OPTIMALITY AND DUALITY FOR NONSMOOTH SEMIDEFINITE MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS USING CONVEXIFICATORS

B.B. Upadhyay[1], Shubham Kumar Singh[2]

This paper deals with a class of nonsmooth semidefinite multiobjective fractional programming problems (in short, (NSMFP)). Using the properties of convexificators, we deduce Fritz John type necessary criteria of optimality for the considered problem (NSMFP). Further, we employ generalized Cottle constraint qualification to derive Karush-Kuhn-Tucker (in short, KKT) type necessary optimality criteria for (NSMFP). We establish sufficient optimality conditions for (NSMFP) using the assumptions of ∂^*-pseudoconvexity and ∂^*-quasiconvexity on the components of the objective function and constraints involved. Moreover, related to (NSMFP), we formulate the Mond-Weir type dual model (in short, (NSMFD)). Furthermore, we establish several duality results (namely, weak, strong, and strict converse duality) relating (NSMFP) and (NSMFD) under ∂^*-pseudoconvex and ∂^*-quasiconvex assumptions. To demonstrate the results derived in this paper, we provide several nontrivial examples. To the best of our knowledge, optimality criteria and duality results for (NSMFP) have not been explored before using convexificators.

Keywords: semidefinite programming, multiobjective optimization, fractional programming problem, convexificators.

MSC2020: 90C22, 90C46, 90C29, 90C32.

1. Introduction

Multiobjective nonlinear semidefinite programming problem may be considered as a generalization of multiobjective nonlinear optimization problem, where the decision variable is considered to be a symmetric positive semidefinite matrix rather than a vector in Euclidean space. Semidefinite programming (in short, (SDP)) has numerous applications in various modern research fields, including control theory [5], combinatorial optimization [16], and eigenvalue optimization [22]. Shapiro [34] derived necessary and sufficient optimality criteria for nonlinear (SDP) problems under convexity assumptions. Forsgren [13] derived similar results for nonlinear semidefinite programming problems, relaxing the assumption of convexity. moreover, Sun et al. [35] and Sun [36] developed some algorithms to solve (SDP) problems. Yamashita and Yabe [41] introduced some numerical methods to solve (SDP) problems and investigated their algorithmic implications.

Nonsmoothness is a common occurrence in many real-world problems, and as such, several concepts associated with nonsmoothness have been developed and studied extensively, see, for instance, [7, 20], and references cited therein. In the field of optimization, to deal with the nonsmooth nature of mathematical programming problems, concepts of generalized derivatives and subdifferentials have been developed and studied extensively. The

[1]Department of Mathematics, Indian Institute of Technology Patna, India, e-mail: bhooshan@iitp.ac.in

[2]Department of Mathematics, Indian Institute of Technology Patna, India, e-mail: shubham_2321ma09@iitp.ac.in

concept of convexificators was introduced by Demyanov [8]. Jeyakumar and Luc [19] introduced a more sophisticated and novel version of convexificators, which is a closed set but can be nonconvex and unbounded. Dutta and Chandra [10] used the term convexifactors instead of convexificators. Further, Dutta and Chandra [11] introduced ∂^*-pseudoconvex functions by employing the idea of convexificators. Golestani and Nobakhtian [17] derived necessary and sufficient optimality criteria for (SDP) problems by employing Abadie's constraint qualification and the notion of convexificator. Lai et al. [21] established necessary and sufficient optimality criteria for nonlinear semidefinite multiobjective programming problems with vanishing constraints using convexificators. Recently, optimality criteria and duality for (SDP) in terms of convexificators have been explored by Mishra et al. [30]. Further, convexificators have been used to extend various results in nonsmooth analysis, see, for instance, [9, 19], and references cited therein.

Fractional programming is a branch of optimization that deals with problems where the objective functions are expressed as ratios of two functions. Fractional programming problems are a significant class of such problems, and they have applications in various fields, including information theory [24], and engineering design [38]. Bector [4] and Schaible [33] have studied duality in fractional programming. Chandra [6], Egudo [12] and Weir [40] explored several dual models for multiobjective fractional programming problems involving generalized convex functions. For further details and updated survey on fractional programming, we refer the reader to [1, 2, 3, 15, 23, 26, 29, 31, 32], and the references cited therein. Gadhi [14] has established necessary as well as sufficient optimality criteria for multiobjective fractional programming problems in terms of convexificators. Recently, Suneja and Kohli [37] have studied duality for multiobjective fractional programming using convexificators.

Motivated by [14, 17, 21, 27, 37], we consider a class of (NSMFP) and establish necessary and sufficient optimality criteria for the considered problem employing the notions of ∂^*-pseudoconvexity and ∂^*-quasiconvexity on the components of the objective function and the constraints involved. Moreover, we formulate Mond-Weir type dual problem and derive weak, strong, and strict converse duality results, that relate the primal and dual problem under generalized convexity assumptions. The results presented in this paper are an extension and generalization of various existing results in the literature, from Euclidean space to the space of symmetric positive semidefinite matrices.

The organization of the article is in the following manner. In Sect. 2, we recall the basic definitions and preliminaries related to semidefinite matrices and convexificators, that will be used in the sequel. In Sect. 3, we establish necessary and sufficient optimality conditions of the considered problem for a feasible solution to be weakly efficient under the assumptions of ∂^*-pseudoconvexity and ∂^*-quasiconvexity. In Sect. 4, we present a formulation of the Mond-Weir type dual problem related to the primal problem and establish weak, strong, and strict converse duality results relating primal and dual problems. Conclusions are drawn and some future research directions are discussed in Sect. 5.

2. Definitions and Preliminaries

Throughout the article, we denote n-dimensional Euclidean space by $\mathbb{R}^n$. Let $v, w \in \mathbb{R}^n$ $(n \geq 2)$. We use the following convention for equalities and inequalities:

$$v = w \Leftrightarrow v_j = w_j, \text{ for all } j = 1, \ldots, n; \quad v < w \Leftrightarrow v_j < w_j, \text{ for all } j = 1, \ldots, n;$$
$$v \leq w \Leftrightarrow v_j \leq w_j, \text{ for all } j = 1, \ldots, n, \text{ and } v_r < w_r \text{ for some } r \in \{1, \ldots, n\}.$$

The set containing every $n \times n$ symmetric matrix is signified by the symbol $\mathbb{S}^n$. Moreover, symbols $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ denote the sets of all symmetric positive semidefinite matrices and symmetric positive definite matrices, respectively. For $M \in \mathbb{S}^n$, we adopt the following notation:

$$M \succeq 0 \iff M \text{ is positive semi-definite}, \quad M \succ 0 \quad \iff M \text{ is positive definite}.$$

For $M,\, N \in \mathbb{S}^n$, the inner product of M and N is defined by $\langle M, N\rangle := M \bullet N = \mathrm{tr}(MN)$. Corresponding to the inner product which is defined above, we deduce the following Frobenius norm, given by $\| M \|_F = \mathrm{tr}(MM)^{1/2} = \left(\sum_{i,j=1}^{n} |m_{ij}|^2\right)^{1/2}$. Let $\mathcal{D}$ be a nonempty subset of $\mathbb{S}^n$, then by $\mathrm{cl}\,\mathcal{D}$, $\mathrm{co}\,\mathcal{D}$, and $\mathrm{cone}\,\mathcal{D}$, we denote the closure of $\mathcal{D}$, convex hull of $\mathcal{D}$, and the convex cone (including the origin) generated by $\mathcal{D}$, respectively.
The negative polar cone of $\mathcal{D}$ is defined as $\mathcal{D}^- := \{V \in \mathbb{S}^n | \langle V, M\rangle \leq 0,\ \forall M \in \mathcal{D}\}$. The strictly negative polar cone of $\mathcal{D}$ is defined as $\mathcal{D}^s := \{V \in \mathbb{S}^n | \langle V, M\rangle < 0,\ \forall M \in \mathcal{D}\}$. The contingent cone $T(\mathcal{D}, M)$, is defined as

$$T(\mathcal{D}, M) := \{U \in \mathbb{S}^n | \exists \beta_n \downarrow 0 \text{ and } U_n \to U, \text{ such that } M + \beta_n U_n \in \mathcal{D},\ \forall n\}.$$

The following definitions are from [11] and [17].

Definition 2.1. For any function $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$, we define $\mathrm{dom}(\Theta) = \{Z \in \mathbb{S}^n | \Theta(Z) \neq \infty\}$. Let $Z \in \mathrm{dom}(\Theta)$. Then *upper and lower Dini derivatives* of function Θ at Z in the direction $V \in \mathbb{S}^n$ are defined in the following manner

$$\Theta^+(Z; V) := \limsup_{t \downarrow 0} \frac{\Theta(Z + tV) - \Theta(Z)}{t}, \quad \Theta^-(Z; V) := \liminf_{t \downarrow 0} \frac{\Theta(Z + tV) - \Theta(Z)}{t}.$$

Definition 2.2. Let us consider a function $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$. We say that Θ has an *upper convexificator* $\partial^*\Theta(Z) \subset \mathbb{S}^n$ at the point Z, if $\partial^*\Theta(Z)$ is closed and for every $V \in \mathbb{S}^n$, we have $\Theta^-(Z; V) \leq \sup_{\xi \in \partial^*\Theta(Z)} \langle \xi, V\rangle$.

Definition 2.3. Let $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ be any function. We say that Θ has a *lower convexificator* $\partial_*\Theta(Z) \subset \mathbb{S}^n$ at Z, if $\partial_*\Theta(Z)$ is closed and for each $V \in \mathbb{S}^n$, we have $\Theta^+(Z; V) \geq \inf_{\xi \in \partial_*\Theta(Z)} \langle \xi, V\rangle$.

Definition 2.4. The function $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ is said to have a *convexificator* $\partial^*\Theta(Z)$ at Z, if it is both an upper and lower convexificator of Θ at Z.

Definition 2.5. The function $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ is said to have *upper regular convexificator* $\partial^*\Theta(Z) \subset \mathbb{S}^n$ at Z, if $\partial^*\Theta(Z)$ is an upper convexificator of Θ at Z and for each $V \in \mathbb{S}^n$, we have $\Theta^+(Z; V) = \sup_{\xi \in \partial^*\Theta(Z)} \langle \xi, V\rangle$.

Definition 2.6. The function $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ is said to have *lower regular convexificator* $\partial_*\Theta(Z) \subset \mathbb{S}^n$ at Z, if $\partial_*\Theta(Z)$ is an lower convexificator of Θ at Z and for each $V \in \mathbb{S}^n$, we have $\Theta^+(Z; V) = \inf_{\xi \in \partial^*\Theta(Z)} \langle \xi, V\rangle$.

Proposition 2.1. Let $\partial^*\Theta(Z)$ be a convexificator of $\Theta : \mathbb{S}^n \to \mathbb{R}$ at Z. Then, for every $\rho \in \mathbb{R}$, $\rho\partial^*\Theta(Z)$ is a convexificator of $\rho\Theta$ at Z.

We recall the following results and definitions from [19].

Proposition 2.2. Suppose that the functions $\Theta, \mathcal{F} : \mathbb{S}^n \to \mathbb{R}$ admit upper convexificators $\partial^*\Theta(Z)$ and $\partial^*\mathcal{F}(Z)$ at $Z \in \mathbb{S}^n$, respectively. Further, assume that one of the convexificators is upper regular at Z. Then $\partial^*\Theta(Z) + \partial^*\mathcal{F}(Z)$ is an upper convexificator of $\Theta + \mathcal{F}$ at Z.

Proposition 2.3. Let $\mathcal{B}$ be a Banach space and $\Theta = (\Theta_1, \ldots, \Theta_n) : \mathcal{B} \to \mathbb{R}^n$, $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ are continuous functions. Moreover, we assume that for each $i = 1, 2, \ldots, n$, Θ_i admits a bounded convexificator $\partial^*\Theta_i\left(\overline{Z}\right)$ and $\mathcal{F}$ admits a bounded convexificator $\partial^*\mathcal{F}(\Theta\left(\overline{Z}\right))$ at

$\Theta\left(\overline{Z}\right)$. If for each $i = 1, \ldots, n$, $\partial^*\Theta_i$ is upper semicontinuous at $\overline{Z}$ and $\partial^*\mathcal{F}$ is upper semicontinuous at $\Theta\left(\overline{Z}\right)$, then the set $\partial^*(\mathcal{F} \circ \Theta)\left(\overline{Z}\right) := \partial^*\mathcal{F}(\Theta\left(\overline{Z}\right))(\partial^*\Theta_1\left(\overline{Z}\right), \ldots, \partial^*\Theta_n\left(\overline{Z}\right))$, is a convexificator of $\mathcal{F} \circ \Theta$ at $\overline{Z}$.

Definition 2.7. Let $\Theta : \mathbb{S}^n \to \mathbb{R}\cup\{\infty\}$ be any function. Assume that for every $Z \in \mathrm{dom}(\Theta)$ and admits a convexificator $\partial^*\Theta(Z)$. Then, Θ is said to be *∂^*-pseudoconvex* at Z if and only if for each $V \in \mathbb{S}^n$, $\Theta(V) < \Theta(Z) \implies \langle \xi, V - Z\rangle < 0, \ \forall \xi \in \partial^*\Theta(Z)$.

Definition 2.8. Let $\Theta : \mathbb{S}^n \to \mathbb{R}\cup\{\infty\}$ be any function. Assume that for every $Z \in \mathrm{dom}(\Theta)$ and admits a convexificator $\partial^*\Theta(Z)$ at Z. Then, Θ is said to be *strictly ∂^*-pseudoconvex* at Z if and only if for all $V(\neq Z) \in \mathbb{S}^n$, $\Theta(V) \leq \Theta(Z) \implies \langle \xi, V - Z\rangle < 0, \ \forall \xi \in \partial^*\Theta(Z)$.

Definition 2.9. Let $\Theta : \mathbb{S}^n \to \mathbb{R}\cup\{\infty\}$ be any function. Assume that for every $Z \in \mathrm{dom}(\Theta)$ and admits a convexificator $\partial^*\Theta(Z)$ at Z. Then, Θ is said to be *∂^*-quasiconvex* at Z if and only if for all $V \in \mathbb{S}^n$, $\Theta(V) \leq \Theta(Z) \implies \langle \xi, V - Z\rangle \leq 0, \ \forall \xi \in \partial^*\Theta(Z)$.

The following definition and results are from [14], which will be used in the sequel.

Definition 2.10. Let S be a convex cone with non-empty interior of a Banach space $\mathcal{B}$. Then, $\Delta_S : \mathcal{B} \to \mathbb{R}$ is defined as $\Delta_S(z) = d(z, S) - d(z, \mathcal{B} \setminus S)$, where $d(z, S)$ denotes the distance between z and S. It can be easily verified that

$$\Delta_S(z) = \begin{cases} d(z, S), & \text{if } z \in \mathcal{B} \setminus S \\ -d(z, \mathcal{B} \setminus S), & \text{if } z \in S \end{cases} \quad \text{where } d(z, S) = \inf\{\| u - z \| : u \in S\}.$$

Proposition 2.4. Let $S \subsetneq \mathcal{B}$ be a closed convex cone with non-empty interior of a Banach space $\mathcal{B}$. Then the function Δ_S is convex, positively homogeneous and Lipschitz. Moreover, it is negative on the interior of S, null on the boundary of S and positive on the exterior of S. That is $\Delta_S(z) > 0$ if $z \in \mathcal{B} \setminus S$, $\Delta_S(z) < 0$ if $z \in Int(S)$ and $\Delta_S(z) = 0$ if $z \in bd(S)$.

Proposition 2.5. Let $S \subset \mathcal{B}$ be a nonempty closed convex cone of Banach space $\mathcal{B}$ with nonempty interior. Then for all $z \in \mathcal{B}$, $0 \notin \partial\Delta_S(z)$.

Proposition 2.6. Let $\Theta : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ be continuous and $\overline{Z} \in \mathrm{dom}(\Theta)$. Suppose that $\partial^*\Theta\left(\overline{Z}\right)$ is closed and that $\partial^*\Theta$ is upper semicontinuous at $\overline{Z}$. Then, $\partial^* f\left(\overline{Z}\right)$ is a convexificator of Θ at $\overline{Z}$.

Corollary 2.1. Let $\Theta : \mathcal{B} \to \mathbb{R}^n$ be any continuous function and for every $i = 1, 2, \ldots, n$, the function Θ_i admits a upper semicontinuous and bounded convexificator $\partial^*\Theta_i\left(\overline{Z}\right)$ at $\overline{Z}$. Let us define $\mathcal{P}(Z) = max\{\Theta_i(Z) : i = 1, \ldots, n\}$ and $\mathcal{A}\left(\overline{Z}\right) = \left\{i : \Theta_i\left(\overline{Z}\right) = \mathcal{P}\left(\overline{Z}\right)\right\}$, Then, $\mathrm{co}\{\partial^*\Theta_i\left(\overline{Z}\right) : i \in \mathcal{A}\left(\overline{Z}\right)\}$ is a convexificator of $\mathcal{P}$ at $\overline{Z}$.

3. Optimality conditions for (NSMFP)

In this section, we formulate a nonsmooth semidefinite multiobjective fractional programming problem (NSMFP) and establish Fritz John type necessary optimality condition for (NSMFP). Moreover, we establish KKT type necessary and sufficient optimality condition. We consider the following multiobjective fractional programming problem:

$$\begin{aligned} &\text{(NSMFP)} \quad \text{Minimize } \mathcal{K}(Z) = \left(\frac{\Theta_1(Z)}{\mathcal{F}_1(Z)}, \ldots, \frac{\Theta_k(Z)}{\mathcal{F}_k(Z)}\right), \\ &\text{subject to } \mathcal{G}_j(Z) \leq 0, \quad j \in \mathcal{J} := \{1, 2, \ldots, m\}, \end{aligned}$$

where, $\Theta_i : \mathbb{S}^n \to \mathbb{R}$, $\mathcal{F}_i : \mathbb{S}^n \to \mathbb{R}$, $i \in \mathcal{I} := \{1, 2, \ldots, k\}$ and $\mathcal{G}_j : \mathbb{S}^n \to \mathbb{R}$, $j \in \mathcal{J}$ are continuous functions. We define the set of all feasible solutions of (NSMFP) as $\mathscr{F} := \{Z \in \mathbb{S}^n : \mathcal{G}_j(Z) \leq 0, \ j \in \mathcal{J}\}$. We assume that for each $Z \in \mathscr{F}$ and for every $i \in \mathcal{I}$, $\Theta_i(Z) \geq 0$ and $\mathcal{F}_i(Z) > 0$.

Now, we provide the definition of weak efficient and local weak efficient solution of (NSMFP), which is a generalization of the definitions given in Miettinen [25] and Suneja and Kohli [37].

Definition 3.1. Let $\overline{Z} \in \mathscr{F}$ be any arbitrary feasible solution of (NSMFP). Then, $\overline{Z}$ is known as a *weak efficient solution* of (NSMFP), if there does not exist any $Z \in \mathscr{F}$, such that $\mathcal{K}_i(Z) < \mathcal{K}_i\left(\overline{Z}\right),\ \forall i \in \mathcal{I}$.

Definition 3.2. Let $\overline{Z} \in \mathscr{F}$ be any arbitrary feasible solution of (NSMFP). Then, $\overline{Z} \in \mathscr{F}$ is said to be a *local weak efficient solution* of (NSMFP), if there exists a neighbourhood $\mathcal{N}$ of $\overline{Z}$, such that for any $Z \in \mathcal{N} \cap \mathscr{F}$, $\mathcal{K}_i(Z) < \mathcal{K}_i\left(\overline{Z}\right),\ \forall i \in \mathcal{I}$ cannot hold.

In the following theorem, we establish Fritz John type necessary optimality criteria for (NSMFP).

Theorem 3.1. Let $\overline{Z}$ be a local weak efficient solution of (NSMFP). Further, suppose that $\Theta_i : \mathbb{S}^n \to \mathbb{R}$, $\mathcal{F}_i : \mathbb{S}^n \to \mathbb{R}$ and $\mathcal{G}_j : \mathbb{S}^n \to \mathbb{R}$ are continuous and admit bounded convexificator $\partial^*\Theta_i\left(\overline{Z}\right)$, $\partial^*\mathcal{F}_i\left(\overline{Z}\right)$ and $\partial^*\mathcal{G}_j\left(\overline{Z}\right)$ at $\overline{Z}$, respectively. Moreover, we assume that for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $\partial^*\Theta_i\left(\overline{Z}\right)$, $\partial^*\mathcal{F}_i\left(\overline{Z}\right)$ and $\partial^*\mathcal{G}_j\left(\overline{Z}\right)$ are upper semicontinuous at $\overline{Z}$. Then, there exists multipliers $(\beta_1^*, \ldots, \beta_k^*) \in \mathbb{R}_+^k$ and $(\mu_1^*, \ldots, \mu_m^*) \in \mathbb{R}_+^m$, such that

$$0 \in \sum_{i=1}^{k} \beta_i^* \left(\partial^*\Theta_i\left(\overline{Z}\right) - \mathcal{K}_i\left(\overline{Z}\right)\partial^*\mathcal{F}_i\left(\overline{Z}\right)\right) + \sum_{j=1}^{m} \mu_j^*\partial^*\mathcal{G}_j\left(\overline{Z}\right), \tag{1}$$

$$\mu_j^*\mathcal{G}_j\left(\overline{Z}\right) = 0,\ j \in \mathcal{J}, \tag{2}$$

$$\mu_j^* \geq 0 \text{ and } \mathcal{G}_j\left(\overline{Z}\right) \leq 0,\ j \in \mathcal{J}. \tag{3}$$

Proof. From the given hypotheses $\overline{Z} \in \mathscr{F}$ is a local weak efficient solution of (NSMFP). Therefore, there exists a neighbourhood $\mathcal{N}$ of $\overline{Z}$, such that $\mathcal{K}(Z) - \mathcal{K}\left(\overline{Z}\right) \in \mathbb{R}^k \setminus -Int\mathbb{R}_+^k$, $\forall Z \in \mathcal{N} \cap \mathscr{F}$. Let us consider the following auxiliary problem (P1):

$$(P_1)\ \text{Minimize}\ \left(\Theta_1(Z) - \mathcal{K}_1\left(\overline{Z}\right)\mathcal{F}_1(Z), \ldots, \Theta_k(Z) - \mathcal{K}_k\left(\overline{Z}\right)\mathcal{F}_k(Z)\right),$$

$$\text{subject to } Z \in \mathscr{F},\ \text{where } \mathcal{K}_i\left(\overline{Z}\right) = \frac{\Theta_i\left(\overline{Z}\right)}{\mathcal{F}_i\left(\overline{Z}\right)},\ i \in \mathcal{I}.$$

We show that $\overline{Z}$ is also a local weak efficient solution of (P1). On contrary, we assume that there exists $Z_1 \in \mathcal{N} \cap \mathscr{F}$, such that $\left(\Theta_i(Z_1) - \mathcal{K}_i\left(\overline{Z}\right)\mathcal{F}_i(Z_1)\right) - \left(\Theta_i\left(\overline{Z}\right) - \mathcal{K}_i\left(\overline{Z}\right)\mathcal{F}_i\left(\overline{Z}\right)\right) < 0$. Since $\Theta_i\left(\overline{Z}\right) - \mathcal{K}_i\left(\overline{Z}\right)\mathcal{F}_i\left(\overline{Z}\right) = 0,$ we have, $\frac{\Theta_i(Z_1)}{\mathcal{F}_i(Z_1)} - \frac{\Theta_i(\overline{Z})}{\mathcal{F}_i(\overline{Z})} < 0$, which contradicts the fact that $\overline{Z}$ is a local weak efficient solution of (NSMFP). Now we formulate another auxiliary problem (P2) and show that $\overline{Z}$ is a local weak efficient solution of (P2).

$$(P2)\ \text{Minimize}\ \Delta_{-Int\ \mathbb{R}_+^k}\left(\psi_1(Z) - \psi_1\left(\overline{Z}\right), \ldots, \psi_k(Z) - \psi_k\left(\overline{Z}\right)\right),$$

$$\text{subject to } Z \in \mathscr{F},\ \text{where } \psi_i := \Theta_i - \mathcal{K}_i\left(\overline{Z}\right)\mathcal{F}_i \text{ for all } i \in \mathcal{I}.$$

Since $\overline{Z}$ is a local weak efficient solution of (P_1), there exists a neighbourhood $\mathcal{N}$ of $\overline{Z}$, such that $\left(\psi_1(Z) - \psi_1\left(\overline{Z}\right), \ldots, \psi_k(Z) - \psi_k\left(\overline{Z}\right)\right) \in \mathbb{R}^k \setminus -Int\ \mathbb{R}_+^k$, for all $Z \in \mathcal{N} \cap \mathscr{F}$. Hence, by Proposition 2.4, we have $\Delta_{-Int\ \mathbb{R}_+^k}(\psi_1(Z) - \psi_1\left(\overline{Z}\right), \ldots, \psi_k(Z) - \psi_k\left(\overline{Z}\right)) \geq 0$. Since $\Delta_{-Int\ \mathbb{R}_+^k}(0) = 0$, it follows that $\overline{Z}$ is a local weak efficient solution of $(P2)$. Now we consider one more auxiliary problem (P3) and show that an efficient solution of $(P2)$ is also an efficient solution of

$$(P3)\ \text{Minimize}\ h(Z) = max\ \left(\mathcal{G}_0(Z) - \mathcal{G}_0\left(\overline{Z}\right), \mathcal{G}_1(Z), \ldots, \mathcal{G}_m(Z)\right)$$

$$\text{subject to } Z \in \mathbb{S}^n,$$

where $\mathcal{G}_0 := \Delta_{-Int\ \mathbb{R}^k_+}\left(\psi_1 - \psi_1\left(\overline{Z}\right), \ldots, \psi_k - \psi_k\left(\overline{Z}\right)\right)$ and $\mathcal{G}_0\left(\overline{Z}\right) = 0$. From Proposition 4.1 in [19], we have $0 \in \overline{co}(\partial^* h\left(\overline{Z}\right))$. Let us consider the following set $\mathcal{J}\left(\overline{Z}\right) := \{j \in \mathcal{J} : \mathcal{G}_j\left(\overline{Z}\right) = 0\}$. Using Corollary 2.1, there exist $\mu_0, \ldots, \mu_m \geq 0$, such that $\mu_0 + \sum_{j\in\mathcal{J}(\overline{Z})} \mu_j = 1$, and $0 \in \mu_0 \partial^* \mathcal{G}_0\left(\overline{Z}\right) + \sum_{j\in\mathcal{J}(\overline{Z})} \mu_j \partial^* \mathcal{G}_j\left(\overline{Z}\right)$. From Proposition 2.3, there exists $\tau^* \in \partial^* \Delta_{-Int\ \mathbb{R}^k_+}(0)$, such that $0 \in \mu_0 \tau^* \circ \left(\partial^* \psi_1\left(\overline{Z}\right), \ldots, \partial^* \psi_k\left(\overline{Z}\right)\right) + \sum_{j\in\mathcal{J}(\overline{Z})} \mu_j \partial^* \mathcal{G}_j\left(\overline{Z}\right)$. Since $\Delta_{-Int\ \mathbb{R}^k_+}$ is a convex Lipschitz function, we get a convex subdifferential $\partial \Delta_{-Int\ \mathbb{R}^k_+}(0)$ as a convexificator of $\Delta_{-Int\ \mathbb{R}^k_+}$ at 0. Hence, we have $\Delta_{-Int\ \mathbb{R}^k_+}(\tau) - \Delta_{-Int\ \mathbb{R}^k_+}(0) \geq \langle \tau^*, \tau \rangle$, $\forall \tau \in \mathbb{R}^k$. Since $\Delta_{-Int\ \mathbb{R}^k_+}(0) = 0$, we have $\langle \tau^*, \tau \rangle \leq \Delta_{-Int\ \mathbb{R}^k_+}(\tau) = -d(\tau, \mathbb{R}^k \backslash -Int\, \mathbb{R}^k_+) \leq 0$, $\forall \tau \in -\mathbb{R}^k_+$. Consequently, $\tau^* \in \mathbb{R}^k_+$. From Proposition 2.5, we deduce that $\tau^* \in \mathbb{R}^k_+ \setminus \{0\}$. Setting $\mu_j = 0$ for $j \notin \mathcal{J}\left(\overline{Z}\right)$, we get

$$0 \in \mu_0 \sum_{i=1}^{k} \rho_i \left(\partial^* \Theta_i\left(\overline{Z}\right) - \mathcal{K}_i\left(\overline{Z}\right) \partial^* \mathcal{F}_i\left(\overline{Z}\right)\right) + \sum_{i=1}^{m} \mu_j \partial^* \mathcal{G}_j\left(\overline{Z}\right), \quad \mu_0 + \sum_{i=1}^{m} \mu_j = 1.$$

For $(\rho_1, \ldots, \rho_k) \in \mathbb{R}^k_+ \backslash \{0\}$ and $(\mu_0, \mu_1, \ldots, \mu_m) \in \mathbb{R}^m_+$. We set $(\beta^*_1, \ldots, \beta^*_k) := \mu_0(\rho_1, \ldots, \rho_k)$, which completes the proof. □

Now we recall the Generalized Cottle constraint qualification from Mishra [30].

Definition 3.3. Let $\overline{Z} \in \mathscr{F}$. Then the generalized Cottle constraint qualification (in short, (GCCQ)) is satisfied at $\overline{Z}$ if

$$\left(\bigcup_{i\in\mathcal{I}^p_{\mathcal{K}}} co\ \partial^* \Theta_i\left(\overline{Z}\right)\right)^s \bigcap \left(\bigcup_{j\in\mathcal{J}(\overline{Z})} co\ \partial^* \mathcal{G}_j\left(\overline{Z}\right)\right)^s \cap \mathbb{S}^n_+ \neq \emptyset,\ \forall p \in \mathcal{I},$$

where $I^p_{\mathcal{K}} := \{1, \ldots, k\} \setminus \{p\}$, and $\mathcal{J}\left(\overline{Z}\right) = \{j \in \mathcal{J} | \mathcal{G}_j\left(\overline{Z}\right) = 0\}$.

In the following theorem, we establish that under (GCCQ) assumption $\beta^* \neq 0$, which is, in fact, KKT type necessary optimality conditions.

Theorem 3.2. Let us assume that $\overline{Z} \in \mathscr{F}$ is a weak efficient solution of (NSMFP) and all the assumptions of Theorem 3.1 hold. Then, there exists multipliers $\beta^* \in \mathbb{R}^k_+$ and $\mu^* \in \mathbb{R}^m_+$, $(\beta^*, \mu^*) \neq 0$, such that (1),(2) and (3) hold. Moreover, if (GCCQ) satisfies at $\overline{Z}$, then $\beta^* \neq 0$.

Proof. Assuming, $\beta_1 = 0$ without loss of generality. Then, from (1) there exists $\xi_i \in \partial^* \Theta_i\left(\overline{Z}\right) - \mathcal{K}_i\left(\overline{Z}\right) \partial^* \mathcal{F}_i\left(\overline{Z}\right),\ i \in \mathcal{I}^1_{\mathcal{K}},\ \zeta_j \in \partial^* \mathcal{G}_i\left(\overline{Z}\right),\ j \in \mathcal{J}^*\left(\overline{Z}\right)$, such that

$$\sum_{i\in\mathcal{I}^1_{\mathcal{K}}} \beta^*_i \xi_i + \sum_{j\in\mathcal{J}^*(\overline{Z})} \mu^*_j \zeta_j = 0. \tag{4}$$

As (GCCQ) satisfies at $\overline{Z}$, we get a $V \in \mathbb{S}^n_+$, such that $\left\langle \sum_{i\in\mathcal{I}^1_{\mathcal{K}}} \beta^*_i \xi_i + \sum_{j\in\mathcal{J}^*(\overline{Z})} \mu^*_j \zeta_j, V \right\rangle < 0$, which is contradiction to (4). Thus $\beta^*_1 > 0$. Proceeding with similar arguments for each $i \in \mathcal{I}$, we can conclude that $\beta^* \neq 0$. □

To establish sufficient optimality criteria, we shall require few additional assumptions on the objective function and constraints.

Theorem 3.3. Let $\overline{Z} \in \mathscr{F}$ and
(1) The functions ψ_i as defined in Theorem 3.1 are ∂^*-pseudoconvex for every $i \in \mathcal{I}$,
(2) The functions $\mu_j \mathcal{G}_j$ are ∂^*-quasiconvex for all $j \in \mathcal{J}$,
(3) The assumptions of Theorem 3.1 hold at $\overline{Z}$.
Then, $\overline{Z}$ is a weakly efficient solution of (NSMFP).

Proof. On the contrary, we assume that $\overline{Z}$ is not a weakly efficient solution of (NSMFP). Following the arguments similar to those in Theorem 3.1, $\overline{Z}$ is not a weakly efficient solution of (P_1). Then, there exists $Z_1 \in \mathscr{F}$, such that $\left(\psi_1(Z_1) - \psi_1\left(\overline{Z}\right), \ldots, \psi_k(Z_1) - \psi_k\left(\overline{Z}\right)\right) \in -Int\ \mathbb{R}^k_+$, $\mathcal{G}_j(Z_1) \leq 0$, $\forall j \in \mathcal{J}$. Consequently, for every $i \in \mathcal{I}$, we have $\psi_i(Z_1) - \psi_i\left(\overline{Z}\right) < 0$. Since ψ_i is ∂^*-pseudoconvex at $\overline{Z}$, therefore we get $\left\langle \xi_i^*, Z_1 - \overline{Z}\right\rangle < 0,\ \forall \xi_i^* \in \partial^*\psi_i\left(\overline{Z}\right).$

$$\text{Since } \rho \in \mathbb{R}^k_+ \setminus \{0\}, \text{ we have } \left\langle \sum_{i=1}^{p} \rho_i \xi_i^*, Z_1 - \overline{Z}\right\rangle < 0,\ \forall \xi_i^* \in \partial^*\psi_i\left(\overline{Z}\right). \tag{5}$$

On the other hand, from (1), there exits $\xi_i^* \in \partial^*\psi_i\left(\overline{Z}\right)$ and $\zeta_j^* \in \partial^*\mathcal{G}_j\left(\overline{Z}\right)$, such that $\sum_{i=1}^p \beta_i \xi^* + \sum_{j=1}^m \mu_j \zeta_j^* = 0$. Now, from (2) and (3), we get $\mu_j \mathcal{G}_j(Z_1) \leq \mu_j \mathcal{G}_j\left(\overline{Z}\right).$ By employing the ∂^*-quasiconvexity of $\mu_j \mathcal{G}_j,\ j \in \mathcal{J}$, we get $\left\langle \zeta_j', Z_1 - \overline{Z}\right\rangle \leq 0,\ \forall \zeta_j' \in \partial^*(\mu_j \mathcal{G}_j)\left(\overline{Z}\right)$ and from Proposition 2.1, it implies that $\langle \mu_j \zeta_j^*, Z_1 - \overline{Z}\rangle \leq 0,\ \forall \zeta_j^* \in \partial^*\mathcal{G}_j\left(\overline{Z}\right),\ j \in \mathcal{J}.$

$$\text{Since } \mu_j \geq 0, \text{ we have } \left\langle \sum_{j=1}^{m} \mu_j \zeta_j^*, Z_1 - \overline{Z}\right\rangle \leq 0,\ \forall \zeta_j^* \in \partial^*\mathcal{G}_j\left(\overline{Z}\right). \tag{6}$$

$$\text{Adding (5) and (6), we get } \left\langle \sum_{i=1}^{p} \rho_i \xi_i^* + \sum_{j=1}^{m} \mu_j \zeta_j^*, Z_1 - \overline{Z}\right\rangle < 0, \text{ which is a contradiction.}$$

□

Remark 3.1. Theorem 3.1 and Theorem 3.3 generalize and extend Theorem 6 and Theorem 7, respectively, of Gadhi [14], from Euclidean space to space of symmetric positive semidefinite matrices.

The following example illustrates the significance of Theorem 3.1 and Theorem 3.3.

Example 3.1. Let us consider the following problem:

$$\text{(P)} \min \mathcal{K}(Z) = \left(\frac{\Theta_1(Z)}{\mathcal{F}_1(Z)}, \frac{\Theta_2(Z)}{\mathcal{F}_2(Z)}, \frac{\Theta_3(Z)}{\mathcal{F}_3(Z)}\right), \text{s.t. } \mathcal{G}_i(Z) \leq 0\ (i = 1, 2),\ Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2_+,$$

$\Theta_1(Z) = z_1 - z_2 + 1,\ \Theta_2(Z) = -z_1{}^2\left(sin\frac{1}{z_1} - 1,\right),\ \Theta_3(Z) = z_1 - z_2 + 1,$
$\mathcal{F}_1(Z) = -z_2 + 1,\ \mathcal{F}_2(Z) = e^{-z_2},\ \mathcal{F}_3(Z) = e^{z_2},\ \mathcal{G}_1(Z) = z_1(z_1 - 1),\ \mathcal{G}_2(Z) = z_2.$ We denote the feasible set of the problem (P) as $\mathscr{F} = \left\{Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} : 0 \leq z_1 \leq 1,\ z_2 \leq 0\right\}.$

Now, we consider $\overline{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then, $\mathcal{K}_1\left(\overline{Z}\right) = 1, \mathcal{K}_2\left(\overline{Z}\right) = 0,\quad \Theta_1(Z) - \mathcal{K}_1\left(\overline{Z}\right)\mathcal{F}_1(Z) = Z_1$. Thus, $\overline{Z}$ is a solution of the problem $\min\{\Theta_1(Z) - \mathcal{K}_1\left(\overline{Z}\right)\mathcal{F}_1(Z) : Z \in \mathscr{F}\}$. Hence, $\overline{Z}$ is a solution of the problem $\min\left\{\frac{\Theta_1(Z)}{\mathcal{F}_1(Z)} : Z \in \mathscr{F}\right\}$. Therefore, $\overline{Z}$ is a weak efficient solution of Problem (P_4). It can be verified that $\partial^*\Theta_1(0) = \left\{\begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}\right\}$, $\partial^*\Theta_2(0) = \left\{\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$, $\partial^*\Theta_3(0) = \left\{\begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}\right\}$, $\partial^*\mathcal{F}_1(0) = \left\{\begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}\right\}$, $\partial^*\mathcal{F}_2(0) = \left\{\begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}\right\}$, $\partial^*\mathcal{F}_3(0) = \left\{\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right\}$, $\partial^*\mathcal{G}_1(0) = \left\{\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$, $\partial^*\mathcal{G}_2(0) = \left\{\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right\}$. We can verify that (GCCQ) is satisfied at $\overline{Z}$. Moreover, for the multipliers $\beta_1 = 1,\ \beta_2 = 1,\ \beta_3 = 1$, and $\mu_1 = 1,\ \mu_2 = 2$, all the hypothesis of Theorem 3.2 are satisfied. It is worthwhile to note that $\overline{Z}$ is a local weak efficient solution of the problem (P).

4. **Duality**

In this section, we formulate a Mond-Weir type dual model (NSMFD) related to (NSMFP). Moreover we derive weak, strong and strict converse duality results relating (NSMFD) and (NSMFP). The Mond-Weir type dual problem (NSMFD) is given as follows:

$$\text{Maximize } \mathcal{K}(U) = \left(\frac{\Theta_1(U)}{\mathcal{F}_1(U)}, \frac{\Theta_2(U)}{\mathcal{F}_2(U)}, \ldots, \frac{\Theta_k(U)}{\mathcal{F}_k(U)} \right),$$

$$\text{subject to } 0 \in \sum_{i=1}^{k} \rho_i(\partial^*\Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U)) + \sum_{i=1}^{m} \gamma_j \partial^* \mathcal{G}_j(U),\ \gamma_j \mathcal{G}_j(U) \geq 0,\ \forall j \in \mathcal{J},$$

where $\rho \in \mathbb{R}^k_+ \setminus \{0\}$, $\gamma \in \mathbb{R}^m_+$, $\sigma_i = (K)_i(U) = \frac{\Theta_i(U)}{\mathcal{F}_i(U)}$, $\forall i \in \mathcal{I}$. The set containing all feasible solutions of the problem (NSMFD) is denoted by $\mathscr{F}_D$ and defined as $\mathscr{F}_D = \{(U, \rho, \gamma, \sigma) \in \mathbb{S}^n \times \mathbb{R}^k_+ \times \mathbb{R}^m_+ \times \mathbb{R}^k : 0 \in \sum_{i=1}^{k} \rho_i(\partial^*\Theta_i(U) - \sigma_i \partial^* g(U)) + \sum_{i=1}^{m} \gamma_j \partial^* \mathcal{G}_j(U), \gamma_j \mathcal{G}_j(U) \geq 0,\ \forall j \in \mathcal{J}\}$.

In the following theorem, we establish the weak duality result for (NSMFD).

Theorem 4.1. Let $Z \in \mathscr{F}$ and $(U, \rho, \gamma, \sigma) \in \mathscr{F}_D$. Further, for every $i \in \mathcal{I}$, we assume that $\partial^*\Theta_i(U)$ is an upper regular convexificator of Θ_i at U and $\partial^*\mathcal{F}_i(U)$ is a lower regular convexificator of $\mathcal{F}_i$ at U. If for $i \in \mathcal{I}$, functions $\Theta_i - \sigma_i \mathcal{F}_i$ are ∂^*-pseudoconvex at U and for $j \in \mathcal{J}$, functions $\gamma_j \mathcal{G}_j$ are ∂^*-quasiconvex at U, then $\mathcal{K}(Z) < \mathcal{K}(U)$ can not hold.

Proof. Using the assumptions of the given theorem and Propositions 2.1 and 2.2, it follows that $\partial^*\Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U)$ is a convexificator of $\Theta_i - \sigma_i \mathcal{F}_i$ at U, for every $i \in \mathcal{I}$. Now, let us assume to the contrary that $\mathcal{K}(Z) < \mathcal{K}(U)$. Then, $\Theta_i(Z) - \sigma_i \mathcal{F}_i(Z) < 0$, $\forall i \in \mathcal{I}$, where $\sigma_i = \mathcal{K}_i(U) = \frac{\Theta_i(U)}{\mathcal{F}_i(U)}$, $\forall i \in \mathcal{I}$. Since $(U, \rho, \gamma, \sigma) \in \mathscr{F}_D$, therefore, there exists $\xi_i \in \partial^*\Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U)$, $\zeta_j \in \partial^* \mathcal{G}_j(U)$, such that

$$\sum_{i=1}^{k} \rho_i \xi_i + \sum_{j=1}^{m} \gamma_j \zeta_j = 0. \tag{7}$$

Since $Z \in \mathscr{F}$ and $(U, \rho, \gamma, \sigma) \in \mathscr{F}_D$, it follows that

$$\Theta_i(Z) - \sigma_i \mathcal{F}_i(Z) < 0 = \Theta_i(U) - \sigma_i \mathcal{F}_i(U),\ \forall i \in \mathcal{I}, \tag{8}$$

$$\gamma_j \mathcal{G}_j(Z) \leq 0 \leq \gamma_j \mathcal{G}_j(U),\ \forall j \in \mathcal{J}. \tag{9}$$

Since $\Theta_i - \sigma_i \mathcal{F}_i$ is ∂^*-pseudoconvex at U, from (8), we get $\langle \xi_i, Z - U \rangle < 0$, $\forall \xi_i \in \partial^*\Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U)$, $\forall i \in \mathcal{I}$.

$$\text{Since } \rho \in \mathbb{R}^k_+ \setminus \{0\}, \left\langle \sum_{i=1}^{k} \rho_i \xi_i, Z - U \right\rangle < 0,\ \forall \xi_i \in \partial^*\Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U),\ \forall i \in \mathcal{I}. \tag{10}$$

Again using ∂^*-quasiconvexity of $\gamma_j \mathcal{G}_j$ and (9), we have $\langle \zeta_j', Z - U \rangle \leq 0, \forall \zeta_j' \in \partial^*(\gamma_j \mathcal{G}_j)(U)$, $\forall j \in \mathcal{J}$. From Proposition 2.1, it follows that $\langle \gamma_j \zeta_j, Z - U \rangle \leq 0$, $\forall \zeta_j \in \partial^* \mathcal{G}_j(U)$, $\forall j \in \mathcal{J}$.

$$\text{Since } \gamma_j \geq 0, \text{ for all } j \in \mathcal{J}, \text{ hence } \left\langle \sum_{j=1}^{m} \gamma_j \zeta_j, Z - U \right\rangle \leq 0,\ \forall \zeta_j \in \partial^* \mathcal{G}_j(U),\ \forall j \in \mathcal{J}. \tag{11}$$

$$\text{Adding (10) and (11) gives us } \left\langle \sum_{i=1}^{k} \rho_i \xi_i + \sum_{j=1}^{m} \gamma_j \zeta_j, Z - U \right\rangle < 0, \text{ which contradicts (7).}$$

□

In the following theorem, we derive a strong duality result for (NSMFD).

Theorem 4.2. *Let Z^* be a weak efficient solution of (NSMFP). Moreover, we assume that all the assumptions of Theorem 3.2 are satisfied. Then, there exists $(\rho^*, \gamma^*, \sigma^*) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^k$, such that $(Z^*, \rho^*, \gamma^*, \sigma^*) \in \mathscr{F}_D$. If for every $Z \in \mathscr{F}$ and for every $(U, \rho, \gamma, \sigma) \in \mathscr{F}_D$, Theorem 4.1 hold, then $(Z^*, \rho^*, \gamma^*, \sigma^*)$ is a weak efficient solution of (NSMFD).*

Proof. Since Z^* is a weak efficient solution of (NSMFP) and all the hypotheses of Theorem 3.2 are satisfied, there exist vectors $\rho^* \in \mathbb{R}^k_+ \setminus \{0\}$ and $\gamma^* \in \mathbb{R}^m_+$, such that (1), (2), and (3) hold. That is,

$$0 \in \sum_{i=1}^{k} \rho_i^* (\partial^* \Theta_i(Z^*) - \Theta_i(Z^*) \partial^* \mathcal{F}_i(Z^*)) + \sum_{j=1}^{m} \gamma_j^* \partial^* \mathcal{G}_j(Z^*),$$

$$\gamma_j^* \mathcal{G}_j(Z^*) = 0,\ \forall j \in \mathcal{J},\quad \gamma_j^* \geq 0,\ \mathcal{G}_j(Z^*) \leq 0,\ \forall j \in \mathcal{J},\quad \mathcal{K}_i(Z^*) = \frac{\Theta_i(Z^*)}{\mathcal{F}_i(Z^*)},\ \forall i \in \mathcal{I}.$$

Hence, $(Z^*, \rho^*, \gamma^*, \sigma^*) \in \mathscr{F}_D$. On contrary, we assume that $(Z^*, \rho^*, \gamma^*, \sigma^*)$ is not a weak efficient solution of (NSMFD). Then, there exists $(U, \rho, \gamma, \sigma) \in \mathscr{F}_D$, such that $\mathcal{K}(Z^*) < \mathcal{K}(U)$, which is a contradiction to Theorem 4.1. □

Remark 4.1. Theorem 4.1 and Theorem 4.2 generalize and extend Theorem 4.1 and Theorem 4.2, respectively, of Suneja and Kohli [37] from Euclidean space to space of symmetric semidefinite matrices.

In the following theorem, we derive strict converse duality for the problem (NSMFD).

Theorem 4.3. Let $\overline{Z} \in \mathscr{F}$ and $(U, \rho, \gamma, \overline{\sigma}) \in \mathscr{F}_D$, such that $\overline{\sigma} = \mathcal{K}(\overline{Z})$. Further, we assume that $\sum_{i=1}^{p} \rho_i(\Theta_i - \sigma_i \mathcal{F}_i)$ is ∂^*-quasiconvex and for every $j \in \mathcal{J}$, $\gamma_j \mathcal{G}_j$ is strictly ∂^*-pseudoconvex and $\Theta_i(U) - \sigma_i \mathcal{F}_i(U) \geq 0$, $i \in \mathcal{I}$. Then $\overline{Z} = U$.

Proof. Let $\overline{Z} \neq U$. Since $(U, \rho, \gamma, \overline{\sigma}) \in \mathscr{F}_D$, we have

$$0 \in \sum_{i=1}^{k} \rho_i (\partial^* \Theta_i(U) - \overline{\sigma}_i \partial^* \mathcal{F}_i(U)) + \sum_{j=1}^{m} \gamma_j \partial^* \mathcal{G}_j(U),\quad \gamma_j \mathcal{G}_j(U) \geq 0,\ \forall j \in \mathcal{J}. \tag{12}$$

Since $\gamma_j \geq 0$ and $\mathcal{G}_j(\overline{Z}) \leq 0$. Therefore, $\gamma_j \mathcal{G}_j(\overline{Z}) \leq \gamma_j \mathcal{G}_j(U)$. Since $\gamma_j \mathcal{G}_j$ is assumed to be strictly ∂^*-pseudoconvex at U, it follows that $\langle \zeta_j', \overline{Z} - U \rangle < 0,\ \ \forall \zeta_j' \in \partial^*(\gamma_j \mathcal{G}_j)(U), \forall j \in \mathcal{J}$. Now, using Proposition 2.1, it implies that $\langle \gamma_j \zeta_j, \overline{Z} - U \rangle < 0,\ \forall \zeta_j \in \partial^* \mathcal{G}_j(U),\ \forall j \in \mathcal{J}$. Since $\gamma_j \geq 0,\ j \in \mathcal{J}$, it follows that

$$\left\langle \sum_{j=1}^{m} \gamma_j \zeta_j, \overline{Z} - U \right\rangle < 0,\ \forall \zeta_j \in \partial^* \mathcal{G}_j(U),\ \forall j \in \mathcal{J}. \tag{13}$$

From (12) and (13), we get $\left\langle \sum_{i=1}^{k} \rho_i \xi_i, \overline{Z} - U \right\rangle > 0,\ \forall \xi_i \in \partial^* \Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U),\ \forall i \in \mathcal{I}$. Then by the ∂^*-quasiconvexity of $\sum_{i=1}^{p} \rho_i(\Theta_i - \sigma_i \mathcal{F}_i)$, we have

$$\sum_{i=1}^{p} \rho_i(\Theta_i(\overline{Z}) - \overline{\sigma}_i \mathcal{F}_i(\overline{Z})) > \sum_{i=1}^{p} \rho_i(\Theta_i(U) - \overline{\sigma}_i \mathcal{F}_i(U)). \tag{14}$$

Since $\Theta(\overline{Z}) = \overline{\sigma}$, the left hand side of (14) becomes zero. Therefore, $\sum_{i=1}^{p} \rho_i(\Theta_i(U) - \overline{\sigma}_i \mathcal{F}_i(U)) < 0$, which is a contradiction to $\Theta_i(U) - \sigma_i \mathcal{F}_i(U) \geq 0$. Hence, $\overline{Z} = U$. □

We present the following example to demonstrate the importance of weak and strong duality results.

Example 4.1. Let us consider the following problem

$$\text{(NP) Minimize} \left(\frac{\Theta_1(Z)}{\mathcal{F}_1(Z)}, \frac{\Theta_2(Z)}{\mathcal{F}_2(Z)} \right),\quad s.t.\ \mathcal{G}_i(Z) \leq 0\ (i = 1, 2),\ \text{where},\ Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2_+,$$

$\Theta_1(Z) = z_1 - z_2 + 1,\ \Theta_2(Z) = -{z_1}^2 \left(sin\frac{1}{z_1} - 1\right),\ \mathcal{F}_1(Z) = e^{z_1},\ \mathcal{F}_2(Z) = e^{-z_2},\ \mathcal{G}_1(Z) = z_1(1+z_1),\ \mathcal{G}_2(Z) = z_2$. The feasible set of (NP) is $\mathscr{F} = \left\{ Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} : z_1 = 0,\ z_2 \leq 0 \right\}$. The Mond-Weir type dual (NPD) corresponding to the problem (NP) is given by

$$\text{(NPD) Maximize } \left(\frac{\Theta_1(Z)}{\mathcal{F}_1(Z)}, \frac{\Theta_1(Z)}{\mathcal{F}_1(Z)}\right),$$

$$\text{subject to } 0 \in \sum_{i=1}^{2} \rho_i(\partial^*\Theta_i(U) - \sigma_i \partial^* \mathcal{F}_i(U)) + \sum_{j=1}^{2} \gamma_j \partial^* \mathcal{G}_j(U), \quad \gamma_j \mathcal{G}_j(U) \geq 0\ (j = 1,2),$$

where $\rho_i > 0,\ \gamma_i \geq 0$ *and* $\sigma_i = \frac{\Theta_i(U)}{\mathcal{F}_i(U)},\ (i = 1,2)$. It can be verified that $(U, \rho_1, \rho_2, \gamma_1, \gamma_2) = (0,1,1,1,1)$ is feasible solution for (NPD), and $\partial^*\Theta_1(0) = \left\{ \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix} \right\}$, $\partial^*\Theta_2(0) = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ $\partial^*\mathcal{F}_1(0) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, $\partial^*\mathcal{F}_2(0) = \left\{ \begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix} \right\}$, $\partial^*\mathcal{G}_1(0) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, $\partial^*\mathcal{G}_2(0) = \left\{ \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right\}$. Moreover, for $v_1 = \frac{\Theta_1(U)}{\mathcal{F}_1(U)} = 1,\ v_2 = \frac{\Theta_1(U)}{\mathcal{F}_1(U)} = 0$, functions $\Theta_1 - v_1\mathcal{F}_1$ and $\Theta_2 - v_2\mathcal{F}_2$ are ∂^*-pseudoconvex at $U = 0$, and $\gamma_1\mathcal{G}_1$ and $\gamma_2\mathcal{G}_2$ are ∂^*-quasiconvex at $U = 0$. We can verify that, for the feasible solution $Z \in \mathscr{F}$ and $(U, \rho_1, \rho_2, \gamma_1, \gamma_2) = (0,1,1,1,1)$, the following holds $\left(\frac{\Theta_1(Z)}{\mathcal{F}_1(Z)}, \frac{\Theta_2(Z)}{\mathcal{F}_2(Z)}\right) \not< \left(\frac{\Theta_1(U)}{\mathcal{F}_1(U)}, \frac{\Theta_2(U)}{\mathcal{F}_2(U)}\right)$.

5. **Conclusions and Future Research Directions**

This paper is devoted to the study of a class of (NSMFP). We have established the KKT-type necessary and sufficient optimality conditions for (NSMFP) under the ∂^*-pseudoconvexity and ∂^*-quasiconvexity assumptions. The necessary and sufficient optimality conditions deduced in this paper extend the necessary as well as sufficient optimality conditions derived by Gadhi [14], from the vector ($\mathbb{R}^n$) to the symmetric positive semidefinite matrix ($\mathbb{S}^n_+$). Moreover, Mond-Weir type dual model (NSMFD) related to (NSMFP) has been formulated. Further, we have established the weak, strong, and strict converse duality results. The duality results deduced in this paper are generalizations of the corresponding results derived by Suneja and Kohli [37] from the setting of Euclidean spaces to the framework of the space of all symmetric positive semidefinite matrices. Non-trivial examples have been presented to demonstrate the significance of the results derived in this paper.

For future work, it would be interesting to explore optimality conditions and duality for (NSMFP) with mixed constraints. Moreover, we intend to study constraint qualifications and optimality conditions for semi-infinite semidefinite programming problems.

Acknowledgments

The authors would like to thank the anonymous referees for their careful reading of the paper and constructivesuggestions,which have substantiallyimproved the paper in itspresen

REFERENCES

[1] *T. Antczak and A. Pitea*, Parametric approach to multitime multiobjective fractional variational problems under (F, ρ)-convexity, Optim. Control Appl. Methods., **37**(2016), 831-847.

[2] *T. Antczak, S.K. Mishra and B.B. Upadhyay*, First order duality for a new class of nonconvex semi-infinite minimax fractional programming problems, J. Adv. Math. Stud., **9**(2016), 132-162.

[3] *T. Antczak, S.K. Mishra and B.B. Upadhyay*, Optimality conditions and duality for generalized fractional minimax programming involving locally Lipschitz (b,ϕ,ψ,ρ)-univex functions, Control Cybern., **47**(2018), 5-32.

[4] *C.R. Bector*, Duality in nonlinear fractional programming, ZOR-Math. Methods Oper. Res., **17**(1973), 183-193.

[5] *S. Boyd L. El Ghaoui, E. Feron and V. Balakrishnan*, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.

[6] *S. Chandra, B.D. Craven and B. Mond*, Vector-valued Lagrangian and multiobjective fractional programming duality, Numer. Funct. Anal. Optim., **11**(1990), 239-254.

[7] *F.H. Clarke*, Optimization and Nonsmooth Analysis, Wiley Interscience, New York 1983.

[8] *V.F. Demyanov*, Convexification and concavification of positively homogeneous functions by the same family of linear functions, Technical Report, University of Pisa, Italy, (1994), 1–11.

[9] *V.F. Demyanov and V. Jeyakumar*, Hunting for a smaller convex subdifferential, J. Glob. Optim., **10**(1997), 305–326.

[10] *J. Dutta and S. Chandra*, Convexifactors, generalized convexity, and optimality conditions, J. Optim. Theory Appl., **113**(2002), 41-64.

[11] *J. Dutta and S. Chandra*, Convexifactors, generalized convexity and vector optimization, Optimization, **53**(2004), 77-94.

[12] *R.R. Egudo*, Multiobjective fractional duality, Bull. Aust. Math. Soc., **37**(1988), 367-378.

[13] *A. Forsgren*, Optimality conditions for nonconvex semidefinite programming, Math. Program., **88**(2000), 105-128.

[14] *N. Gadhi*, Necessary and sufficient optimality conditions for fractional multi-objective problems, Optimization, **57**(2008), 527-537.

[15] *A. Ghosh, B.B. Upadhyay and I.M. Stancu-Minasian*, Pareto efficiency criteria and duality for multiobjective fractional programming problems with equilibrium constraints on Hadamard manifolds, Mathematics, **11**(2023), 3469.

[16] *M.X. Goemans*, Semidefinite programming in combinatorial optimization, Math. Program., **79**(1997), 143-161.

[17] *M. Golestani and S. Nobakhtian*, Optimality conditions for nonsmooth semidefinite programming via convexificators, Positivity, **19**(2015), 221-236.

[18] *J.B. Hiriart-Urruty*, Tangent cones, generalized gradients and mathematical programming in Banach spaces, Math. Oper. Res., **4**(1979), 79-97.

[19] *V. Jeyakumar and D.T. Luc*, Nonsmooth calculus, minimality, and monotonicity of convexificators, J. Optim. Theory Appl.,**101**(1999), 599-621.

[20] *V. Jeyakumar and D.T. Luc*, Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization, SIAM J. Control Optim., **36**(1998), 1815-1832.

[21] *K.K. Lai, M. Hassan, S.K. Singh, J.K. Maurya and S.K. Mishra*, Semidefinite multiobjective mathematical programming problems with vanishing constraints using convexificators, Fractal Fract., **6**(2022), 3.

[22] *A.S. Lewis and M.L. Overton*, Eigenvalue optimization, Acta Numer., **5**(1996), 149-190.

[23] *J.C. Liu*, Optimality and duality for multiobjective fractional programming involving nonsmooth pseudoinvex functions, Optimization, **37**(1996), 27-39.

[24] *B. Meister and W. Oettli*, On the capacity of a discrete, constant channel, Inf. Control., **11**(1967), 341-351.

[25] *K. Miettinen*, Nonlinear Multiobjective Optimization, Springer Science & Business Media, 1999.

[26] *S.K. Mishra and B.B. Upadhyay*, Efficiency and duality in nonsmooth multiobjective fractional programming involving η-pseudolinear functions, Yugosl. J. Oper. Res., **22**(2012), 3-18.

[27] *S.K. Mishra and B.B. Upadhyay*, Duality in nonsmooth multiobjective programming involving

η-pseudolinear functions, Indian J. Indust. Appl. Math., **3**(2012), 152-161.
[28] *S.K. Mishra and B.B. Upadhyay*, Nonsmooth minimax fractional programming involving η-pseudolinear functions, Optimization, **63**(2014), 775-788.
[29] *S.K. Mishra and B.B. Upadhyay*, Pseudolinear Functions and Optimization, Chapman and Hall/CRC, London, 2019.
[30] *S.K. Mishra, R. Kumar, V. Laha and J.K. Maurya*, Optimality and duality for semidefinite multiobjective programming problems using convexificators, J. Appl. Numer. Optim., **4**(2022), 103-118.
[31] *S. Nobakhtian*, Optimality and duality for nonsmooth multiobjective fractional programming with mixed constraints, J. Glob. Optim., **41**(2008), 103–115.
[32] *A. Pitea and M. Postolache*, Minimization of vectors of curvilinear functionals on the second order jet bundle: sufficient efficiency conditions, Optim. Lett., **6**(2012), 1657-1669.
[33] *S. Schaible*, Fractional programming, Handbook of Global Optimization, Kluwer Academic Publishers, (1995), 495-608.
[34] *A. Shapiro*, First and second order analysis of nonlinear semidefinite programs, Math. Program., **77**(1997), 301-320.
[35] *D. Sun, J. Sun and L. Zhang*, The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming, Math. Program., **114**(2008), 349-391.
[36] *J. Sun*, On methods for solving nonlinear semidefinite optimization problems, Numer. Algebra Control Optim., **1**(2011), 1-14.
[37] *S.K. Suneja and B. Kohli*, Duality for multiobjective fractional programming problem using convexifactors, Math. Sci., **7**(2013), 1-8.
[38] *J.F. Tsai*, Global optimization of nonlinear fractional programming problems in engineering design, Eng. Optim. **37**(2005), 399-409.
[39] *B.B. Upadhyay, T. Antczak, S.K. Mishra and K. Shukla*, Nondifferentiable generalized minimax fractional programming under (ϕ, ρ)-invexity, Yugosl. J. Oper. Res., **32**(2022), 3-27.
[40] *T. Weir*, A duality theorem for a multiple objective fractional optimization problem, Bull. Aust. Math., **34**(1986), 415-425.
[41] *H. Yamashita and H. Yabe*, A survey of numerical methods for nonlinear semidefinite programming, J. Oper. Res. Soc. Jpn., **58**(2015), 24-60.