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ANALYTICALLY NON-CLASSIC DESIGN OF A PIEZOELECTRIC RESONATOR

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Nowadays, with regard to many different applications of piezoelectric high power ultrasonic transducers in many different fields of industries, the correct design of these tools is of crucial importance. In typical analytical methods, in the absence of energy loss assumption, using the equations of wave transmission, the dimensions of different parts of transducer consisting of backing and matching parts are being determined in a specific frequency. But with regard to high frequencies that ultrasonic transducers are working on (>20 kHz), the effect of energy loss can be dominant and this part of energy loss is converted to heat. In this paper, considering the damping parameter in the wave equation, differential equations of damped longitudinal displacement and stress have been obtained and with considering the boundary conditions and using separation of variable techniques, differential equations are solved. Present vibrational energy loss in all parts of ultrasonic transducer, the dimensional equations of all part of this tool have been presented. The obtained new analytical formulations are the functions of damping parameter and some other parameters, respectively, and by correct measurement of these values and parameters, the dimensions of different parts of the piezoelectric ultrasonic transducers can be determined very accurately and reliably.

Keywords: High power ultrasonic transducers, damping, piezoelectric, wave transmission in solid material

1. Introduction

Nowadays, with regard to many different applications of piezoelectric high power ultrasonic transducers in many different fields of industries, the correct design of these tools is crucially important. Use of ultrasonic transducers for various applications of atomizers [1], ultrasonic cleaning [2], sonochemistry and sonoluminescence [3], medical ultrasounds [4], ultrasonic peening and welding devices [5], ultrasonic assisted forming [6], ultrasonic motor [7], ultrasonic lubrication [8] and ultrasonic nondestructive evaluations [9] etc. has long been a field of interest. In typical analytical methods, with the absence of energy loss assumption, using the equations of wave transmission, the dimensions of different parts of transducer consisting of backing and matching parts are being determined in a specific frequency.

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Considering high frequencies that ultrasonic transducers are working on (>20 kHz), the effect of the energy loss can be dominant and this part of energy loss is converted to heat [10-15].

In classical or one-dimensional design procedure of ultrasonic transducers, based on wave transmission theory, the total length of transducer is equal to half-length of wave [16-18]. It should be mentioned that the one dimensional formulation is only valid for thin rods with diameter<<λ/4.

In this study, the transducer has a node and two anti-nodes along the transducer, which are solved by differential equation and consideration of boundary conditions (equality of displacement and force), then, the length of parts of the transducer has been determined [19-20].

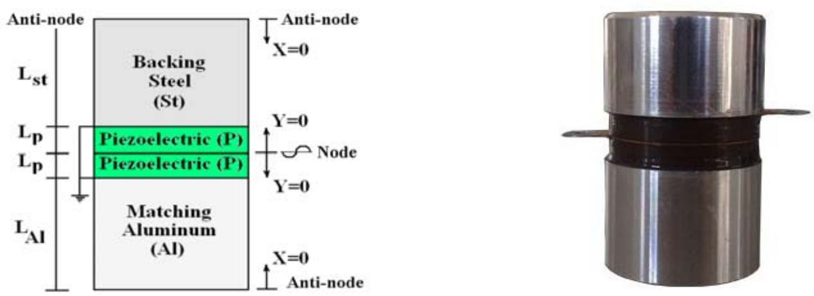

Fig. 1. Piezoelectric ultrasonic transducer

In this paper, unlike classical formulations, damping parameter has been considered in wave equation and by using imaginary algebraic and the separation of variables techniques, differential equations are solved and the relation of damped displacement has been extracted.

According to the extracted relations and with considering boundary conditions, novel relations for determining the correct length of parts are obtained. New relations are the function of different parameters or material properties, for example, the module of elasticity, material density, sound speed and damping parameters. These parameters can be determined from the experimental measurements or standard tables.

2. The differential equation of wave transmission

The differential equation of wave transmission or longitudinal displacement is obtained based on the vibration of all parts of transducer and boundary conditions of which these elements vibrate.

Relations of displacement in backing and matching are calculated from the vibration of damped free-free cylindrical rod, and according to the vibration of damped clamped-free cylindrical rod, the vibration equation of the two piezoelectric rings are calculated. In non-classical differential equations of transducer's element, viscous damping model has been used.

2.1. Used model for energy loss

Viscous damping model as the energy loss model in solving differential equations of damped motion of wave in solid materials has been considered.

Viscous damping model is one of the simplest models in structural analysis and vibration of damped systems [21]. In this model, free longitudinal vibration (as governing equation of its motion based on wave equation) has been seen and considering the viscous damping, the main damped differential equation is extracted.

Figure 1 shows the cylindrical rod that the length of which equals nλ/2 and vibrates in one of the resonance vibrational modes. Figure 1 exhibits one of the vibrating elements, all forces and stresses of which are shown on this schematic figure. According to the damped longitudinal vibration of this rod, the differential equation of this type of vibration can be calculated as follows [21-23]:

$$\frac{\partial^2 U}{\partial t^2} + \delta \frac{\partial U}{\partial t} - c^2 \frac{\partial^2 U}{\partial x^2} = 0 \tag{1}$$

Where U is the particle displacement, t is the time, x is a coordinate, c is the infinite sound speed in bar, δ is the energy loss factor in unit mass (equivalent viscous damping coefficient).

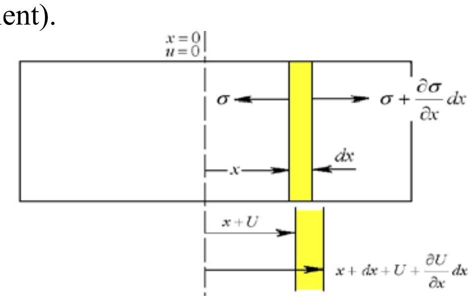

Fig. 2. Damped vibration of cylindrical rod in the resonance condition with viscous damping as energy loss model

3. Cylindrical ultrasonic transducer design

Considering figure (1) and the location of node between two contact surfaces of piezoelectric, transducer's set would be divided to the symmetrical parts. The first part is composed of backing and upper piezoelectric and the second part is composed of matching and the beneath piezoelectric.

With the given length of piezoelectric ring, only matching length and backing length are left for calculation. The design of the first and the second half of transducer is completely similar, so in this paper, the design of one half (backing and piezoelectric) has been presented.

Between steel backing and piezoelectric, the values of displacement and force of backing and piezoelectric are equal:

$$U_{st}\Big|_{x=L_{St}} = U_P\Big|_{x=L_P} \tag{2}$$

$$\sigma_{St} A_{st}\Big|_{x=L_{St}} = \sigma_P A_P\Big|_{x=L_P} \qquad (3)$$

A is the cross-sectional area, σ is the stress, L is the length. Subscripts St and P are steel backing and piezoelectric, respectively.

3.1. The displacement equation of steel backing

The differential equation of displacement of steel backing is the base for equation (1) and using the separation of variable techniques, the longitudinal damped displacement equation has been assumed as follow:

$$U(x,t)=f(x)g(t) \qquad (4)$$

Introducing equation 4 into equation 1 and separating the variables, it follows:

$$f(x)\frac{d^2 g(t)}{dt^2} + \delta f(x)\frac{dg(t)}{dt} - c^2 g(t)\frac{d^2 f(x)}{dx^2} + k^2 f(x)g(t) - k^2 f(x)g(t) = 0 \qquad (5)$$

$$f(x)\left[\frac{d^2 g(t)}{dt^2} + \delta\frac{dg(t)}{dt} - k^2 g(t)\right] + g(t)\left[k^2 f(x) - c^2\frac{d^2 f(x)}{dx^2}\right] = 0 \qquad (6)$$

$$\frac{d^2 g(t)}{dt^2} + \delta\frac{dg(t)}{dt} - k^2 g(t) = 0 \qquad (7)$$

$$k^2 f(x) - c^2\frac{d^2 f(x)}{dx^2} = 0 \qquad (8)$$

Adding and subtracting the term of $k^2 f(x)g(t)$ is for separating the variables as shown above [15]. More to the point, k is a constant as follows:

Where α and ω are constant and the values of these constants will be determined considering the boundary conditions. Let us solve the temporal equations (7) and the space equation (8) using the traditional method. After all the calculations, this equation can be presented as:

$$g(t) = Ae^{-\frac{\delta}{2}t}e^{\left(\sqrt{\delta^2+4k^2}/2\right)t} + Be^{-\frac{\delta}{2}t}e^{\left(-\sqrt{\delta^2+4k^2}/2\right)t} \qquad (9)$$

$$f(x) = Ce^{\frac{kx}{c}} + De^{-\frac{kx}{c}} \qquad (10)$$

A, B, C and D are constant. These constants will be determined by applying boundary and initial conditions. The space-time solution U(x, t) is then:

$$U(x,t) = \left(Ae^{-\frac{\delta}{2}t}e^{\left(\sqrt{\delta^2+4k^2}/2\right)t} + Be^{-\frac{\delta}{2}t}e^{\left(-\sqrt{\delta^2+4k^2}/2\right)t}\right)\left(Ce^{\frac{kx}{c}} + De^{-\frac{kx}{c}}\right) \qquad (11)$$

According to the resonance condition in backing part, appropriate boundary condition for steel backing is [21]:

$$\forall t\ ;\ \frac{\partial U}{\partial x}(0,t) = 0\ ;\ \frac{\partial U}{\partial x}(L,t) = 0 \qquad (12)$$

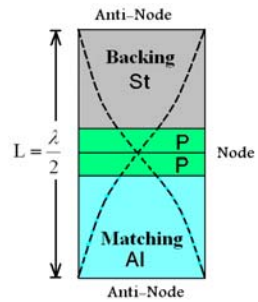

Fig. 3. Location of nodes and anti-nodes

Applying these boundary conditions to the solution (10) yields:

$$C = D \neq 0 \tag{13}$$

$$e^{\frac{kL}{c}} - e^{-\frac{kL}{c}} = 0 \tag{14}$$

Let us consider a complex k in equation (15):

$$-e^{\frac{L}{c}(\alpha + j\omega)} + e^{\frac{L}{c}(\alpha + j\omega)} = e^{-\alpha\frac{L}{c}}\left(-\cos\frac{\omega L}{c} + j\sin\frac{\omega L}{c}\right) + e^{\alpha\frac{L}{c}}\left(\cos\frac{\omega L}{c} + j\sin\frac{\omega L}{c}\right) = 0 \tag{15}$$

Equation (16) becomes:

$$2\cos\left(\frac{\omega L}{c}\right)\left[\frac{e^{\alpha\frac{L}{c}} - e^{-\alpha\frac{L}{c}}}{2}\right] + 2j\sin\left(\frac{\omega L}{c}\right)\left[\frac{e^{\alpha\frac{L}{c}} + e^{-\alpha\frac{L}{c}}}{2}\right] = 0 \tag{16}$$

With simplifying equation (17):

$$\cos\left(\frac{\omega L}{c}\right)\sinh\left(\frac{\alpha L}{c}\right) + j\sin\left(\frac{\omega L}{c}\right)\cosh\left(\frac{\alpha L}{c}\right) = 0 \tag{17}$$

Separating the real and imaginary parts:

$$\cos\left(\frac{\omega L}{c}\right)\sinh\left(\frac{\alpha L}{c}\right) = 0 \ , \ \sin\left(\frac{\omega L}{c}\right)\cosh\left(\frac{\alpha L}{c}\right) = 0 \tag{18}$$

$$k = \alpha + j\omega \tag{19}$$

According to non-trivial solution, constants α and ω. can be obtained as follow:

$$\alpha = 0 \ , \omega = \omega_n = \frac{\mathrm{n}\pi \mathrm{c}}{\mathrm{L}}; \ \mathrm{n} = 1, 2, 3, \ldots, \infty \Rightarrow \mathrm{k_n} = \frac{\mathrm{n}\pi \mathrm{c}}{\mathrm{L}} j \tag{20}$$

So, the relation between damped natural frequency Ω_n. and non-damped natural frequency ω_n. can be given as:

$$\Omega_n = \sqrt{\omega_n^2 - (\delta/2)^2} = \omega_n\sqrt{1 - \xi^2} \tag{21}$$

It is the damped normal angular frequency as opposed to the non-damped normal angular frequency ω_n. which characterizes the vibrations when δ= 0 and equation (1) is reduced to the standard wave equation. In this relation, $\xi = \delta / 2\omega_n$ is the damping ratio. Grouping all these results, the solution of space (11) is particularized for the mode shapes $f_n(x)$ defined for each n mode:

$$f_n(x) = 2C_n \cos\left(\frac{\mathrm{n}\pi}{\mathrm{L}} x\right) = 2C_n \cos\left(\frac{\omega_n}{\mathrm{c}} x\right) \tag{22}$$

The temporal response of n mode, $g_n(t)$ can also be expressed in the form of the following:

$$g_n(t) = e^{-\frac{\delta}{2}t}\left[A_n e^{j\Omega_n t} + B_n e^{-j\Omega_n t}\right] \tag{23}$$

Thus:

$$U(x,t) = \sum_{n=1}^{\infty} f_n(x) g_n(t) \tag{24}$$

It necessarily follows those two initial conditions for backing part as follows:

$$U(x,0) = D(x) \ ; \ \partial U/\partial \mathrm{t}\,(x,0) = 0 \tag{25}$$

$U(x,0) = D(x)$ is the initial displacement and $\partial U/\partial \mathrm{t}\,(x,0)$ is the initial velocity. By assuming $A'_n = 2A_n C_n$, $B'_n = 2B_n C_n$, it follows that:

$$U(x,0) = \sum_{n=1}^{\infty} (A'_n + B'_n) Cos\left(\frac{n\pi}{L} x\right) = D(x) \tag{26}$$

$$\frac{\partial U}{\partial t}(x,0) = \sum_{n=1}^{\infty}\left[j(A'_n - B'_n)\Omega_n - \frac{\delta}{2}(A'_n + B'_n)\right] \times \cos\left(\frac{n\pi}{L} x\right) = 0 \tag{27}$$

To determine the constants A'_n. and B'_n, it is enough to use the properties of orthogonality, which amounts to breaking up the initial displacements and speed into a Fourier series of cosine and to identify them term by term. Equations (26) and (27) yield, respectively:

$$A'_n = \frac{j\Omega_n + \frac{\delta}{2}}{j\Omega_n L} \int_0^L D(x) \cos\left(\frac{n\pi}{L} x\right) dx \tag{28}$$

$$B'_n = \frac{j\Omega_n - \frac{\delta}{2}}{j\Omega_n L} \int_0^L D(x) \cos\left(\frac{n\pi}{L} x\right) dx \tag{29}$$

According to equations (24), (28) and (29), the displacement equation of backing has been concluded as follows:

$$U(x,t)=\sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t}\left\{\int_0^L \cos\left[\frac{n\pi}{L}x\right]\frac{2D(x)}{L}dx\right\}\times\left[\cos(\Omega_n t)+\frac{\delta}{2\Omega_n}\sin(\Omega_n t)\right]Cos\left(\frac{n\pi}{L}x\right) \quad 30)$$

Figure 4 illustrates the time history of mode *n* represented in equation (30) which is a damped sinusoid representing a dissipation of energy during movement.

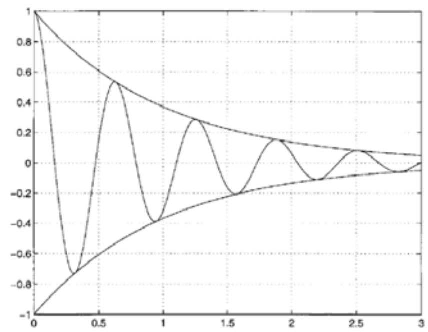

Fig. 4. Time history of mode n

Initial displacement of all points of backing part is as follows:

$$D(x)=u_{\circ}\cos\left(\frac{\pi}{L}x\right) \quad (31)$$

So, the displacement relation of backing part has been written as follows:

$$U(x,t)=\sum_{n=1}^{\infty} \begin{matrix} e^{-\frac{\delta}{2}t}\left\{\frac{2u_{\circ}}{L}\int_0^L \cos\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}x\right)dx\right\}\times \\ \left[\cos(\Omega_n t)+\frac{\delta}{2\Omega_n}\sin(\Omega_n t)\right]Cos\left(\frac{\pi}{L}x\right)\end{matrix} \quad (32)$$

$$U(x,t)=u_{\circ}e^{-\frac{\delta}{2}t}\left[\cos(\Omega_n t)+\frac{\delta}{2\Omega_n}\sin(\Omega_n t)\right]Cos\left(\frac{\pi}{L}x\right) \quad (33)$$

3.2. Displacement equation of piezoelectric

Like all parts of ultrasonic transducer, the vibration of piezoelectric is longitudinal and the differential equation of wave transmission of this part is equation (1), too.

$$\forall t \;;\; \mathrm{U}(0,\mathrm{t})=0 \;;\; \frac{\partial \mathrm{U}}{\partial \mathrm{x}}(L,t)=0 \quad (34)$$

In t=0, the displacement of piezoelectric is as follows:

$$D(x)=u'_{\circ}\sin\left(\frac{\pi}{2L}x\right) \quad (35)$$

To introduce boundary and initial conditions into the differential equation of piezoelectric for extracting the piezoelectric displacement, the following equation would be:

$$U(x,t)=\sum_{n=1}^{\infty}\begin{matrix} e^{-\frac{\delta}{2}t}\left\{\frac{2u_\circ}{L}\int_0^L \sin\left(\frac{\pi}{2L}x\right)\sin\left(\frac{\pi}{2L}x\right)dx\right\}\times \\ \left[\cos(\Omega_n t)+\frac{\delta}{2\Omega_n}\sin(\Omega_n t)\right]Cos\left(\frac{\pi}{2L}x\right)\end{matrix} \quad (36)$$

$$U(x,t)=u'_\circ e^{-\frac{\delta}{2}t}\left[\cos(\Omega_n t)+\frac{\delta}{2\Omega_n}\sin(\Omega_n t)\right]\sin\left(\frac{\pi}{2L}x\right) \quad (37)$$

3.3. Stress relations in components of ultrasonic transducer

According to high frequency that any ultrasonic transducer works on, also very quick variations of strain, better and precision relation for stress in all parts of transducer can be written as follows [22, 23]:

$$\sigma = Y\varepsilon + \delta'\dot{\varepsilon} \quad (38)$$

Where σ is stress, Y is the module of elasticity, ε is strain, $\dot{\varepsilon}$ is the rate of strain and δ' is viscous damping coefficient per mass unit. Equation (38) is only applicable for the metallic materials and for piezoelectric materials; stress relation is different; therefore, with regard to the piezoelectricity properties, the stress relation of piezoelectric is as follows [13, 15]:

$$\sigma = \frac{\varepsilon_{33}}{s_{33}^E} + \frac{d_{33}}{s_{33}^E}E \quad (39)$$

Where ε_{33} is the piezoelectric axial strain, s_{33}^E is the compliance matrix under a constant electric field and d_{33} is the piezoelectric constant, thus [13, 15]:

$$\sigma = Y_{33}^E(\varepsilon_{33} - Ed_{33}) \quad (40)$$

Where ε_{33}. is equal to strain caused by mechanical force and an electric field $\varepsilon_{33} - Ed_{33}$ is the strain caused by only mechanical force and σ is the piezoelectric stress. In vibrating piezoelectric in the case of open circuit, stress is as [13, 15]:

$$\sigma = Y_{33}^D\varepsilon_{33} \quad (41)$$

Y_{33}^D is young modulus under open circuit conditions.

If piezoelectric is connected to a constant voltage source (For example short circuit) and this condition is vibrating, the stress relation of piezoelectric will be as:

$$\sigma = Y_{33}^E\varepsilon_{33} \quad (42)$$

Y_{33}^{E} is young modulus under short circuit conditions [13, 15].

By viscous damping assumption as the energy loss model of piezoelectric, in the open circuit conditions, the relation of the stress will be written as:

$$\sigma = Y_{33}^{D}(\varepsilon_{33} + \delta'_{p}\dot{\varepsilon}_{33}) = Y_{33}^{E}(\varepsilon_{33} - Ed_{33}) + \delta'_{p}\dot{\varepsilon}_{33} \tag{43}$$

And in short circuit conditions, the relation of stress will be written as:

$$\sigma = Y_{33}^{E}(\varepsilon_{33} + \delta'_{p}\dot{\varepsilon}_{33}) \tag{44}$$

3.4. Stress relation for backing part of transducer

According to equation (33), the displacement relation of backing part can be written as:

$$U_{st}\Big|_{x=L_{St}} = \frac{u_{\circ St}}{2}\cos\left(\frac{\omega_{nSt}}{c_{St}}L_{St}\right)\left[-\frac{\mu_2^{St}}{j\Omega_{nSt}}e^{\mu_1^{St}t} + \frac{\mu_1^{St}}{j\Omega_{nSt}}e^{\mu_2^{St}t}\right] \tag{45}$$

According to equation (45), the value of stress in backing part between piezoelectric and backing parts can be concluded as:

$$\sigma_{st}A_{st}\Big|_{x=L_{St}} = -A_{st}Y_{st}u_{\circ St}\frac{\omega_{nSt}}{2c_{St}}\sin\left(\frac{\omega_{nSt}}{c_{St}}L_{St}\right)\left[\begin{array}{l}-\dfrac{\mu_2^{St}Y_{st} + \mu_2^{St}\mu_1^{St}\delta'_{st}}{jY_{st}\Omega_{nSt}}e^{\mu_1^{St}t} + \\ \dfrac{\mu_1^{St}Y_{st} + \mu_1^{St}\mu_1^{St}\delta'_{st}}{jY_{st}\Omega_{nSt}}e^{\mu_2^{St}t}\end{array}\right] \tag{46}$$

μ_1^{St} and μ_2^{St} have been used only for the simplicity in the equations of (45) and (46), thus:

$$\mu_1^{St} = -\frac{\delta_{st}}{2} + j\Omega_{nSt} \tag{47}$$

$$\mu_2^{St} = -\frac{\delta_{st}}{2} - j\Omega_{nSt} \tag{48}$$

3.5. Stress relation for piezoelectric part of transducer

According to the equation (37), the displacement of piezoelectric is as follows:

$$U_{P}\Big|_{x=L_{P}} = \frac{u_{\circ P}}{2}\sin\left(\frac{\omega_{nP}}{c_{P}}L_{P}\right)\left[-\frac{\mu_2^{P}}{j\Omega_{nP}}e^{\mu_1^{P}t} + \frac{\mu_1^{P}}{j\Omega_{nP}}e^{\mu_2^{P}t}\right] \tag{49}$$

According to the equation (49), the force relation of piezoelectric between piezoelectric and backing can be written as follows:

$$\sigma_{P}A_{P}\Big|_{x=L_{P}} = A_{P}Y_{P}u_{\circ P}\frac{\omega_{nP}}{2c_{P}}\cos\left(\frac{\omega_{nP}}{c_{P}}L_{P}\right)\times\left[\begin{array}{l}-\dfrac{\mu_2^{P}Y_{33} + \mu_2^{P}\mu_1^{P}\delta'_{P}}{jY_{33}\Omega_{nP}}e^{\mu_1^{P}t} + \\ \dfrac{\mu_1^{P}Y_{33} + \mu_1^{P}\mu_1^{P}\delta'_{P}}{jY_{33}\Omega_{nP}}e^{\mu_2^{P}t}\end{array}\right] \tag{50}$$

μ_1^p and μ_2^p have been used only for the simplicity in the equations (45) and (46), thus:

$$\mu_1^p = -\frac{\delta_p}{2} + j\Omega_{nP} \tag{51}$$

$$\mu_2^P = -\frac{\delta_P}{2} - j\Omega_{nP} \tag{52}$$

Between the piezoelectric surface and the backing surface, the amplitude of displacement and force are equal.

$$L_{St} = \frac{c_{St}}{\sqrt{\Omega_{nSt}^2 + \left(\frac{\delta_{St}}{2}\right)^2}} Arc\tan\left\{\begin{array}{l} \dfrac{A_P Y_{33} c_{St} \sqrt{\Omega_{nP}^2 + \left(\frac{\delta_P}{2}\right)^2}}{A_{St} Y_{St} c_P \sqrt{\Omega_{nSt}^2 + \left(\frac{\delta_{St}}{2}\right)^2}} \times \\ \dfrac{\delta_{St}\left[\delta_P - \frac{2\delta'_P}{Y_{33}}\sqrt{\Omega_{nP}^2 + \left(\frac{\delta_P}{2}\right)^2}\right]}{\delta_P\left[\delta_{St} - \frac{2\delta'_{St}}{Y_{St}}\sqrt{\Omega_{nSt}^2 + \left(\frac{\delta_{St}}{2}\right)^2}\right]} \times \dfrac{1}{\tan\left[\frac{L_P}{c_P}\sqrt{\Omega_{nP}^2 + \left(\frac{\delta_P}{2}\right)^2}\right]} \end{array}\right\} \tag{53}$$

If $\Omega_{nSt} = \Omega_{nP} = \Omega_n$ the final equation of backing length part will be as follow:

$$L_{St} = \frac{c_{St}}{\sqrt{\Omega_n^2 + \left(\frac{\delta_{St}}{2}\right)^2}} Arc\tan\left\{\begin{array}{l} \dfrac{A_P Y_{33} c_{St} \sqrt{\Omega_n^2 + \left(\frac{\delta_P}{2}\right)^2}}{A_{St} Y_{St} c_P \sqrt{\Omega_n^2 + \left(\frac{\delta_{St}}{2}\right)^2}} \times \\ \dfrac{\delta_{St}\left[\delta_P - \frac{2\delta'_P}{Y_{33}}\sqrt{\Omega_n^2 + \left(\frac{\delta_P}{2}\right)^2}\right]}{\delta_P\left[\delta_{St} - \frac{2\delta'_{St}}{Y_{St}}\sqrt{\Omega_n^2 + \left(\frac{\delta_{St}}{2}\right)^2}\right]} \times \dfrac{1}{\tan\left[\frac{L_P}{c_P}\sqrt{\Omega_n^2 + \left(\frac{\delta_P}{2}\right)^2}\right]} \end{array}\right\} \tag{54}$$

Value of δ. can be obtained from equation (55).

$$\delta = 2\xi\omega_n \tag{55}$$

Value of δ'. can be obtained from the reference to [23].

$$\delta' = 2\xi\omega_n Y = \delta Y \tag{56}$$

According to the equation (54), it can be clearly seen that when damping value of parts is ignored, extracted relations can be reduced to non-damped classical formulations. Because the classical formulation is the base of this paper, and also the relations presented in the paper are logically mathematical formulations, and moreover the classical formulations cab be achieved by omitting damping in the given relations, this itself can be described as a proof for the mathematics given in this study.

The design and determination of matching length of ultrasonic transducer (L_{Al} : Aluminum matching length) is absolutely similar to determining the backing length of transducer and the final relation of this part of transducer will be obtained as follows:

$$L_{Al}=\frac{c_{Al}}{\sqrt{\Omega_n^2+\left(\frac{\delta_{Al}}{2}\right)^2}}Arc\tan\left\{\begin{array}{l}\dfrac{A_P Y_{33} c_{Al}\sqrt{\Omega_n^2+\left(\frac{\delta_P}{2}\right)^2}}{A_{Al} Y_{Al} c_P\sqrt{\Omega_n^2+\left(\frac{\delta_{Al}}{2}\right)^2}}\times \\ \dfrac{\delta_{Al}\left[\delta_P-\dfrac{2\delta'_P}{Y_{33}}\sqrt{\Omega_n^2+\left(\frac{\delta_P}{2}\right)^2}\right]}{\delta_P\left[\delta_{Al}-\dfrac{2\delta'_{Al}}{Y_{Al}}\sqrt{\Omega_n^2+\left(\frac{\delta_{Al}}{2}\right)^2}\right]}\times\dfrac{1}{\tan\left[\dfrac{L_P}{c_P}\sqrt{\Omega_n^2+\left(\frac{\delta_P}{2}\right)^2}\right]}\end{array}\right\} \qquad (57)$$

In these relations, different parameters as sound speed, material density, module of elasticity and damping parameters can be extracted from the standard table or can be measured through the experimental investigations.

4. Conclusions

In this paper with regard to the principles and techniques of mathematics in vibration of continuous systems, differential equation of damped longitudinal rod has been proved and obtained, and according to the separation of variable techniques and complex algebraic methods and also by using appropriate boundary conditions, the differential equation has been obtained and solved analytically. Thus, considering these relations, analytically design of different parts of transducer such as backing and matching parts is done. Designing relations are the functions of physical and mechanical properties of the materials used in ultrasonic transducers and damping coefficients, too. These novel design and producer would lead to accurate determination of all parts' lengths than analytically classical producer.

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