. Introduction

The classical numerical methods for solving first-order ordinary differential equations (ODE) with initial values are based on approximations derived from the Taylor expansions of the unknown function. The simplest (and oldest) method is named after Euler and is based on the first order Taylor approximation of the function we are looking for. However, the Euler method is not as accurate as engineering problems require, so that more precise methods are needed to solve Cauchy problems for ODE's and systems of ODE's. Today, the most used such a method, specially on solving large systems of 1-st order ODE's with constant coefficients, is known as "Runge-Kutta method" (RK), which in fact is a class of methods. The most applied version belonging to this class is the 4-th order RK method ([1], [4]). Other methods are also known, which belong to families of linear multistep methods, like the explicit "Adams–Bashforth" methods or implicit "Adams-Moulton" methods, having various orders of accuracy ([2], [5]). Linear multi-step methods using higher-order derivatives have been developed (such as Störmer method) as well ([6]).

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Other methods use adaptative step sizes in order to improve the accuracy of approximation, by changing the step size during the integration. The implicit methods are not mentioned in this paper, since the proposed method is an explicit one and it will be compared with methods of the same category.

The method proposed in this paper combines the versatility of Taylor approximations with the accuracy given by the exact analytic solutions of linear ODE of 1'st order. One consequence of this feature is that for some classes of ODE's, the order of the approximation gets higher, using the same method. This could be an advantage over the classical methods.

2. First order ODE – first method

One considers the Cauchy problem stated as follows: find the function $y = y(x)$ satisfying the differential equation and the initial condition given below:

$$y' = f(x,y),\ \ y(x_0) = y_0 \text{ , where } f \in C^2(D),\ D \subset R^2. \tag{1}$$

The first proposed method consists of taking the linear term from the Taylor expansion of $f(x,y)$ with respect to the variable y:

$$f(x,y) \cong f(x,y_0) + (y - y_0)\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f(x,y_0) + (y - y_0)\, f'_{y_0}. \tag{2}$$

One can observe that Euler method considered only the first term of (2) where, in addition, the value of $f(x,y)$ was taken in $x = x_0$.

By performing the function change

$$u(x) = y(x) - y_0 \tag{3}$$

the equation (1) becomes a linear inhomogeneous ODE of form:

$$u' - f'_{y_0} \cdot u = f(x,y_0),\ u(x_0) = 0. \tag{4}$$

The general solution of above equation can be expressed as

$$u(x) = \int_{x_0}^{x} f(\xi, y_0) e^{-f'_{y_0}(\xi - x)} \cdot d\xi. \tag{5}$$

According to (3), it follows that

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y_0) e^{-f'_{y_0}(\xi - x)} d\xi . \tag{6}$$

Since the interval $[x_0, x]$ is in general small (it represents the step of the numerical algorithm), a mean value theorem can be applied on the integral in (6):

$$y(x) \cong y_0 + f(\xi_M, y_0) \int_{x_0}^{x} e^{-f'_{y_0}(\xi - x)} d\xi \text{, where } \xi_M = \frac{x + x_0}{2} . \tag{7}$$

By computing the integral, one gets finally the approximate solution on $[x_0, x]$:

$$y(x) \cong y_0 + f\left(\frac{x + x_0}{2}, y_0\right) \frac{e^{f'_{y_0}(x - x_0)} - 1}{f'_{y_0}} \text{, where } f'_{y_0} \neq 0 . \tag{8}$$

Remarks:

1) Since the function f is supposed to be of class C^2, its partial derivatives can be easily found and the above expression can be implemented without difficulties in the computing codes.

2) In the case when $f'_{y_0} = 0$, one can observe that the last fraction in (8) has a finite limit:

$$\lim_{f'_{y_0} \to 0} \frac{e^{f'_{y_0}(x - x_0)} - 1}{f'_{y_0}} = x - x_0 \tag{9}$$

(in other words, $f'_{y_0} = 0$ is an apparent singularity). The approximate solution becomes:

$$y(x) \cong y_0 + (x - x_0)\, f\left(\frac{x + x_0}{2}, y_0\right) . \tag{10}$$

The trivial case when $f'_y \equiv 0$ can be easily treated using the above formula, which is similar but not identical to the formulas given by other known methods, like improved Euler, second order Runge-Kutta or Heun's method.

3) For the linear ODE of first order, the rule applies as follows:

$$y' = p(x) \cdot y + q(x),\ y(x_0) = y_0 \tag{11}$$

$$f(x, y) = p(x) \cdot y + q(x),\ f'_y = p(x) \tag{12}$$

$$y(x) \cong y_0 + \left[p\left(\frac{x + x_0}{2}\right) \cdot y_0 + q\left(\frac{x + x_0}{2}\right)\right] \frac{e^{p(x_0)(x-x_0)} - 1}{p(x_0)} . \tag{13}$$

This expression approximates the closed form solution

$$y(x) = y_0 + \int_{x_0}^{x} q(\xi)\, e^{-p(\xi)(\xi - x)}\, d\xi \tag{14}$$

wherein the integral cannot be computed analytically and numerical approaches were needed involving thus a higher volume of computations.

The accuracy of approximation

In the sequel, the Taylor expansions of the exact solution and the approximate one about $x = x_0$ will be analyzed.

The Taylor series of function $y(x)$ about $x = x_0$ has the form:

$$\begin{aligned} y(x) &= y_0 + y'(x_0)(x - x_0) + \frac{1}{2} y''(x_0)(x - x_0)^2 + \\ &+ \frac{1}{6} y'''(x_0)(x - x_0)^3 + \dots . \end{aligned} \tag{15}$$

Using (1), the above expansion can be rewritten as:

$$\begin{aligned} y(x) &= y_0 + (x - x_0) f_0 + \frac{(x - x_0)^2}{2} \cdot \frac{d}{dx} f(x,y)\Big|_0 + \\ &+ \frac{(x - x_0)^3}{6} \cdot \frac{d^2}{dx^2} f(x,y)\Big|_0 + \dots \end{aligned} \tag{16}$$

where

$$f_0 = f(x_0, y_0) . \tag{17}$$

By using the basic rules of Calculus, the total derivatives of $f(x,y)$ with respect to x will get the following explicit forms:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} = f'_x + f\, f'_y \tag{18}$$

$$\frac{d^2 f}{dx^2} = f''_{xx} + 2 f\, f''_{xy} + f^2 f''_{yy} + \left(f'_x + f\, f'_y\right) f'_y . \tag{19}$$

It follows that the Taylor expansion of $y(x)$ about $x = x_0$ gets the form:

$$y(x) = y_0 + (x - x_0) f_0 + \frac{(x-x_0)^2}{2}\left[f'_{x_0} + f_0 f'_{y_0} \right] + \\ + \frac{(x-x_0)^3}{6}\left[f''_{xx_0} + 2 f_0 f''_{xy_0} + f_0^2 f''_{yy_0} + f'_{y_0}\left(f'_{x_0} + f_0 f'_{y_0} \right)\right] + \dots \tag{20}$$

A similar analysis will be performed on the approximate solution (8). The exponential function has the expansion:

$$e^{f'_{y_0}(x-x_0)} = 1 + (x-x_0) f'_{y_0} + \frac{1}{2}(x-x_0)^2 f'^2_{y_0} + \\ + \frac{1}{6}(x-x_0)^3 f'^3_{y_0} + \dots \tag{21}$$

so that the corresponding term in (8) reduces to

$$\frac{e^{f'_{y_0}(x-x_0)} - 1}{f'_{y_0}} = x - x_0 + \frac{1}{2}(x-x_0)^2 f'_{y_0} + \frac{1}{6}(x-x_0)^3 f'^2_{y_0} + \dots \tag{22}$$

The other term which has to be expanded is

$$f\left(\frac{x+x_0}{2}, y_0\right) \cong f(x_0, y_0) + \left(\frac{x+x_0}{2} - x_0\right)\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + \\ + \frac{1}{2}\left(\frac{x+x_0}{2} - x_0\right)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} + \dots \tag{23}$$

$$f\left(\frac{x+x_0}{2}, y_0\right) \cong f_0 + \frac{1}{2}(x-x_0) f'_{x_0} + \frac{1}{4}(x-x_0)^2 f''_{xx_0} + \dots \tag{24}$$

Introducing in (8), one yields:

$$y(x) = y_0 + (x-x_0) f_0 + \frac{1}{2}(x-x_0)^2\left[f'_{x_0} + f_0 f'_{y_0} \right] + \\ + \frac{1}{6}(x-x_0)^3\left[\frac{3}{2}\left(f''_{xx_0} + f'_{x_0} f'_{y_0} \right) + f_0 f'^2_{y_0} \right] + \dots \tag{25}$$

By comparing to (8), one can observe that the differences occur at the term of order 3, which means that the accuracy of the method is of order 2.

The next example will illustrate the accuracy of the proposed method, by comparing the results with those given by other known methods (like Euler and Runge-Kutta of order 4).

The following Cauchy problem is considered:

$$e^{2x} y' = 2(x+2) y^3, \;\; y(0) = \frac{1}{\sqrt{5}}. \tag{26}$$

The equation is of separable variables and admits a closed analytical solution, such that it is easy to compare the approximate solutions with the exact one:

$$y(x) = \frac{e^x}{\sqrt{2x+5}}. \tag{27}$$

For the given example, one derives:

$$f(x, y) = 2(x+2) e^{-2x} y^3, \;\; f(x_n, y_n) = 2(x_n + 2) e^{-2x_n} y_n^3 \tag{28}$$

$$f'_{y_n} = \left.\frac{\partial f}{\partial y}\right|_{(x_n, y_n)} = 6(x_n + 2) e^{-2x_n} y_n^2. \tag{29}$$

The recurrent formula derived from (8) is

$$y_{n+1} = y_n + \frac{1}{f'_{y_n}} f\left(\frac{x_{n+1} + x_n}{2}, y_n\right)\left[e^{f'_{y_n}(x_{n+1} - x_n)} - 1\right]. \tag{30}$$

A constant step of $h = x_{n+1} - x_n = 0.05$ was chosen for the given example. In the Table 1 and Fig.1, the numerical results and the graphical representation are shown.

Table 1

Comparative results of numerical solutions of Cauchy problem (26) using the 1-st proposed method and other known methods

x_n	y_n - exact	y_n – proposed (1)	y_n – RK	y_n - Euler
0	0.447214	0.447214	0.447214	0.447214
0.1	0.484650	0.484667	0.484649	0.483765
0.2	0.525608	0.525651	0.525608	0.523528
0.3	0.570419	0.570497	0.570419	0.566742
0.4	0.619446	0.619571	0.619446	0.613653
0.5	0.673088	0.673277	0.673087	0.66451
0.6	0.731781	0.732056	0.73178	0.719563
0.7	0.796006	0.796396	0.796005	0.779054
0.8	0.866291	0.866834	0.86629	0.84321
0.9	0.943215	0.943962	0.943214	0.912233
1.0	1.027414	1.02843	1.027413	0.986289
1.1	1.119587	1.120957	1.119585	1.065491
1.2	1.220499	1.222337	1.220497	1.149888

1.3	1.330994	1.333447	1.330991	1.23944
1.4	1.451995	1.455257	1.45199	1.334002
1.5	1.584516	1.588844	1.584511	1.433308

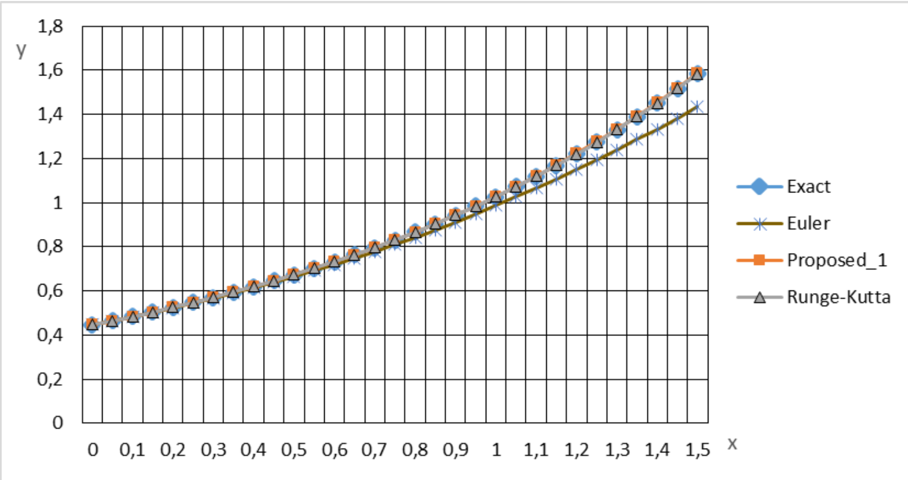

Fig. 1. Graphical representation of the solution of problem (26) using the 1-st proposed method and other known methods

3. First order ODE – second method

Unlike the first proposed method, the second one uses the expansion of the unknown solution in Taylor series of two variables, taking into consideration only the linear terms:

$$f(x,y) \cong f(x_0,y_0) + (x-x_0)\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + (y-y_0)\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \tag{31}$$

or, in shorter notations:

$$f(x,y) \cong f_0 + (x-x_0)\, f'_{x_0} + (y-y_0)\, f'_{y_0}\,. \tag{32}$$

A new linear inhomogeneous ODE of first order is obtained:

$$y' - f'_{y_0}\, y = f_0 + (x-x_0)\, f'_{x_0} - y_0\, f'_{y_0} \tag{33}$$

whose general analytical solution can be written as

$$y(x) = e^{f'_{y_0}x}\left\{C_0 + \int\left[f_0 + (x-x_0)\, f'_{x_0} - y_0 f'_{y_0}\right] e^{-f'_{y_0}x} dx\right\}. \tag{34}$$

By computing the integral, one yields:

$$y(x) = C_0 e^{f'_{y_0} x} - (x - x_0) \frac{f'_{x_0}}{f'_{y_0}} + \left(y_0 - \frac{f'_{x_0}}{f'^2_{y_0}} - \frac{f_0}{f'_{y_0}} \right). \tag{35}$$

The constant C_0 will be determined by applying the initial value and, finally, a new approximate solution of problem (1) is found:

$$y(x) \cong y_0 - (x - x_0) \frac{f'_{x_0}}{f'_{y_0}} + \left(f_0 + \frac{f'_{x_0}}{f'_{y_0}} \right) \frac{e^{f'_{y_0}(x - x_0)} - 1}{f'_{y_0}}, \ f'_{y_0} \neq 0. \tag{36}$$

One can see again that $f'_{y_0} = 0$ is an apparent singularity, when the approximate solution becomes:

$$y(x) \to y_0 + (x - x_0) f_0, \text{ when } f'_{y_0} \to 0 \tag{37}$$

which is the well-known Euler approximation.

The accuracy of approximation

Using the expansion of the exponential term (22), the Taylor expansion of the approximate solution (36) gets the form:

$$\begin{aligned} y(x) \cong y_0 + (x - x_0) f_0 + \frac{1}{2}(x - x_0)^2 \left(f_0 f'_{y_0} + f'_{x_0} \right) + \\ + \frac{1}{6}(x - x_0)^3 \left(f_0 f'^2_{y_0} + f'_{x_0} f'_{y_0} \right) + \dots . \end{aligned} \tag{38}$$

By comparing with the exact expansion (20), the terms up to order 2 are identical and, in addition, part of the term of order 3 is the same. However, the accuracy of this method is the same as the previous one, namely is of order 2.

From the point of view of the volume of computational work, this method involves the additional calculation of the partial derivative f'_x.

For the example given in (28), the following expressions are used:

$$f(x_n, y_n) = 2(x_n + 2) e^{-2x_n} y_n^3 \tag{39}$$

$$\begin{aligned} f'_{x_n} &= \left. \frac{\partial f}{\partial x} \right|_{(x_n, y_n)} = -2 y_n^3 (2x_n + 3) e^{-2x_n} \\ f'_{y_n} &= \left. \frac{\partial f}{\partial y} \right|_{(x_n, y_n)} = 6(x_n + 2) e^{-2x_n} y_n^2 . \end{aligned} \tag{40}$$

The recurrent formula derived from (36) used for numerical calculations is:

$$y_{n+1} = y_n - \left(x_{n+1} - x_n\right)\frac{f'_{x_n}}{f'_{y_n}} + \\ + \left[f\left(x_n, y_n\right) + \frac{f'_{x_n}}{f'_{y_n}} \right] \frac{e^{f'_{y_n}\left(x_{n+1} - x_n\right)} - 1}{f'_{y_n}}. \tag{41}$$

The results obtained by applying the second proposed method of approximation are presented in the Table 2 and in Fig.2, in comparison with the exact solution and 4-th order Runge-Kutta approximation.

As in the previous example, a constant step of $h = x_{n+1} - x_n = 0.05$ was chosen. Since the method has the same order of accuracy like the previous one, namely order 2, the results are comparable, as expected. The 4-th order Runge-Kutta method remains however the most precise one, having the disadvantage of a higher volume of computational work.

4. Second order ODE

In general, the 2-nd order ODE are integrated numerically by transforming them first into a system of two 1-st order ODE's on which an appropriate method of integrating is applied. In the next, an approximate method which applies directly on the 2-nd order ODE is proposed, using the same idea used in the case of the 1-st order ODE's.

Table 2

Comparative results of numerical solutions of Cauchy problem (26) using the 2-nd proposed method and other known methods

x_n	y_n - exact	y_n – proposed (2)	y_n – RK
0	0.447214	0.447213	0.447214
0.1	0.484650	0.484672	0.484649
0.2	0.525608	0.525662	0.525608
0.3	0.570419	0.570517	0.570419
0.4	0.619446	0.619603	0.619446
0.5	0.673088	0.673323	0.673087
0.6	0.731781	0.732122	0.73178
0.7	0.796006	0.796488	0.796005
0.8	0.866291	0.866959	0.86629
0.9	0.943215	0.944129	0.943214
1	1.027414	1.028653	1.027413
1.1	1.119587	1.121254	1.119585
1.2	1.220499	1.222730	1.220497

1.3	1.330994	1.333965	1.330991
1.4	1.451995	1.455940	1.45199
1.5	1.584516	1.589741	1.584511

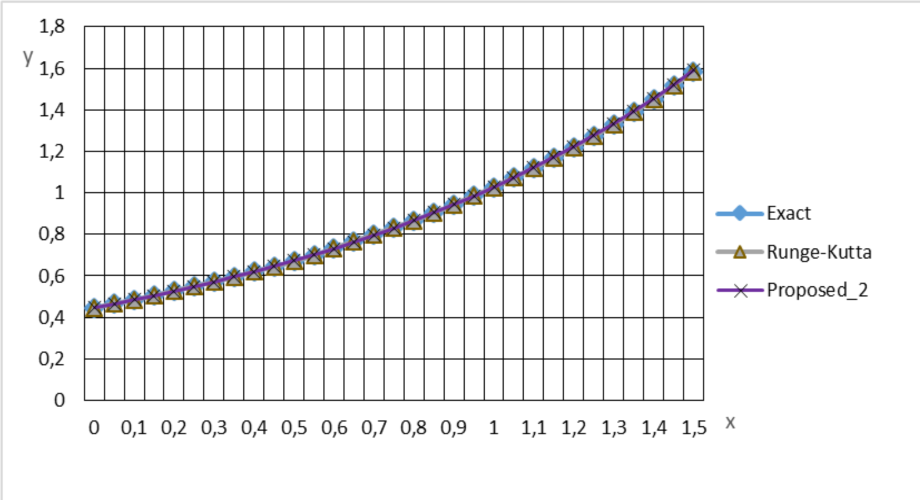

Fig. 2. Graphical representation of the solution of problem (26) using the 2-nd proposed method and other known methods

Let us consider a 2-nd order Cauchy problem written in the form:

$$y'' = f(x, y, y'),\ y(x_0) = y_0,\ y'(x_0) = y'_0 . \tag{42}$$

The Taylor expansion of $f(x, y, y')$ with respect to y and y' has the form:

$$f(x, y, y') = f(x, y_0, y'_0) + (y - y_0) \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0, y'_0)} + \\ + (y' - y'_0) \left.\frac{\partial f}{\partial y'}\right|_{(x_0, y_0, y'_0)} + \dots . \tag{43}$$

Considering only the linear terms and using short notations, one can write:

$$f(x, y, y') \cong f(x, y_0, y'_0) + (y - y_0) f'_{y_0} + (y' - y'_0) f'_{y'_0} . \tag{44}$$

The second term in the right-hand side can be also approximated by using Euler's rule:

$$y - y_0 \cong \left.\frac{dy}{dx}\right|_0 (x - x_0) = y'_0 (x - x_0) . \tag{45}$$

Introducing in (44), one yields:

$$f\left(x,y,y'\right)\cong f\left(x,y_0,y'_0\right)+\left(x-x_0\right)y'_0\, f'_{y_0}+\left(y'-y'_0\right)f'_{y'_0}\,. \tag{46}$$

With the function change:

$$z\left(x\right)=y'-y'_0 \tag{47}$$

the 2-nd order differential equation (42) gets the following approximate form, which is an inhomogeneous linear 1-st order ODE:

$$z'-z\cdot f'_{y'_0}=f\left(x,y_0,y'_0\right)+\left(x-x_0\right)y'_0\, f'_{y_0},\quad z\left(x_0\right)=0\,. \tag{48}$$

The solution of the above equation can be written as

$$z\left(x\right)=\int_{x_0}^{x}\left[f\left(\xi,y_0,y'_0\right)+\left(\xi-x_0\right)y'_0\, f'_{y_0}\right]e^{-f'_{y'_0}\left(\xi-x\right)}d\xi \tag{49}$$

and applying again a mean value theorem in the integral, one yields:

$$\begin{aligned} z\left(x\right)= & f\left(\xi_M,y_0,y'_0\right)\int_{x_0}^{x}e^{-f'_{y'_0}\left(\xi-x\right)}d\xi\ + \\ & + y'_0\, f'_{y_0}\int_{x_0}^{x}\left(\xi-x_0\right)e^{-f'_{y'_0}\left(\xi-x\right)}d\xi \end{aligned} \tag{50}$$

where ξ_M is given by (7). After computing the integrals, the expression of $z\left(x\right)$ becomes:

$$\begin{aligned} z\left(x\right)= & f\left(\xi_M,y_0,y'_0\right)\frac{e^{f'_{y'_0}\left(x-x_0\right)}-1}{f'_{y'_0}}\ + \\ & + y'_0\, f'_{y_0}\,\frac{e^{f'_{y'_0}\left(x-x_0\right)}-\left(x-x_0\right)f'_{y'_0}-1}{f'^{\,2}_{y'_0}}\,. \end{aligned} \tag{51}$$

By replacing $\xi_M=\dfrac{x+x_0}{2}$, the approximate expression of $y'\left(x\right)$ becomes:

$$y'(x) \cong y'_0 + f\left(\frac{x+x_0}{2}, y_0, y'_0\right)\frac{e^{f'_{y'_0}(x-x_0)}-1}{f'_{y'_0}} + \\ + y'_0\, f'_{y_0}\,\frac{e^{f'_{y'_0}(x-x_0)}-(x-x_0)\,f'_{y'_0}-1}{f'^{\,2}_{y'_0}}\,. \tag{52}$$

A second integration yields:

$$y(x) \cong y_0 + y'_0(x-x_0) + \int_{x_0}^{x} f\left(\frac{\xi+x_0}{2}, y_0, y'_0\right)\frac{e^{f'_{y'_0}(\xi-x_0)}-1}{f'_{y'_0}}\,d\xi + \\ + \frac{y'_0\, f'_{y_0}}{f'^{\,2}_{y'_0}}\int_{x_0}^{x}\left[e^{f'_{y'_0}(\xi-x_0)}-(\xi-x_0)\,f'_{y'_0}-1\right]d\xi\ . \tag{53}$$

Following the same judgment and computing the integrals, the final expression of the approximate solution of problem (42) is found:

$$y(x) \cong y_0 + y'_0(x-x_0) + \\ + f\left(\frac{x+3x_0}{4}, y_0, y'_0\right)\frac{e^{f'_{y'_0}(x-x_0)}-f'_{y'_0}(x-x_0)-1}{f'^{\,2}_{y'_0}} + \\ + \frac{y'_0\, f'_{y_0}}{f'^{\,3}_{y'_0}}\left[e^{f'_{y'_0}(x-x_0)}-\frac{(x-x_0)^2}{2}\,f'^{\,2}_{y'_0}-(x-x_0)\,f'_{y'_0}-1\right]. \tag{54}$$

Remark:

Like in the case of the 1-st order ODE analyzed, all the expressions containing $f'_{y'_0}$ at denominator have finite limits as $y'_0 \to 0$ (apparent singularity). Thus:

$$\lim_{f'_{y0}\to 0}\frac{e^{f'_{y0}(x-x_0)}-1}{f'_{y0}} = x-x_0\,,\quad \lim_{f'_{y'_0}\to 0}\frac{e^{f'_{y0}(x-x_0)}-(x-x_0)f'_{y0}-1}{f'^{\,2}_{y0}} = \frac{(x-x_0)^2}{2}$$

$$\lim_{f'_{y'_0}\to 0}\frac{e^{f'_{y'0}(x-x_0)}-\frac{(x-x_0)^2}{2}\,f'^{\,2}_{y'0}-(x-x_0)f'_{y'0}-1}{f'^{\,3}_{y'0}} = \frac{(x-x_0)^3}{6}\,. \tag{55}$$

The accuracy of approximation

The Taylor expansion of the exact solution $y(x)$ about $x = x_0$:

$$y(x) = y_0 + y'_0(x - x_0) + y''_0 \frac{(x-x_0)^2}{2} + y'''_0 \frac{(x-x_0)^3}{6} + \dots \tag{56}$$

will be compared with the similar expansion of the approximate solution (54). The series (56) can be re-written as:\

$$y(x) = y_0 + y'_0(x - x_0) + f(x_0, y_0, y'_0)\frac{(x-x_0)^2}{2} + y'''_0 \frac{(x-x_0)^3}{6} + \dots . \tag{57}$$

The third derivative of $y(x)$ has the expression:

$$\begin{aligned} y'''(x) &= \frac{d}{dx} f(x, y, y') = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} = \\ &= f'_x + y' f'_y + f(x, y, y') f'_{y'} \end{aligned} \tag{58}$$

so that the expansion (57) becomes:

$$\begin{aligned} y(x) &\cong y_0 + y'_0(x - x_0) + f_0 \frac{(x-x_0)^2}{2} + \\ &+ \left(f'_{x_0} + y'_0 f'_{y_0} + f_0 f'_{y'_0}\right)\frac{(x-x_0)^3}{6} + \dots \end{aligned} \tag{59}$$

where

$$f_0 = f(x_0, y_0, y'_0). \tag{60}$$

For the approximate solution (54), the terms containing the exponential function have the known expansions:

$$\frac{e^{f'_{y'_0}(x-x_0)} - \frac{(x - x_0)^2}{2} f'^2_{y'_0} - (x - x_0) f'_{y'_0} - 1}{f'^3_{y'_0}} = \frac{(x - x_0)^3}{6} + \dots \tag{61}$$

$$\frac{e^{f'_{y'_0}(x-x_0)} - (x - x_0) f'_{y'_0} - 1}{f'^2_{y'_0}} = \frac{(x - x_0)^2}{2} + f'_{y'_0} \frac{(x - x_0)^3}{6} + \dots . \tag{62}$$

On the other hand, the following first order approximation applies:

$$f\left(\frac{x+3x_0}{4}, y_0, y'_0\right) \cong f\left(x_0, y_0, y'_0\right) + \left(\frac{x+3x_0}{4} - x_0\right)\frac{\partial f}{\partial x}\bigg|_0 = \\ = f_0 + \frac{x-x_0}{4} f'_{x_0}. \tag{63}$$

Introducing all these results in (54), the following Taylor expansion of the approximate solution is found:

$$y(x) = y_0 + y'_0(x-x_0) + f_0\frac{(x-x_0)^2}{2} + \\ + \frac{(x-x_0)^3}{6}\left(f_0 f'_{y'_0} + y'_0 f'_{y_0} + \frac{3}{4} f'_{x_0}\right) + \ldots . \tag{64}$$

Comparing (64) with (59), one can see that the expansions differ by a quantity of order 3:

$$\varepsilon = y_{\text{exact}} - y_{\text{approx}} = f'_{x_0}\frac{(x-x_0)^3}{24} + \text{O}\left(x-x_0\right)^4. \tag{65}$$

In the particular case when $f'_{x_0} \equiv 0$, the proposed approximation becomes completely of order 3.

Example: $y''(x) + 4y(x) = 0;\ \ y(0)=1,\ \ y'(0)=0$.

The exact solution is $y(x) = \cos 2x$.

For the approximate solution, one writes:

$y'' = -4y \Rightarrow f(x,y,y') = -4y,\ f'_y = -4,\ f'_{y'} = 0,\ f'_x = 0$.

Since $f'_{y'} \equiv 0$, the formulas (55) apply. The recurrent expression for numerical computation, derived from (54) and wherein the above expressions are introduced, will read:

$$y_{n+1} = y_n + y'_n(x_{n+1}-x_n) - 4y_n\frac{(x_{n+1}-x_n)^2}{2} - 4y'_n\frac{(x_{n+1}-x_n)^3}{6} \tag{66}$$

where

$$y'_{n+1} = y'_n - 4y_n(x_{n+1}-x_n) - 4y'_n\frac{(x_{n+1}-x_n)^2}{2}. \tag{67}$$

For computation, a constant step of $h = x_{n+1} - x_n = 0.1$ was chosen. The results are presented in the Table 3 and Fig.3, in comparison with the exact solution. Since $f'_{x_0} \equiv 0$, according to the above analysis, the approximation is of order 3 and thus, the Runge-Kutta method of 4-th order is more precise. On the other hand, the proposed method is much easier to be implemented than RK algorithm, involving fewer explicit formulas for computation (the differential equation doesn't need to be transformed into a system of two 1-st order ODE's).

In the Table 3, the 1-st order derivatives of the function we are looking for are also presented, in comparison with the exact values, along with the relative errors regarding the approximate solution.

Table 3

Comparative results of exact and approximate numerical solutions of Cauchy problem (42)

x_n	y_n - exact	y_n – proposed	Error	y'_n - exact	y'_n – proposed
0	1.000000	1.000000	0.000000	0.000000	0.000000
0.4	0.696707	0.694988	0.001719	-1.434712	-1.443605
0.8	-0.029200	-0.034518	0.005319	-1.999147	-2.006575
1.2	-0.737394	-0.743338	0.005944	-1.350926	-1.344714
1.6	-0.998295	-0.998684	0.000389	0.116748	0.138526
2	-0.653644	-0.644412	0.009232	1.513605	1.537980
2.4	0.087499	0.103500	0.016001	1.992329	1.999153
2.8	0.775566	0.788619	0.013053	1.262533	1.239973
3.2	0.993185	0.992605	0.000580	-0.233098	-0.276688
3.6	0.608351	0.590657	0.017695	-1.587336	-1.625224
4	-0.145500	-0.172136	0.026636	-1.978716	-1.982185

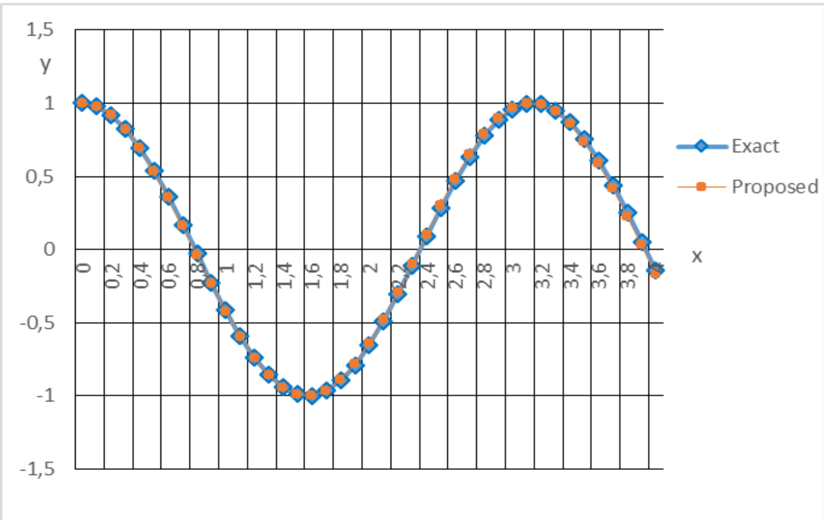

Fig. 3. Graphical representation of the solution of problem (42), showing the exact and the approximate solution

5. Conclusions

An alternate method of numerical integration of first order ODEs (Cauchy problem) was presented in two variants, based on the linearization of the expression which doesn't contain the derivative of the unknown function. Both variants have the order 2 of approximation and, by taking a smaller step of advancing, the accuracy of solution can be improved.

The advantage versus Runge-Kutta of 4-th order (which is more precise at the same size of the step) consists in the simplicity of implementing on computer. The method was adapted also for 2'nd order differential equations, obtaining the order 2 of accuracy, but it was shown that for some classes of equations, the approximation becomes of order 3. The advantage is that the proposed method applies directly to 2'nd order equations, without the need of transforming them into systems of first order.

R E F E R E N C E S

[1]. *D.W. Zingg, T.T. Chisholm,* Runge–Kutta methods for linear ordinary differential equations, Applied Numerical Mathematics 31, pp. 227–238, 1999.

[2]. *J.C. Butcher*, Numerical Methods for Ordinary Differential Equations, Wiley, New York, 2003.

[3]. *J.D. Lambert*, Computational Methods in Ordinary Differential Equations, Wiley, New York, 1973.

[4]. *D.H. Griffiths, D.J. Higham*, Numerical Methods for Ordinary Differential Equations, Springer-Verlag London Limited, 2010.

[5]. *K. Atkinson, W. Han, D. Stewart*, Numerical Solution of Ordinary Differential Equations, John Wiley & Sons, Inc., Hoboken, New Jersey, 2009.

[6]. *N.S. Bakhvalov*, Numerical methods: analysis, algebra, ordinary differential equations, MIR, 1977 (translated from Russian).