MODULE BIPROJECTIVE AND MODULE BIFLAT BANACH ALGEBRAS

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Dedicated to Professor Alireza Medghalchi

In this paper we define module biprojectivity and module biflatness for a Banach algebra which is a Banach module over another Banach algebra with compatible actions, and comparing to the classical notion of biprojectivity and biflatness, we show that these are more natural concepts if one tries to generalize the classical results on function algebras on groups to semigroups. As a typical example, we show that for an inverse semigroup S with the set of idempotents E, the semigroup algebra $\ell^1(S)$, as an $\ell^1(E)$-module, is module biprojective if and only if an appropriate group homomorphic image of S is finite. Also we show that $\ell^1(S)$ is module biflat if and only if S is amenable.

Keywords: Banach modules; Inverse semigroup; Module amenable; Module biflat; Module biprojective; Module derivation.

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1. Introduction

For a discrete semigroup S, $\ell^\infty(S)$ is the Banach algebra of bounded complex-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let l_af and r_af denote the left and the right translations of f by a, that is $(l_af)(s) = f(as)$ and $(r_af)(s) = f(sa)$, for each $s \in S$. Then a linear functional $m \in (\ell^\infty(S))^*$ is called a *mean* if $\|m\| = \langle m, 1\rangle = 1$; m is called a *left* (*right*) *invariant mean* if $m(l_af) = m(f)$ ($m(r_af) = m(f)$, respectively) for all $s \in S$ and $f \in \ell^\infty(S)$. A discrete semigroup S is called *amenable* if there exists a mean m on $\ell^\infty(S)$ which is both left and right invariant (see [8]). An *inverse semigroup* is a discrete semigroup S such that for each $s \in S$, there is a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. Elements of the form ss^* are called *idempotents* of S. For an inverse semigroup S, a left invariant mean on $\ell^\infty(S)$ is right invariant and vice versa.

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-module is inner, that is $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-module X, where $H^1(\mathcal{A}, X^*)$ is the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X^*. This concept was introduced by B. E. Johnson in [13]. Also $\mathcal{A}$ is called

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super-amenable (*contractible*) if $H^1(\mathcal{A}, X) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X (see [7, 21]).

The second author in [1] introduced the concept of module amenability for Banach algebras which are Banach modules over another Banach algebra with compatible actions, and showed that for an inverse semigroup S with set of idempotents E, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$, if and only if S is amenable. This generalizes the celebrated theorem of Johnson for a discrete group G (valid also for locally compact groups) which states that the group algebra $\ell^1(G)$ is amenable if and only if G is amenable. Pourmahmood in [16] introduced the concept of module super-amenability (contractibility) and showed that for an inverse semigroup S, the semigroup algebra $\ell^1(S)$ is module super-amenable if and only if the group homomorphic image $S/\approx$ is finite, where $s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$ (see also [3] and [4]).

Biprojective Banach algebras were introduced by A. Ya. Helemskii in [9]. Later he has studied biprojectivity and biflatness of the Banach algebras in more details in [10, Chapters IV and VII]. It follows from [7, Proposition 2.8.41] that a biprojective Banach algebra is biflat, but the converse is not true. For instance, ℓ^1-group algebra of integer numbers $\mathbb{Z}$ is biflat but not biprojective. For an infinite-dimensional Hilbert space $\mathcal{H}$, the Banach algebra $A(\mathcal{H}\widehat{\otimes}\mathcal{H})$ consisting of norm limits of all sequences of finite rank operators on $\mathcal{H}\widehat{\otimes}\mathcal{H}$ is biflat, but not biprojective [21, Example 4.3.25]. If $\mathcal{K}$ is a non-empty, locally compact space, then $C_0(\mathcal{K})$ is biprojective if and only if $\mathcal{K}$ is discrete [7, Proposition 4.2.31], but if $\mathcal{K}$ is an infinite compact space, then $C(\mathcal{K})$ is not biprojective [7, Corollary 5.6.3]. In general each commutative biprojective Banach algebra has a discrete character space and the converse holds for all commutative Banach algebra [21, Exercise 4.3.5]. It is shown by Selivanov in [22], that for any a non-zero Banach space E, the nuclear algebra $E\widehat{\otimes}E^*$ is biprojective (see [7, Corollary 2.8.43]).

In part two of this paper, we define the module biprojectivity and module biflatness of a Banach algebra $\mathcal{A}$ which is a Banach $\mathfrak{A}$-module with compatible actions on another Banach algebra $\mathfrak{A}$ and find their relation with module amenability [1] and module super-amenability [16]. In particular, we show that when $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from left then under some mild conditions, module biprojectivity (biflatness) of $\mathcal{A}$ implies biprojectivity (biflatness) of the quotient Banach algebra $\mathcal{A}/J$, where J is the closed ideal of $\mathcal{A}$ generated by $(a \cdot \alpha)b - a(\alpha \cdot b)$ for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Also we show that, under some conditions, biprojectivity (biflatness) of a Banach algebra implies its module biprojectivity (biflatness), but the converse is not true.

Let P be a partially ordered set. For $p \in P$, we set $(p] = \{x : x \leq p\}$ and $[p) = \{x : p \leq x\}$. Then P is called *locally finite* if $(p]$ is finite for all $p \in P$, and *locally C-finite* for some constant $C > 1$ if $|(p]| < C$ for all $p \in P$. A partially ordered set P which is locally C-finite, for some constant C is called *uniformly locally finite*. Y. Choi in [6, Theorem 6.1] proved that if S is a Clifford semigroup, then the group algebra $\ell^1(S)$ is biflat if and only if $(E, \leq)$ is uniformly locally finite, and each maximal subgroup of S is amenable. Later, P. Ramsden generalized this result to any discrete semigroup S [18]. He also showed that for any discrete semigroup S, $\ell^1(S)$ is biprojective if and only if S is uniformly locally finite and and all maximal subgroups of S are finite.

In part three of this paper, we prove that if S is an inverse semigroup with the set of idempotents E, then $\ell^1(S)$ is module biprojective, as an $\ell^1(E)$-module, if and only if an appropriate group homomorphic image $S/\approx$ of S is finite. This could be considered as the module version (for inverse semigroups) of a result of Helemskii [11] which asserts that for a discrete group G, $\ell^1(G)$ is biprojective if and only if G is finite (see also [7, Theorem 3.3.32]). Finally we show that $\ell^1(S)$ is $\ell^1(E)$-module biflat if and only if S is amenable. This also can be regarded as the module version of a result of Helemskii [10] which states that for any locally compact group G, the group algebra $L^1(G)$ is biflat if and only if G is amenable.

2. **Module biprojectivity and module biflatness**

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b,\ (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x) = (\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x) = (a\cdot\alpha)\cdot x,\ (\alpha\cdot x)\cdot a = \alpha\cdot(x\cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X})$$

and similarly for the right or two-sided actions. Then we say that $\mathcal{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A}, x \in \mathcal{X})$$

then $\mathcal{X}$ is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If $\mathcal{X}$ is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is $\mathcal{X}^*$, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on $\mathcal{X}^*$ are defined by

$$\langle \alpha \cdot f, x\rangle = \langle f, x \cdot \alpha\rangle,\ \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X}, f \in \mathcal{X}^*)$$

and similary for the right actions. Let $\mathcal{Y}$ be another $\mathcal{A}$-$\mathfrak{A}$-module, then a $\mathcal{A}$-$\mathfrak{A}$-module homomorphism from $\mathcal{X}$ to $\mathcal{Y}$ is a norm-continuous map $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ with $\varphi(x \pm y) = \varphi(x) \pm \varphi(y)$ and

$$\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x),\ \varphi(x \cdot \alpha) = \varphi(x) \cdot \alpha,\ \varphi(a \cdot x) = a \cdot \varphi(x), \varphi(x \cdot a) = \varphi(x) \cdot a,$$

for $x, y \in \mathcal{X}, a \in \mathcal{A}$, and $\alpha \in \mathfrak{A}$.
Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \quad (\alpha \in \mathfrak{A}, a, b \in \mathcal{A}).$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then so are the dual space $\mathcal{A}^*$ and the second dual space $\mathcal{A}^{**}$. If moreover $\mathcal{A}$ is a commutative $\mathfrak{A}$-module, then $\mathcal{A}^*$ and the $\mathcal{A}^{**}$ are commutative $\mathcal{A}$-$\mathfrak{A}$-modules. Also the canonical embedding $\hat{}: \mathcal{A} \to \mathcal{A}^{**}; a \mapsto \hat{a}$ is an $\mathfrak{A}$-module homomorphism.

Consider the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$. It is well known that $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach algebra with respect to the canonical multiplication map defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd \qquad (a, b, c, d \in \mathcal{A})$$

and extended by bi-linearity and continuity [7]. Then $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with canonical actions. Let I be the closed ideal of the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ generated by elements of the form

$$\{a\cdot\alpha\otimes b - a\otimes\alpha\cdot b \mid \alpha\in\mathfrak{A}, a,b\in\mathcal{A}\}. \tag{1}$$

Consider the map $\omega\in\mathcal{L}(\mathcal{A}\widehat{\otimes}\mathcal{A},\mathcal{A})$ defined by $\omega(a\otimes b)=ab$ and extended by linearity and continuity. Let J be the closed ideal of $\mathcal{A}$ generated by

$$\omega(I)=\{(a\cdot\alpha)b-a(\alpha\cdot b)\mid\alpha\in\mathfrak{A}, a,b\in\mathcal{A}\}. \tag{2}$$

Then the module projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}\cong(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ and the quotient Banach algebra $\mathcal{A}/J$ are both Banach $\mathcal{A}$-modules and Banach $\mathfrak{A}$-modules. Also, $\mathcal{A}/J$ is always $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically. We have $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^*=\mathcal{L}_{\mathfrak{A}}(\mathcal{A},\mathcal{A}^*)$ where the right hand side is the space of all $\mathfrak{A}$-module homomorphism from $\mathcal{A}$ to $\mathcal{A}^*$[20]. Also the map $\widetilde{\omega}\in\mathcal{L}(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A},\mathcal{A}/J)$ defined by $\widetilde{\omega}(a\otimes b+I)=ab+J$ extends to an $\mathfrak{A}$-module homomorphism. Moreover, $\widetilde{\omega}^*$ and $\widetilde{\omega}^{**}$, the first and second adjoints of $\widetilde{\omega}$ are $\mathcal{A}$-$\mathfrak{A}$-module homomorphisms. Let $\Box$ and $\Diamond$ be the first and second Arens products on the second dual space $\mathcal{A}^{**}$, then $\mathcal{A}^{**}$ is a Banach algebra with respect to both of these products [7, Theorem 2.6.15].

Let $\mathcal{A}$ and $\mathfrak{A}$ be as in the above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D:\mathcal{A}\longrightarrow\mathcal{X}$ is called a *module derivation* if

$$D(a\pm b)=D(a)\pm D(b),\ \ D(ab)=D(a)\cdot b+a\cdot D(b)\quad(a,b\in\mathcal{A}),$$

and

$$D(\alpha\cdot a)=\alpha\cdot D(a),\quad D(a\cdot\alpha)=D(a)\cdot\alpha\quad(a\in\mathcal{A},\alpha\in\mathfrak{A}).$$

Although D is not necessary linear, but still its boundedness implies its norm continuity (since it preserves subtraction). When $\mathcal{X}$ is commutative $\mathfrak{A}$-module, each $x\in\mathcal{X}$ defines a module derivation

$$D_x(a)=a\cdot x-x\cdot a\quad(a\in\mathcal{A}).$$

These are called *inner* module derivations. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $\mathcal{X}$, each module derivation $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ is inner [1]. Similarly, $\mathcal{A}$ is called *module super-amenable* (*contractible*) if each module derivation $D:\mathcal{A}\longrightarrow\mathcal{X}$ is inner [16].

Let I and J be the closed ideals defined in (1) and (2), respectively. Then $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ is not always an $\mathcal{A}/J$-module unless $\mathcal{A}$ is a commutative $\mathfrak{A}$-module. Let L be the closed ideal generated by elements of the form $(a\cdot\alpha)c\otimes b-ac\otimes\alpha\cdot b$ for all $a,b,c\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$. Then $\langle\omega(L)\rangle^-=\langle\omega(I)\rangle^-=J$ but I is not in general equal to L. If $\mathcal{A}$ has a bounded approximate identity (e_j), then for each $a,b\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ we have

$$\|[(a\cdot\alpha)e_j\otimes b-ae_j\otimes\alpha\cdot b]-[a\cdot\alpha\otimes b-a\otimes\alpha\cdot b]\|\longrightarrow 0.$$

So, $I\subseteq L$. Also

$$\|[(a\cdot\alpha\otimes b-a\otimes\alpha\cdot b)(c\otimes e_j)]-[(a\cdot\alpha)c\otimes b-ac\otimes\alpha\cdot b]\|\longrightarrow 0$$

for all $a,b,c\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$. Hence $L\subseteq I$. Therefore $L=I$.

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if there is a continuous linear functional f on $\mathfrak{A}$ such that $\alpha\cdot a=f(\alpha)a$ ($a\cdot\alpha=f(\alpha)a$), for each $\alpha\in\mathfrak{A}$ and $a\in\mathcal{A}$ (see also [2]). The following lemma is proved in [5, Lemma 3.13].

Lemma 2.1. *If $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from the left or right and $\mathcal{A}/J$ has a right bounded approximate identity, then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $f(\alpha)a - a\cdot\alpha \in J$.*

We show that when $\mathcal{A}$ has a bounded approximate identity, then $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ is $\mathcal{A}/J$-module if $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from left or right. For the case of the trivial left action, consider the following actions

$$(a+J)\cdot(b\otimes c+I) = ab\otimes c+I, \quad (b\otimes c+I)\cdot(a+J) = b\otimes ca+I.$$

For $a, b, c \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we have

$$[a\cdot\alpha - f(\alpha)a]\cdot(b\otimes c) = (a\cdot\alpha)b\otimes c - f(\alpha)ab\otimes c = (a\cdot\alpha)b\otimes c - ab\otimes\alpha\cdot c \in I.$$

Thus left action is well-defined. Similarly, one can show that the right action is also well-defined. Here and subsequently, when we consider $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ as an $\mathcal{A}/J$-module, we have supposed that the above conditions are satisfied.

Recall that a Banach algebra $\mathcal{A}$ is called *biprojective* if ω has a bounded right inverse which is an $\mathcal{A}$-bimodule homomorphism, and is called *biflat* if ω^* has a bounded left inverse which is an $\mathcal{A}$-bimodule homomorphism.

Definition 2.1. *A Banach algebra $\mathcal{A}$ is called module biprojective (as an $\mathfrak{A}$-module) if $\widetilde{\omega}$ has a bounded right inverse which is an $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism.*

Definition 2.2. *A Banach algebra $\mathcal{A}$ is called module biflat (as an $\mathfrak{A}$-module) if $\widetilde{\omega}^*$ has a bounded left inverse which is an $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism.*

Proposition 2.1. *Assume that $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left and $\mathcal{A}/J$ has an identity. If $\mathcal{A}$ is biprojective, then $\mathcal{A}$ is module biprojective.*

Proof. Suppose that ρ is the the bounded right inverse of ω and $e+J$ is identity of $\mathcal{A}/J$. Define $\widetilde{\rho}: \mathcal{A}/J \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ via

$$\widetilde{\rho}(a+J) := (\rho(e)+I)\cdot(a+J) \quad (a\in\mathcal{A}).$$

For each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, we have

$$\begin{aligned}\widetilde{\rho}(\alpha\cdot(a+J)) &= (\rho(e)+I)\cdot(\alpha\cdot a+J) = (\rho(e)+I)\cdot(f(\alpha)a+J)\\ &= f(\alpha)(\rho(e)+I)\cdot(a+J) = \alpha\cdot\widetilde{\rho}(a+J),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\rho}((a+J)\cdot\alpha) &= (\rho(e)+I)\cdot(a\cdot\alpha+J)\\ &= (\rho(e)+I)\cdot(a+J)\cdot\alpha\\ &= \widetilde{\rho}(a+J)\cdot\alpha.\end{aligned}$$

Obviously $\widetilde{\rho}$ is a $\mathcal{A}/J$-bimodule homomorphism. Hence $\widetilde{\rho}$ is a $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism. Now for each $a \in \mathcal{A}$ we have

$$\begin{aligned}(\widetilde{\omega}\circ\widetilde{\rho})(a+J) &= \widetilde{\omega}((\rho(e)+I)\cdot(a+J)) = \widetilde{\omega}(\rho(e)\cdot a+I)\\ &= \omega(\rho(e))a+J = ea+J = a+J.\end{aligned}$$

Therefore $\widetilde{\rho}$ is a bounded right inverse for $\widetilde{\omega}$. □

Proposition 2.2. *Assume that $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left and $\mathcal{A}/J$ has an identity. If $\mathcal{A}$ is biflat, then $\mathcal{A}$ is module biflat.*

Proof. It is straightforward to show that $\widetilde{\omega}\circ\pi_1=\pi_2\circ\omega$, where $\pi_1:\mathcal{A}\widehat{\otimes}\mathcal{A}\longrightarrow(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ and $\pi_2:\mathcal{A}\longrightarrow\mathcal{A}/J$ are projection maps. Assume that θ is the the bounded left inverse of ω^*. Define $\widetilde{\theta}:(\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^*\longrightarrow(\mathcal{A}/J)^*$ via

$$(\widetilde{\theta}(\phi))(a+J):=[\theta(\phi\circ\pi_1)](a)\quad(a\in\mathcal{A}),$$

where ϕ is a functional in $((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^*$. As in the proof of Proposition 2.1, it is easily verfied that $\widetilde{\theta}$ is a well-defined and $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism. If f is a bounded functional on $\mathcal{A}/I$ and $a\in\mathcal{A}$, then

$$\begin{aligned}[(\widetilde{\theta}\circ\widetilde{\omega}^*)(f)](a+J)&=(\widetilde{\theta}(\widetilde{\omega}^*)(f))(a+J)=[(\widetilde{\theta}(\widetilde{\omega}^*(f)\circ\pi_1)](a)\\&=[(\theta(f\circ\widetilde{\omega}\circ\pi_1)](a)=[(\theta(f\circ\pi_2\circ\omega)](a)\\&=[(\theta(\omega^*(f\circ\pi_2))](a)=(f\circ\pi_2)(a)=f(a+J).\end{aligned}$$

Therefore $\widetilde{\theta}$ is a bounded left inverse for $\widetilde{\omega}^*$. □

In section 3, we give some examples of Banach algebras which are module biprojective (biflat), but not biprojective (biflat). However, in the upcoming proposition we show that module biprojectivity (biflatness) of $\mathcal{A}$ with some conditions implies biprojectivity (biflatness) of $\mathcal{A}/J$.

We say that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\alpha_j\}$ in $\mathfrak{A}$ such that $\|\alpha_j\cdot a-a\|\to0$ and $\|a\cdot\alpha_j-a\|\to0$, for each $a\in\mathcal{A}$.

Proposition 2.3. *Let $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left and $\mathcal{A}/J$ be a commutative $\mathfrak{A}$-module such that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. If $\mathcal{A}$ is module biprojective, then $\mathcal{A}/J$ is biprojective.*

Proof. Suppose that $\widetilde{\rho}$ is the the bounded right inverse of $\widetilde{\omega}$. We show that the map

$$\overline{\omega}:(\mathcal{A}/J)\widehat{\otimes}(\mathcal{A}/J)\longrightarrow\mathcal{A}/J;\ ((a+J)\otimes(b+J)\mapsto ab+J)$$

has a right inverse. Consider the map

$$\Gamma:(\mathcal{A}\widehat{\otimes}\mathcal{A})/ker(\pi\otimes\pi)\longrightarrow(\mathcal{A}/J)\widehat{\otimes}(\mathcal{A}/J);\ (a\otimes b+ker(\pi\otimes\pi)\mapsto(a+J)\otimes(b+J)),$$

where $\pi:\mathcal{A}\longrightarrow\mathcal{A}/J$ is the projection map. We have $I\subseteq ker(\pi\otimes\pi)$ because for each $a,b\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$

$$\begin{aligned}(\pi\otimes\pi)(a\cdot\alpha\otimes b-a\otimes\alpha\cdot b)&=(a\cdot\alpha+J)\otimes(b+J)-(a+J)\otimes(\alpha\cdot b+J)\\&=(f(\alpha)a+J)\otimes(b+J)-(a+J)\otimes(f(\alpha)b+J)\\&=f(\alpha)(a+J)\otimes(b+J)-f(\alpha)(a+J)\otimes(b+J)\\&=0.\end{aligned}$$

Hence the map

$$\Phi:(\mathcal{A}\widehat{\otimes}\mathcal{A})/I\longrightarrow(\mathcal{A}\widehat{\otimes}\mathcal{A})/ker(\pi\otimes\pi)\quad(x+I\mapsto x+ker(\pi\otimes\pi))$$

is well-defined. We put $\overline{\rho}=\Gamma\circ\Phi\circ\widetilde{\rho}$. Since $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$, it follows that $\overline{\rho}$ is $\mathbb{C}$-linear. Now if $\widetilde{\rho}(a+J)=\sum_{i=1}^n x_i\otimes y_i+I$, where

$x_i, y_i \in \mathcal{A}$, then

$$\begin{aligned}
\langle \overline{\omega} \circ \overline{\rho}, a + J \rangle &= \langle \overline{\omega} \circ \Gamma \circ \Phi \circ \widetilde{\rho}, a + J \rangle \\
&= \langle \overline{\omega} \circ \Gamma \circ \Phi, \sum_{i=1}^{n} x_i \otimes y_i + I \rangle \\
&= \langle \overline{\omega} \circ \Gamma, \sum_{i=1}^{n} x_i \otimes y_i + ker(\pi \otimes \pi) \rangle \\
&= \langle \overline{\omega}, \sum_{i=1}^{n} (x_i + J) \otimes (y_i + J) \rangle. \\
&= \sum_{i=1}^{n} \omega(x_i \otimes y_i) + J = \widetilde{\omega}(\sum_{i=1}^{n} (x_i \otimes y_i) + I) \\
&= \widetilde{\omega} \circ \widetilde{\rho}(a + J) = a + J.
\end{aligned}$$

Therefore $\overline{\rho}$ is right inverse of $\overline{\omega}$. □

In analogy with Proposition 2.3, we have the following parallel result for the biflatness.

Proposition 2.4. *Suppose that $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left, $\mathcal{A}/J$ is a commutative $\mathfrak{A}$-module, and $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. If $\mathcal{A}$ is module biflat, then $\mathcal{A}/J$ is biflat.*

Proof. Assume that Φ, Γ and $\overline{\omega}$ are as the above. Suppose that $\widehat{\rho}$ is the the bounded left inverse of $\widetilde{\omega}^*$. We prove that the map $\overline{\omega}^* : (\mathcal{A}/J)^* \longrightarrow (\mathcal{A}/J \widehat{\otimes} \mathcal{A}/J)^*$ has a left inverse. From the proof of Proposition 2.3 we see that $\overline{\omega} \circ \Gamma \circ \Phi = \widetilde{\omega}$. Now, for each $\varphi \in (\mathcal{A}/J)^*$ we have

$$\begin{aligned}
(\widehat{\rho} \circ (\Gamma \circ \Phi)^* \circ \overline{\omega}^*)(\varphi) &= (\widehat{\rho} \circ (\Gamma \circ \Phi)^*)(\varphi \circ \overline{\omega}) \\
&= \widehat{\rho}(\varphi \circ \overline{\omega} \circ \Gamma \circ \Phi) \\
&= (\widehat{\rho} \circ \widetilde{\omega}^*)(\varphi) = \varphi.
\end{aligned}$$

Therefore the map $\widehat{\rho} \circ (\Gamma \circ \Phi)^*$ is a left inverse of $\overline{\omega}^*$ which is $\mathbb{C}$-linear (see again the proof of Proposition 2.3). □

In the above Propositions, note that if $\mathcal{A}/J$ has a bounded approximate identity, then $\mathcal{A}/J$ is a commutative $\mathfrak{A}$-module (see Lemma 2.1). One can easily show that module biprojectivity implies module biflatness. In section 3, we shall give an example of a Banach algebra which is module biflat but not module biprojective.

Let X, Y and Z be Banach $\mathcal{A}/J$-$\mathfrak{A}$-modules. Then the short exact sequence

$$\{0\} \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow \{0\} \tag{3}$$

is *admissible* if ψ has a bounded right inverse which is $\mathfrak{A}$-module homomorphism, and *splits* if ψ has a bounded right inverse which is a $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism. Obviously, the short exact sequence (3) is admissible if and only if φ has a bounded left inverse which is $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism. We set $K = \ker \widetilde{\omega}$. If $\mathcal{A}/J$ has a bounded approximate identity, then the following sequences are exact.

$$\{0\} \longrightarrow K \xrightarrow{i} (\mathcal{A} \widehat{\otimes} \mathcal{A})/I \xrightarrow{\widetilde{\omega}} \mathcal{A}/J \longrightarrow \{0\} \tag{4}$$

$$\{0\} \longrightarrow (\mathcal{A}/J)^* \xrightarrow{\widetilde{\omega}^*} (\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^* \xrightarrow{i^*} K^* \longrightarrow \{0\} \tag{5}$$

Definition 2.3. *A bounded net* $\{\widetilde{\xi}_j\}$ *in* $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ *is called a module approximate diagonal if* $\widetilde{\omega}_{\mathcal{A}}(\widetilde{\xi}_j)$ *is a bounded approximate identity of* $\mathcal{A}/J$ *and*

$$\lim_j \|\widetilde{\xi}_j \cdot a - a \cdot \widetilde{\xi}_j\| = 0 \quad (a \in \mathcal{A}).$$

An element $\widetilde{E} \in (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ *is called a module virtual diagonal if*

$$\widetilde{\omega}_{\mathcal{A}}^{**}(\widetilde{E}) \cdot a = \widetilde{a}, \quad \widetilde{E} \cdot a = a \cdot \widetilde{E} \quad (a \in \mathcal{A}),$$

where $\widetilde{a} = a + J^{\perp\perp}$.

Lemma 2.2. *With the above notations:*

(i) *If* $\mathcal{A}/J$ *has an identity, the exact sequences* (4) *and* (5) *are admissible;*

(ii) *If* $\mathcal{A}/J$ *is a commutative* $\mathfrak{A}$*-module and* $\mathcal{A}/J$ *has a bounded approximate identity* (*or* $\mathfrak{A}$ *acts trivially on* $\mathcal{A}$ *from the left and* $\mathcal{A}/J$ *has a bounded approximate identity*), *then the exact sequence* (5) *is admissible.*

Proof. Suppose that $\mathcal{A}/J$ has an identity $e + J$, the map

$$\widetilde{\rho} : \mathcal{A}/J \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})/I; \quad \widetilde{\rho}(a + J) = e \otimes a + I$$

is an $\mathfrak{A}$-module homomorphism and right inverse for $\widetilde{\omega}$. Also, we can show that $\widetilde{\rho}^*$ is left inverse $\widetilde{\omega}^*$ which is an $\mathfrak{A}$-module homomorphism. This complete the proof of part (i). For the part (ii), assume that $(e_j + J)$ is a bounded approximate identity for $\mathcal{A}/J$. Since the net $(e_j \otimes e_j + I)$ is bounded, it has a weak* cluster point M in $((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^{**}$. Define the map $\Psi : ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^* \longrightarrow (\mathcal{A}/J)^*$ via

$$\langle \Psi(\rho), a + J\rangle = \langle M, (a + J) \cdot \rho\rangle, \quad \rho \in ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^*, \ (a \in \mathcal{A}).$$

Since $\mathcal{A}/J$ is commutative $\mathfrak{A}$-module, Ψ is $\mathfrak{A}$-module homomorphism. Now for each $\rho \in (\mathcal{A}/J)^*, a \in \mathcal{A}$, we have

$$\begin{aligned}\langle (\Psi \circ \widetilde{\omega}^*)(\rho), a + J, \rangle &= \langle M, (a + J) \cdot (\rho \circ \widetilde{\omega})\rangle \\ &= \lim_j \langle (a + J) \cdot (\rho \circ \widetilde{\omega}), e_j \otimes e_j + I\rangle \\ &= \lim_j \langle \rho \circ \widetilde{\omega}, e_j \otimes e_j a + I\rangle \\ &= \lim_j \langle \rho, e_j e_j a + J\rangle = \langle \rho, a + J\rangle.\end{aligned}$$

Therefore Ψ is left inverse $\widetilde{\omega}^*$. □

Note that in the above Lemma, if $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left and $\mathcal{A}/J$ has a bounded approximate identity, then $\mathcal{A}/J$ is a commutative $\mathfrak{A}$-module by Lemma 2.1.

The second author in [1, Proposition 2.2] showed that if $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module which is module amenable then it has a bounded approximate identity. The converse is not true for the semigroup algebra even for the classical case of groups. For the free group $\mathbb{F}_2$ on two generators, the group algebra $\ell^1(\mathbb{F}_2)$ has a bounded approximate identity and even an identity, but it is not amenable [7, Example 3.3.62]. The following theorem shows when the module amenability of a Banach algebra $\mathcal{A}$ is equivalent to its module biflatness.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and let $\mathcal{A}/J$ be a commutative Banach $\mathfrak{A}$-module. Suppose $\mathcal{A}$ has a bounded approximate identity. Then $\mathcal{A}$ is $\mathfrak{A}$-module amenable if and only if $\mathcal{A}$ is module biflat.*

Proof. Suppose that $\mathcal{A}$ is module amenable. Since $\mathcal{A}/J$ is commutative $\mathfrak{A}$-module, $\mathcal{A}$ has a module virtual diagonal $\widetilde{E} \in ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^{**}$ [1, Theorem 2.1]. Define the map $\Psi : ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^{*} \longrightarrow (\mathcal{A}/J)^{*}$ via

$$\langle \Psi(\rho), a + J\rangle = \langle \widetilde{E}, (a + J) \cdot \rho\rangle, \quad \rho \in ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^{*}, \ (a \in \mathcal{A}).$$

It is easy to show that Ψ is $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism. Now, similar to the proof of Lemma 2.2, Ψ is a left inverse for $\widetilde{\omega}^{*}$.

Conversely, let (e_j) be a bounded approximate identity for $\mathcal{A}$ and $\widetilde{\varphi}$ is a $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism such that $\widetilde{\varphi}\circ\widetilde{\omega}^{*} = I_{(\mathcal{A}/J)^{*}}$. Suppose that the net $(e_j \otimes e_j + I)$ converge weak* to E in $((\mathcal{A}\widehat{\otimes}\mathcal{A})/I)^{**}$. Hence, $\widetilde{E} = (\widetilde{\varphi}^{*} \circ \widetilde{\omega}^{**})(E)$ is a module virtual diagonal for $\mathcal{A}$. Therefore $\mathcal{A}$ is module amenable. □

Recall that an element $\mathcal{M} \in \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is called a *module diagonal* if $\widetilde{\omega}(\mathcal{M})$ is an identity of $\mathcal{A}/J$ and $a \cdot \mathcal{M} = \mathcal{M} \cdot a$, for all $a \in \mathcal{A}$. The following theorem shows that under some conditions module super-amenability of a Banach algebra $\mathcal{A}$ is equivalent to its module biprojectivity.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and let $\mathcal{A}/J$ be a commutative Banach $\mathfrak{A}$-module. Suppose $\mathcal{A}$ has a bounded approximate identity. Then $\mathcal{A}$ is module super-amenable if and only if $\mathcal{A}/J$ has an identity and $\mathcal{A}$ is module biprojective.*

Proof. Assume that $\mathcal{A}$ is module super-amenable. Then $\mathcal{A}/J$ has an identity by [16, proposition 3.2]. Also $\mathcal{A}$ has a module diagonal $\mathcal{M}$ [16, Theorem 3.5]. Define $\widetilde{\rho} : \mathcal{A}/J \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})/I$ by $\widetilde{\rho}(a + J) := a \cdot \mathcal{M}$, for all $a \in \mathcal{A}$. Since $a \cdot \mathcal{M} = \mathcal{M} \cdot a$, we can show that $\widetilde{\rho}$ is a $\mathcal{A}/J$- $\mathfrak{A}$-module homomorphism. Also

$$(\widetilde{\omega} \circ \widetilde{\rho})(a + J) = \widetilde{\omega}((a + J) \cdot \mathcal{M}) = (a + J) \cdot \widetilde{\omega}(\mathcal{M}) = a + J.$$

Therefore $\widetilde{\rho}$ is a bounded right inverse for $\widetilde{\omega}$.

Conversely, if $e + J$ is an identity for $\mathcal{A}/J$ and $\widetilde{\rho}$ is a bounded right inverse for $\widetilde{\omega}$ which is $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism, then it is easy to show $\widetilde{\rho}(e + J)$ is a module diagonal for $\mathcal{A}$. Now the module super-amenability of $\mathcal{A}$ follows from [16, Theorem 3.5]. □

3. **Module biprojectivity and module biflatness of semigroup algebras**

In this section we find conditions on a (discrete) inverse semigroup S such that the semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module biprojective and biflat, where E (or E_S) is the set of idempotents of S. Throughout this section S is an inverse semigroup with the set of idempotents E, where the order of E is defined by

$$e \leq d \Longleftrightarrow ed = e \quad (e, d \in E).$$

Since E is a commutative subsemigroup of S [12, Theorem V.1.2], actually a semilattice, $\ell^1(E)$ could be regarded as a commutative subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions [1].

Here we let $\ell^1(E)$ acts on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s,\ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

In this case, the ideal J (see section 2) is the closed linear span of

$$\{\delta_{set} - \delta_{st} \quad s,t \in S, e \in E\}.$$

We consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \ \ (s,t \in S).$$

For an inverse semigroup S, the quotient $S/\approx$ is a discrete group (see [2] and [16]). Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S [14] of S [17]. In particular, S is amenable if and only if $S/\approx$ is amenable [8, 14]. As in [19, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. With the notations of section 2, $\ell^1(S)/J$ is a commutative $\ell^1(E)$-bimodule with the following actions:

$$\delta_e \cdot (\delta_s + J) = \delta_s + J,\ (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s \in S, e \in E).$$

Let $k \in \mathbb{N}$. Recall that E satisfies condition D_k [8] if given $f_1, f_2, ..., f_{k+1} \in E$ there exist $e \in E$ and i, j such that

$$1 \leq i < j \leq k+1, f_i e = f_i, f_j e = f_j.$$

Duncan and Namioka in [8, Theorem 16] proved that for any inverse semigroup S, $\ell^1(S)$ has a bounded approximate identity if and only if E satisfies condition D_k for some k. Helemskii showed in [11, Theorem 51] that for any locally compact group G, $L^1(G)$ is biprojective if and only if G is compact (see also [7, Theorem 3.3.32]). The following Theorem is the module version of Helemskii's result for inverse semigroups.

Theorem 3.1. *Let S be an inverse semigroup with the set of idempotents E. If E satisfies condition D_k for some k, then $\ell^1(S)$ is module biprojective as an $\ell^1(E)$-module with trivial left action if and only if S has a finite maximal group homomorphic image G_S.*

Proof. Since G_S is a (discrete) group, $\ell^1(S)/J \cong \ell^1(G_S)$ has an identity. Also $\ell^1(S)$ has a bounded approximate identity [8, Theorem 16]. Hence by Theorem 2.2, $\ell^1(S)$ is module biprojective if and only if $\ell^1(S)$ is module super-amenable. It follows from [16, Theorem 3.7] that $\ell^1(S)$ is module super-amenable if and only if $G_S = S/\approx$ is finite (see also [3, Theorem 3.2]). □

Corollary 3.1. *Let S be an inverse semigroup. If G_S is infinite, then $\ell^1(S)$ is not biprojective.*

Proof. This follows from Proposition 2.1 and Theorem 3.1. □

Theorem 3.2. *Let S be an inverse semigroup with the set of idempotents E. If E satisfies condition D_k for some k, then $\ell^1(S)$ is module biflat as an $\ell^1(E)$-module with trivial left action if and only if S is amenable.*

Proof. Since $\ell^1(S)/J \cong \ell^1(S/\approx)$ has identity and $\ell^1(S)$ has a bounded approximate identity, by Theorem 2.1, $\ell^1(S)$ is module biflat if and only if $\ell^1(S)$ is module amenable. It follows from [1, Theorem 3.1] that $\ell^1(S)$ is module amenable if and only if S is amenable. □

Corollary 3.2. *Let S be an inverse semigroup. If $\ell^1(S)$ is biflat, then S is amenable.*

Proof. This is a consequence of Proposition 2.2 and Theorem 3.2. □

Note that the above Corollary is also a direct consequence of [18, Theorem 3.7].

Example 3.1. Let $\mathbb{N}$ be the commutative semigroup of positive integers. It is known that $\ell^1(\mathbb{N})$ with pointwise multiplication is biprojective, but with convolution it is not biprojective [7, Example 4.1.42]. Now consider $(\mathbb{N}, \vee)$ with maximum operation $m \vee n = \max(m, n)$, then each element of $\mathbb{N}$ is an idempotent, hence $\mathbb{N}/\approx$ is the trivial group with one element. Thus $\ell^1(\mathbb{N})$ is module biprojective (as an $\ell^1(\mathbb{N})$-module) by Theorem 3.1. Since $\mathbb{N}/\approx$ is amenable, $\mathbb{N}$ is amenable. Therefore $\ell^1(\mathbb{N})$ is module biflat by Theorem 3.2. Also we know for an idempotent e in an inverse semigroup S, $(e] = \{f \in E : fe = ef = f\}$. For $S = \mathbb{N}$, for each $n \in \mathbb{N}$ we have $(n] = \{m \in \mathbb{N} : m \geq n\}$. Hence $\mathbb{N}$ is not uniformly locally finite (even not locally finite, see the introduction), so $\ell^1(\mathbb{N})$ is not biflat, so it is not biprojective [18]. We note that $\ell^1(\mathbb{N})$ with pointwise multiplication is biprojective [7, Example 4.1.42 (vii)].

Example 3.2. Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by a and b, that is

$$\mathcal{C} = \{a^m b^n : m, n \geq 0\}, \ \ (a^m b^n)^* = a^n b^m.$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{a^n b^n : n = 0, 1, ...\}$ which is totally ordered with the following order

$$a^n b^n \leq a^m b^m \Longleftrightarrow m \leq n.$$

It is shown in [2] that $\mathcal{C}/\approx$ is isomorphic to the group of integers $\mathbb{Z}$, hence $\mathcal{C}$ is amenable. Therefore $\ell^1(\mathcal{C})$ is module biflat, but not module biprojective. It is easy to see $E_{\mathcal{C}}$ is not uniformly locally finite, so $\ell^1(\mathcal{C})$ is neither biprojective nor biflat.

Example 3.3. Let S be an amenable E-unitary inverse semigroup with infinite number of idempotents (see [12] and [15]). Then $\ell^1(S)$ is module biflat. As a concrete example, the free inverse semigroup $FI(\{x\})$ on a singleton is an amenable, E-unitary inverse semigroup with an infinite number of idempotents, hence $\ell^1(FI(\{x\}))$ is not amenable (see [8]). Also it is easy to see that $FI(\{x\})$ is not uniformly locally finite, hence it is not biflat [18].

Example 3.4. Let G be a group, and let I be a non-empty set. Then for $S = \mathcal{M}(G, I)$, the *Brandt inverse semigroup* corresponding to G and the index set I, it is shown in [16] that $S/\approx$ is the trivial group. Therefore $\ell^1(S)$ is module biprojective. But if index set I is infinite, then $\ell^1(S)$ is not amenable [8, Theorem 12]. Clearly S is uniformly locally finite, indeed $(e] = \{0, e\}$, for each idempotent e of S. Also each maximal subgroup of S at an idempotent is isomorphic to G, hence $\ell^1(S)$ is biflat if and only if G is amenable and $\ell^1(S)$ is biprojective if and only if G is finite [18, Theorem 3.7]. In particular, for any infinite group G, $\ell^1(S)$ is module biprojective, but it is not biprojective.

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