EXISTENCE AND UNIQUENESS RESULTS FOR (k,ψ)-FRACTIONAL ORDER QUADRATIC INTEGRAL EQUATIONS

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The current paper mainly discusses the existence and uniqueness of solutions for a new kind of nonlinear quadratic-(k,ψ)*-fractional integral equations (NQ-*(k,ψ)*-FIEs, for short). Under different assumptions, we first prove the existence of the solutions by using the well known fixed point theorem attributed to Dhage, while the basic tool for the uniqueness of solutions for the mentioned problem, relies on Banach's contraction theorem. Ultimately, suitable examples are given to test the feasibility of our obtained results. In this way, our results represent generalized versions of some recent interesting contributions.*

Keywords: (k,ψ)-fractional integral, Quadratic equations, Fixed point theorems, Existence, Uniqueness.

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1. Introduction

Undoubtedly, nonlinear integral equations (NIEs) theory is one of the most essential subject in the mathematical analysis area. Various types of NIEs have been studied by many researches in different spaces. In particular, nonlinear quadratic integral equations (NQIEs) are widely used in many mathematical models or real-world applications. For detailed information about the mentioned area, we refer the interest readers to [7, 9, 8, 1].

From the modern literature, it is also observed that the topic of fractional differential equations (FDEs) received an overwhelming interest from many scholars, due to their importance in understanding the dynamic memory of many real-world phenomena. In this regards, a wide variety of fractional integrals (FIs) have been defined and extensively studied by many researchers. Some typical kinds of FI operators are Rieman-Liouville (R-L)-FI, k-R-L-FI, Hadamard FI, Katugampola FI, and R-L-FI of a function with respect to another function. For more details on the applications of fractional calculus, the reader is directed to the books of Benchohra *et al.* [3], Herrmann [10], Hilfer [11], Kilbas *et al.* [12] and Samko *et al.* [20].

Due to this diversity of FI operators, the authors of [15] proposed a new FI which unifies the aforesaid FIs and other recent FIs into a single form. This FI is known as (k,ψ) Riemann-Liouville. The appropriate choice of the parameters involved in its definition helps in the modeling of physical phenomenon and makes the approach more suitable from the application point of view. These types of operators are becoming more and more popular as time goes on. For some information, we refer the reader to the papers of Chu *et al.* [4],

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Diaz *et al.* in [5], Kucche *et al.* [13], Mubeen *et al.* [14], Salim *et al.* [19, 16, 17, 18] and Sousa *et al.* [21].

However, to the best of our knowledge, the existence and uniqueness of solutions for quadratic-integral equations including the (k,ψ)-fractional operator has not been investigated. As a consequence, the previously mentioned works inspired us to continue on the same path. In precise terms, we employ fixed-point theorem involving product of operators as well as the well-known Banach fixed-point theorem (BFPT) for investigating and creating the required circumstances for the existence and uniqueness of solutions for a new kind of nonlinear quadratic-(k,ψ)-fractional integral equations stated as follows:

$$\begin{aligned} x(\tau) = & \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau)-\psi(\nu))^{\frac{\alpha_1}{\mathrm{k}_1}-1}}{\mathrm{k}_1\Gamma_{\mathrm{k}_1}(\alpha_1)} \mathrm{f}_1(\nu, x(\nu))\mathrm{d}\nu \\ & \times \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{\mathrm{k}_2}-1}}{\mathrm{k}_2\Gamma_{\mathrm{k}_2}(\alpha_2)} \mathrm{f}_2(\nu, x(\nu))\mathrm{d}\nu \\ & + \sum_{i=1}^{m} \int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau)-\psi_i(\nu))^{\frac{\beta_i}{\mathrm{k}_i^\star}-1}}{\mathrm{k}_i^\star\Gamma_{\mathrm{k}_i^\star}(\beta_i)} \mathrm{g}_i(\nu, x(\nu))\mathrm{d}\nu,\ \tau \in \Delta, \end{aligned} \tag{1.1}$$

where $\Delta = [\tau_0, \tau_f], 0 \leq \tau_0 < \tau_f < \infty, \psi, \phi, \psi_i : \Delta \to \mathbb{R}$ are increasing functions with $\psi'(\tau), \phi'(\tau), \psi_i'(\tau) \neq 0$ for all $\tau \in \Delta$, $i = 1, \ldots, m \in \mathbb{N}$, $\alpha_1 \in (0, \mathrm{k}_1), \alpha_2 \in (0, \mathrm{k}_2), \beta_i \in (0, \mathrm{k}_i^\star)$, $i = 1, \ldots, m$. The assumptions on the functions $\mathrm{f}_1, \mathrm{f}_2$, and g_i; $i = 1, \ldots, m$ will be stated precisely in Section **3**.

A brief outline of the paper is as follows. In Section **2**, we review some essential facts that are used to obtain our main results. In Section **3**, we use Dhage and Banach fixed point theorems to derive the required results. Section **4** contains particular cases. Also, we give appropriate examples to illustrate the given theory.

2. Essential preliminaries

Let us begin this section with some basic definitions and fundamental results that will be used throughout the paper.

Let $0 \leq \tau_0 < \tau_f < \infty, \Delta = [\tau_0, \tau_f]$, and $\psi : \Delta \to \mathbb{R}$ be an increasing function with $\psi'(\tau) \neq 0$ for all $\tau \in \Delta$.

By $\mathbb{X} = C(\Delta, \mathbb{R})$ we denote the Banach space of all continuous functions from Δ into $\mathbb{R}$ with the norm

$$\|x\|_{\mathbb{X}} = \sup_{t\in\Delta} |x(\tau)|,$$

and a multiplication in $\mathbb{X}$ by

$$(xy)(\tau) = x(\tau)y(\tau).$$

Clearly $\mathbb{X}$ is a Banach algebra with respect to above supremum norm and the multiplication in it.

Definition 2.1 ([15]). *For $x \in L^1(\Delta, \mathbb{R})$ and $\mathrm{k} \in (0, +\infty)$, the (k,ψ)-Riemann-Liouville fractional integral of order $\alpha > 0$ of the function x is given by*

$$^{\mathrm{k}}\mathbb{I}_{\tau_0^+}^{\alpha;\psi} x(\tau) = \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau)-\psi(\nu))^{\frac{\alpha}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\alpha)} x(\nu)\mathrm{d}\nu,$$

where Γ_{k} is the k*-Gamma function defined by*

$$\Gamma_k(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-\frac{t^{\mathrm{k}}}{\mathrm{k}}} \mathrm{d}t.$$

Lemma 2.1. *Let $\alpha \in (0,1), \mathrm{k} \in (0,\infty)$. If $x \in C(\Delta,\mathbb{R})$, then ${}^{\mathrm{k}}\mathbb{I}^{\alpha,\psi}_{a^+}x \in C(\Delta,\mathbb{R})$. Furthermore, we have*

$$\left\|{}^{\mathrm{k}}\mathbb{I}^{\alpha,\psi}_{\tau_0^+}x\right\|_{\mathbb{X}} \leq \frac{(\psi(\tau_f)-\psi(\tau_0))^{\frac{\alpha}{\mathrm{k}}}}{\Gamma_{\mathrm{k}}(\alpha+\mathrm{k})}\|x\|_{\mathbb{X}}.$$

Proof. The proof can be derived easily by a simple calculation from Definition **2**.1. □

The following hybrid fixed point theorem for three operators in a Banach algebra $\mathbb{E}$ due to Dhage [6] will be used to prove the existence result for the NQ-(k, ψ)-FIE (**1**.1).

Lemma 2.2. *Let Ω be a closed, convex, bounded and nonempty subset of a Banach algebra $\mathbb{E}$, and let $\mathbb{S}_1, \mathbb{S}_3 : \mathbb{E} \longrightarrow \mathbb{E}$ and $\mathbb{S}_2 : \Omega \longrightarrow \mathbb{E}$ be three opertors such that*

(*a*) *$\mathbb{S}_1$ and $\mathbb{S}_3$ are Lipschitzian with Lipschitz constants δ and ρ, respectively;*
(*b*) *$\mathbb{S}_2$ is compact and continuous;*
(*c*) *$x = \mathbb{S}_1 x \mathbb{S}_2 y + \mathbb{S}_3 x \Rightarrow x \in \Omega$ for all $y \in \Omega$;*
(*d*) *$\delta\mathbb{M} + \rho < 1$ where $\mathbb{M} = \left\|\mathbb{S}_2(\Omega)\right\|$.*

Then, the operator equation $\mathbb{S}_1 x \mathbb{S}_2 x + \mathbb{S}_3 x = x$ has a solution in Ω.

3. Principal outcomes

Let us evoke some essential assumptions which are required to prove the existence of solutions for the problem mentioned.

(H1) The functions $\mathrm{f}_1, \mathrm{f}_2$, and g_i; $i = 1, \ldots, m$, are continuous.

(H2) There exist positive constants $\mathbb{L}_{\mathrm{f}_1}, \mathbb{L}_{\mathrm{g}_i}$, $i = 1, \ldots, m$, such that

$$|\mathrm{f}_1(\tau,x) - \mathrm{f}_1(\tau,y)| \leq \mathbb{L}_{\mathrm{f}_1}|x-y|,$$

and

$$|\mathrm{g}_i(\tau,x) - \mathrm{g}_i(\tau,y)| \leq \mathbb{L}_{\mathrm{g}_i}|x-y|,\ i = 1,\ldots,m,$$

for all $\tau \in \Delta$ and $x, y \in \mathbb{R}$.

(H3) There exists a function $\mu \in C(\Delta, \mathbb{R}^+)$ and a continuous nondecreasing function $\omega : [0,\infty) \longrightarrow (0,\infty)$ such that

$$|\mathrm{f}_2(\tau,x)| \leq \mu(\tau)\omega(|x|),$$

for any $x \in \mathbb{R}$ and $\tau \in \Delta$.

(H4) There exists a constant $\xi > 0$ such that

$$\xi \geq \frac{\mathbb{L}_{\mathrm{f}_1}\mathrm{f}_1^\star\mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1}\mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2}\mu^\star\omega(\xi) + \sum_{i=1}^m \mathbb{L}_{\mathrm{g}_i}\mathrm{g}_i^\star\mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star}}{1-\left(\mathbb{L}_{\mathrm{f}_1}\mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1}\mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2}\mu^\star\omega(\xi) + \sum_{i=1}^m \mathbb{L}_{\mathrm{g}_i}\mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star}\right)}$$

and

$$\mathbb{L}_{\mathrm{f}_1}\mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1}\mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2}\mu^\star\omega(\xi) + \sum_{i=1}^m \mathbb{L}_{\mathrm{g}_i}\mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star} < 1,$$

where

$$\mathrm{f}_1^\star := \sup_{\tau\in\Delta}|\mathrm{f}_1(\tau,0)|,\ \mathrm{g}_i^\star := \sup_{\tau\in\Delta}|\mathrm{g}_i(\tau,0)|, i = 1,\ldots,m;$$

$$\mu^\star := \sup_{\tau\in\Delta}|\mu(\tau)|,\ \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1} := \frac{(\psi(\tau_f)-\psi(\tau_0))^{\frac{\alpha_1}{\mathrm{k}_1}}}{\Gamma_{\mathrm{k}_1}(\alpha_1+\mathrm{k}_1)},$$

$$\mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2} := \frac{(\phi(\tau_f)-\phi(\tau_0))^{\frac{\alpha_2}{\mathrm{k}_2}}}{\Gamma_{\mathrm{k}_2}(\alpha_2+\mathrm{k}_2)},\ \mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star} := \frac{(\psi_i(\tau_f)-\psi_i(\tau_0))^{\frac{\beta_i}{\mathrm{k}_i^\star}}}{\Gamma_{\mathrm{k}_i^\star}(\beta_i+\mathrm{k}_i^\star)}, i = 1,\ldots,m.$$

(H5) The function f_2 is Lipschitzian in its second argument, namely, there exists $\mathbb{L}_{f_2} > 0$, so that the inequality

$$|f_2(\tau, x) - f_2(\tau, y)| \leq \mathbb{L}_{f_2}|x - y|,$$

holds for all $\tau \in \Delta$ and $x, y \in \mathbb{R}$.

(H6) There exist two nonnegative constants $\mathcal{C}_1, \mathcal{C}_2$ such that

$$|f_1(\tau, x)| \leq \mathcal{C}_1, \quad |f_2(\tau, x)| \leq \mathcal{C}_2, \text{ for all } x \in \mathbb{R} \text{ and } \tau \in \Delta.$$

Our first preliminary result is based on Dhage fixed point theorem with three operators.

Theorem 3.1. *Presume that the assumptions (H1)-(H4) are fulfilled. Then our considered problem* (**1**.1) *has at least one solution.*

Proof. Construct a bounded, closed, convex and nonempty set Θ_ξ of $\mathbb{X}$ as follows

$$\Theta_\xi = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} \leq \xi\}.$$

Define three operators $\mathbb{S}_1, \mathbb{S}_3 : \mathbb{X} \longrightarrow \mathbb{X}$ and $\mathbb{S}_2 : \Theta_\xi \longrightarrow \mathbb{X}$ by

$$\mathbb{S}_1 x(\tau) = \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau) - \psi(\nu))^{\frac{\alpha_1}{k_1}-1}}{k_1\Gamma_{k_1}(\alpha_1)} f_1(\nu, x(\nu))d\nu, \ \tau \in \Delta,$$

$$\mathbb{S}_2 x(\tau) = \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau) - \phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} f_2(\nu, x(\nu))d\nu, \ \tau \in \Delta,$$

and

$$\mathbb{S}_3 x(\tau) = \sum_{i=1}^{m} \int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau) - \psi_i(\nu))^{\frac{\beta_i}{k_i^\star}-1}}{k_i^\star\Gamma_{k_i^\star}(\beta_i)} g_i(\nu, x(\nu))d\nu, \ \tau \in \Delta.$$

Clearly, the operators $\mathbb{S}_1, \mathbb{S}_2$, and $\mathbb{S}_3$ are well defined due to (H1) and Lemma **2**.1. Moreover, the NQ-(k, ψ)-FIE (**1**.1) can be written in the operator form as

$$x(\tau) = \mathbb{S}_1 x(\tau)\mathbb{S}_2 x(\tau) + \mathbb{S}_3 x(\tau), \ \tau \in \Delta.$$

Thus, the existence of a solution for the NQ-(k, ψ)-FIE (**1**.1) is equivalent to the existence of a fixed point for operator $\mathbb{T} := \mathbb{S}_1\mathbb{S}_2 + \mathbb{S}_3$.

Presently, we shall check that the operators $\mathbb{S}_1, \mathbb{S}_2$ and $\mathbb{S}_3$ satisfy all the conditions of Lemma **2**.2. For ease of understanding, we split the proof into the following series of steps.

Step 1: In the first step, we show that $\mathbb{S}_1$ and $\mathbb{S}_3$ are Lipschitzian on $\mathbb{X}$.
Let $x, y \in \mathbb{X}$ and $\tau \in \Delta$. Then, in view of the hypothesis (H2), we have

$$\begin{aligned}
|\mathbb{S}_1 x(\tau) - \mathbb{S}_1 y(\tau)| &\leq \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau) - \psi(\nu))^{\frac{\alpha_1}{k_1}-1}}{k_1\Gamma_{k_1}(\alpha_1)} |f_1(\nu, x(\nu)) - f_1(\nu, y(\nu))|d\nu \\
&\leq \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau) - \psi(\nu))^{\frac{\alpha_1}{k_1}-1}}{k_1\Gamma_{k_1}(\alpha_1)} \mathbb{L}_{f_1}|x(\nu) - y(\nu)|d\nu \\
&\leq \frac{\mathbb{L}_{f_1}(\psi(\tau_f) - \psi(\tau_0))^{\frac{\alpha_1}{k_1}}}{\Gamma_{k_1}(\alpha_1 + k_1)} \|x - y\|_{\mathbb{X}}.
\end{aligned}$$

We deduce that

$$\|\mathbb{S}_1 x - \mathbb{S}_1 y\|_{\mathbb{X}} \leq \mathbb{L}_{f_1}\mathbb{K}_{\psi,\alpha_1,k_1}\|x - y\|_{\mathbb{X}}.$$

As a consequence, $\mathbb{S}_1$ is a Lipschitzian on $\mathbb{X}$ with Lipschitz constant $\delta = \mathbb{L}_{f_1}\mathbb{K}_{\psi,\alpha_1,k_1}$.
Also, for any $x, y \in \mathbb{X}$ and $\tau \in \Delta$. Then, from assumption (H4), we obtain

$$\begin{aligned}
&|\mathbb{S}_3 x(\tau) - \mathbb{S}_3 y(\tau)| \\
&\quad \leq \sum_{i=1}^{m} \int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau)-\psi_i(\nu))^{\frac{\beta_i}{k_i^\star}-1}}{k_i^\star \Gamma_{k_i^\star}(\beta_i)} |g_i(\nu, x(\nu)) - g_i(\nu, y(\nu))| d\nu \\
&\quad \leq \sum_{i=1}^{m} \mathbb{L}_{g_i} \int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau)-\psi_i(\nu))^{\frac{\beta_i}{k_i^\star}-1}}{k_i^\star \Gamma_{k_i^\star}(\beta_i)} |x(\nu) - y(\nu)| d\nu \\
&\quad \leq \sum_{i=1}^{m} \frac{\mathbb{L}_{g_i}(\psi_i(\tau_f)-\psi_i(\tau_0))^{\frac{\beta_i}{k_i^\star}}}{\Gamma_{k_i^\star}(\beta_i + k_i^\star)} \|x-y\|_{\mathbb{X}}.
\end{aligned}$$

Hence, we have

$$\|\mathbb{S}_3 x - \mathbb{S}_3 y\|_{\mathbb{X}} \leq \sum_{i=1}^{m} \mathbb{L}_{g_i}\mathbb{K}_{\psi_i,\beta_i,k_i{}^\star} \|x-y\|_{\mathbb{X}}.$$

From this it follows that $\mathbb{S}_3$ is Lipschitzian on $\mathbb{X}$ with Lipschitz constant $\rho = \sum_{i=1}^{m} \mathbb{L}_{g_i}\mathbb{K}_{\psi_i,\beta_i,k_i{}^\star}$.

Step 2: In this step, we show that the operator $\mathbb{S}_2$ is compact and continuous on Θ_ξ. First of all, observe that the continuity of $\mathbb{S}_2$ follows from the continuity of f_2. Next we will prove that the set $\mathbb{S}_2(\Theta_\xi)$ is a uniformly bounded in $\mathbb{X}$. Indeed, for all $x \in \Theta_\xi$, we derive from (H3) that

$$\begin{aligned}
|\mathbb{S}_2 x(\tau)| &\leq \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} |f_2(\nu, x(\nu))| d\nu \\
&\leq \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} \mu(\nu)\omega\big(|x(\nu)|\big) d\nu \\
&\leq \mu^\star \omega\big(\|x\|_{\mathbb{X}}\big) \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} d\nu \\
&\leq \frac{(\phi(\tau_f)-\phi(\tau_0))^{\frac{\alpha_2}{k_2}}}{\Gamma_{k_2}(\alpha_2+k_2)} \mu^\star \omega\big(\|x\|_{\mathbb{X}}\big) \\
&\leq \mu^\star \omega\big(\xi\big)\mathbb{K}_{\phi,\alpha_2,k_2}.
\end{aligned}$$

Thus,

$$\|\mathbb{S}_2 x\| \leq \mu^\star \omega\big(\xi\big)\mathbb{K}_{\phi,\alpha_2,k_2}.$$

This shows that $\mathbb{S}_2$ is uniformly bounded on Θ_ξ.
Now, we will show that $\mathbb{S}_2$ maps bounded sets into equicontinuous set of $\mathbb{X}$. For this purpose,

let $x \in \Theta_\xi$ and $\tau, \tau^\star \in \Delta$ with $\tau < \tau^\star$. One gets

$$\begin{aligned}
&|\mathbb{S}_2 x(\tau) - \mathbb{S}_2 x(\tau^\star)| \\
&\le \int_{\tau_0}^{\tau} \frac{\phi'(\nu)\left[(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{\mathrm{k}_2}-1} - (\phi(\tau^\star)-\phi(\nu))^{\frac{\alpha_2}{\mathrm{k}_2}-1}\right]}{\mathrm{k}_2\Gamma_{\mathrm{k}_2}(\alpha_2)} |\mathrm{f}_2(\nu, x(\nu))| \mathrm{d}\nu \\
&\quad + \int_{\tau}^{\tau^\star} \frac{\phi'(\nu)(\phi(\tau^\star)-\phi(\nu))^{\frac{\alpha_2}{\mathrm{k}_2}-1}}{\mathrm{k}_2\Gamma_{\mathrm{k}_2}(\alpha_2)} |\mathrm{f}_2(\nu, x(\nu))| \mathrm{d}\nu \\
&\le \frac{\mu^\star \omega(\xi)}{\alpha_2 \Gamma_{\mathrm{k}_2}(\alpha_2)} \left[(\phi(\tau)-\phi(\tau_0))^{\frac{\alpha_2}{\mathrm{k}_2}} - (\phi(\tau^\star)-\phi(\tau_0))^{\frac{\alpha_2}{\mathrm{k}_2}} + 2(\phi(\tau^\star)-\phi(\tau))^{\frac{\alpha_2}{\mathrm{k}_2}}\right] \\
&\le 2\frac{\mu^\star \omega(\xi)}{\Gamma_{\mathrm{k}_2}(\alpha_2+1)} (\phi(\tau^\star)-\phi(\tau))^{\frac{\alpha_2}{\mathrm{k}_2}}.
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in \Theta_\xi$ as $\tau^\star \to \tau$. As a consequence of the Ascoli-Arzelà theorem, $\mathbb{S}_2$ is a completely continuous operator on Θ_ξ.

Step 3: The hypothesis (c) of Lemma **2**.2 is satisfied.
Let $x \in \mathbb{X}$ and $y \in \Theta_\xi$ be arbitrary elements such that $x = \mathbb{S}_1 x \mathbb{S}_2 y + \mathbb{S}_3 x$. Then we have

$$\begin{aligned}
|x(\tau)| \le &\int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau)-\psi(\nu))^{\frac{\alpha_1}{\mathrm{k}_1}-1}}{\mathrm{k}_1\Gamma_{\mathrm{k}_1}(\alpha_1)} \big(|\mathrm{f}_1(\nu, x(\nu)) - \mathrm{f}_1(\nu, 0)| + |\mathrm{f}_1(\nu, 0)|\big) \mathrm{d}\nu \\
&\times \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{\mathrm{k}_2}-1}}{\mathrm{k}_2\Gamma_{\mathrm{k}_2}(\alpha_1)} \mu(\nu)\omega\big(|x(\nu)|\big) \mathrm{d}\nu \\
&+ \sum_{i=1}^{m} \int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau)-\psi_i(\nu))^{\frac{\beta_i}{\mathrm{k}_i^\star}-1}}{\mathrm{k}_i^\star \Gamma_{\mathrm{k}_i^\star}(\beta_i)} \big(|\mathrm{g}_i(\nu, x(\nu)) - \mathrm{g}_i(\nu, 0)| \\
&+ |\mathrm{g}_i(\nu, 0)|\big) \mathrm{d}\nu.
\end{aligned}$$

One can adopt the same technique as we did in Step 1 to get the following estimation

$$\|x\| \le \frac{\mathbb{L}_{\mathrm{f}_1} \mathrm{f}_1^\star \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1} \mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2} \mu^\star \omega\big(\xi\big) + \sum_{i=1}^{m} \mathbb{L}_{\mathrm{g}_i} \mathrm{g}_i^\star \mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star}}{1 - \big(\mathbb{L}_{\mathrm{f}_1} \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1} \mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2} \mu^\star \omega\big(\xi\big) + \sum_{i=1}^{m} \mathbb{L}_{\mathrm{g}_i} \mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star}\big)}.$$

Step 4: Finally we show that $\delta\mathbb{M} + \rho < 1$, that is, (d) of Lemma **2**.2 holds.
Since

$$\mathbb{M} = \|\mathbb{S}_2(\Theta_\xi)\| = \sup_{x \in \Theta_\xi} \Big\{ \sup_{\tau \in \Delta} |\mathbb{S}_2 x(\tau)| \Big\} \le \mu^\star \omega\big(\xi\big) \mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2},$$

then

$$\begin{aligned}
\mathbb{L}_{\mathrm{f}_1} \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1} \mathbb{M} + \sum_{i=1}^{m} \mathbb{L}_{\mathrm{g}_i} \mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star} &\le \mathbb{L}_{\mathrm{f}_1} \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1} \mu^\star \omega\big(\xi\big) \mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2} + \sum_{i=1}^{m} \mathbb{L}_{\mathrm{g}_i} \mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star} \\
&< 1,
\end{aligned}$$

with $\delta = \mathbb{L}_{\mathrm{f}_1} \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1}, \rho = \sum_{i=1}^{m} \mathbb{L}_{\mathrm{g}_i} \mathbb{K}_{\psi_i,\beta_i,\mathrm{k}_i{}^\star}$. Thus the operators $\mathbb{S}_1, \mathbb{S}_2$ and $\mathbb{S}_3$ satisfy all the desired conditions of Lemma **2**.2 and hence the operator equation $x = \mathbb{S}_1 x \mathbb{S}_2 x + \mathbb{S}_3 x$ has a solution in Θ_ξ. As a result, the NQ-(k, ψ)-FIE (**1**.1) has a solution on Δ. □

In the next theorem we prove the existence of a unique solution of the problem (**1**.1) by applying the contraction mapping principle.

Theorem 3.2. *Under the assumptions (H1), (H2), (H5), and (H6) are satisfied. Then the problem* (**1**.1) *has a unique solution $x \in \mathbb{X}$ provided that the following condition holds:*

$$\left[\left(\mathcal{C}_1\mathbb{L}_{f_2} + \mathcal{C}_2\mathbb{L}_{f_1}\right)\mathbb{K}_{\psi,\alpha_1,k_1}\mathbb{K}_{\phi,\alpha_2,k_2} + \sum_{i=1}^{m}\mathbb{L}_{g_i}\mathbb{K}_{\psi_i,\beta_i,k_i{}^\star}\right] < 1. \tag{3.1}$$

Proof. We consider the operator $\mathbb{T}$ defined by $\mathbb{T} : \mathbb{X} \to \mathbb{X}$;

$$\begin{aligned}
\mathbb{T}x(\tau) = &\int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau)-\psi(\nu))^{\frac{\alpha_1}{k_1}-1}}{k_1\Gamma_{k_1}(\alpha_1)} f_1(\nu, x(\nu))d\nu \\
&\times \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} f_2(\nu, x(\nu))d\nu \\
&+ \sum_{i=1}^{m}\int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau)-\psi_i(\nu))^{\frac{\beta_i}{k_i^\star}-1}}{k_i^\star\Gamma_{k_i^\star}(\beta_i)} g_i(\nu, x(\nu))d\nu.
\end{aligned}$$

The Banach contraction principle will be used to prove that $\mathbb{T}$ has a unique fixed point. For this reason, we shall show that $\mathbb{T}$ is a contraction with respect to the supremum norm $\|.\|_{\mathbb{X}}$. Indeed, let $x, y \in \mathbb{X}$ and $\tau \in \Delta$. Then, one can write

$$\begin{aligned}
|\mathbb{T}x(\tau) - \mathbb{T}y(\tau)| \leq &\int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} |f_2(\nu, x(\nu))|d\nu \\
&\times \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau)-\psi(\nu))^{\frac{\alpha_1}{k_1}-1}}{k_1\Gamma_{k_1}(\alpha_1)} |f_1(\nu, x(\nu)) - f_1(\nu, y(\nu))|d\nu \\
&+ \int_{\tau_0}^{\tau} \frac{\psi'(\nu)(\psi(\tau)-\psi(\nu))^{\frac{\alpha_1}{k_1}-1}}{k_1\Gamma_{k_1}(\alpha_1)} |f_1(\nu, x(\nu))|d\nu \\
&\times \int_{\tau_0}^{\tau} \frac{\phi'(\nu)(\phi(\tau)-\phi(\nu))^{\frac{\alpha_2}{k_2}-1}}{k_2\Gamma_{k_2}(\alpha_2)} |f_2(\nu, x(\nu)) - f_2(\nu, y(\nu))|d\nu \\
&+ \sum_{i=1}^{m}\int_{\tau_0}^{\tau} \frac{\psi_i'(\nu)(\psi_i(\tau)-\psi_i(\nu))^{\frac{\beta_i}{k_i^\star}-1}}{k_i^\star\Gamma_{k_i^\star}(\beta_i)} |g_i(\nu, x(\nu)) \\
&- g_i(\nu, y(\nu))|d\nu.
\end{aligned}$$

As we did earlier, using the assumptions (H2), (H5), and (H6), one can get

$$\|\mathbb{T}x - \mathbb{T}y\|_{\mathbb{X}} \leq \left[\left(\mathcal{C}_1\mathbb{L}_{f_2} + \mathcal{C}_2\mathbb{L}_{f_1}\right)\mathbb{K}_{\psi,\alpha_1,k_1}\mathbb{K}_{\phi,\alpha_2,k_2} + \sum_{i=1}^{m}\mathbb{L}_{g_i}\mathbb{K}_{\psi_i,\beta_i,k_i{}^\star}\right] \|x - y\|_{\mathbb{X}}.$$

From the condition (**3**.1), we may infer that $\mathbb{T}$ is a contraction operator. As an outcome of the Banach fixed point theorem, we can conclude that the operator $\mathbb{T}$ has a unique fixed point, which corresponds to the unique solution of the problem (**1**.1). □

4. Special cases and supportive examples

Our proposed problem (**1**.1) covers many of the corresponding problems in the literature [2, 14], which are considered special cases, for example:

- Letting $\mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_i^\star=1, \psi(\tau)=\phi(\tau)=\psi_i(\tau)=\tau$, $i=1,\ldots,m$, the problem (**1**.1) reduces to the classical Rieman-Liouville quadratic equation of fractional order:

$$x(\tau)=\int_{\tau_0}^{\tau}\frac{(\tau-\nu)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{f}_1(\nu,x(\nu))\mathrm{d}\nu\int_{\tau_0}^{\tau}\frac{(\tau-\nu)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mathrm{f}_2(\nu,x(\nu))\mathrm{d}\nu$$
$$+\sum_{i=1}^{m}\int_{\tau_0}^{\tau}\frac{(\tau-\nu)^{\beta_i-1}}{\Gamma(\beta_i)}\mathrm{g}_i(\nu,x(\nu))\mathrm{d}\nu,\ \tau\in\Delta.$$

- Taking $\mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_i^\star=1, \psi(\tau)=\phi(\tau)=\psi_i(\tau)=\sigma(\tau)=\frac{1}{1+e^{-\tau}}$, $i=1,\ldots,m$, the convenience of the sigmoid function is its derivative $\sigma'(\tau)=\sigma(\tau)(1-\sigma(\tau))$. So, the problem (**1**.1) takes the following form:

$$x(\tau)=\int_{\tau_0}^{\tau}\frac{\sigma(\nu)(1-\sigma(\nu))(\sigma(\tau)-\sigma(\nu))^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{f}_1(\nu,x(\nu))\mathrm{d}\nu$$
$$\times\int_{\tau_0}^{\tau}\frac{\sigma(\nu)(1-\sigma(\nu))(\sigma(\tau)-\sigma(\nu))^{\alpha_2-1}}{\Gamma(\alpha_2)}\mathrm{f}_2(\nu,x(\nu))\mathrm{d}\nu$$
$$+\sum_{i=1}^{m}\int_{\tau_0}^{\tau}\frac{\sigma(\nu)(1-\sigma(\nu))(\sigma(\tau)-\sigma(\nu))^{\beta_i-1}}{\Gamma(\beta_i)}\mathrm{g}_i(\nu,x(\nu))\mathrm{d}\nu,\ \tau\in\Delta.$$

- Let $\mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_i^\star=1, \psi(\tau)=\phi(\tau)=\psi_i(\tau)=\frac{\tau^\rho}{\rho}$, $\rho>0$, $i=1,\ldots,m$. Then, the problem (**1**.1) reduces to the Katugampola quadratic equation of fractional order:

$$x(\tau)=\rho^{1-\alpha_1}\int_{\tau_0}^{\tau}\frac{\nu^{\rho-1}(\tau^\rho-\nu^\rho)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{f}_1(\nu,x(\nu))\mathrm{d}\nu$$
$$\times\rho^{1-\alpha_2}\int_{\tau_0}^{\tau}\frac{\nu^{\rho-1}(\tau^\rho-\nu^\rho)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mathrm{f}_2(\nu,x(\nu))\mathrm{d}\nu$$
$$+\sum_{i=1}^{m}\rho^{1-\beta_i}\int_{\tau_0}^{\tau}\frac{\nu^{\rho-1}(\tau^\rho-\nu^\rho)^{\beta_i-1}}{\Gamma(\beta_i)}\mathrm{g}_i(\nu,x(\nu))\mathrm{d}\nu,\ \tau\in\Delta.$$

- If $\mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_i^\star=1, \psi(\tau)=\phi(\tau)=\psi_i(\tau)=\ln\tau$, $i=1,\ldots,m$, then the considered problem (**1**.1) reduces to the standard Hadamard quadratic equation of fractional order:

$$x(\tau)=\int_{\tau_0}^{\tau}\frac{(\ln(\tau)-\ln(\nu))^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{f}_1(\nu,x(\nu))\frac{\mathrm{d}\nu}{\nu}$$
$$\times\int_{\tau_0}^{\tau}\frac{(\ln(\tau)-\ln(\nu))^{\alpha_2-1}}{\Gamma(\alpha_2)}\mathrm{f}_2(\nu,x(\nu))\frac{\mathrm{d}\nu}{\nu}$$
$$+\sum_{i=1}^{m}\int_{\tau_0}^{\tau}\frac{(\nu)(\ln(\tau)-\ln(\nu))^{\beta_i-1}}{\Gamma(\beta_i)}\mathrm{g}_i(\nu,x(\nu))\frac{\mathrm{d}\nu}{\nu},\ \tau\in\Delta.$$

- Let $\mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_i^\star=1, \psi(\tau)=\phi(\tau)=\psi_i(\tau)=e^{-\lambda\tau}$, $i=1,\ldots,m$. Then, the nonlinear quadratic integral equation specified in (**1**.1) takes the following form:

$$x(\tau)=\int_{\tau_0}^{\tau}\frac{-\lambda e^{-\lambda\nu}(e^{-\lambda\tau}-e^{-\lambda\nu})^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{f}_1(\nu,x(\nu))\mathrm{d}\nu$$
$$\times\int_{\tau_0}^{\tau}\frac{-\lambda e^{-\lambda\nu}(e^{-\lambda\tau}-e^{-\lambda\nu})^{\alpha_2-1}}{\Gamma(\alpha_2)}\mathrm{f}_2(\nu,x(\nu))\mathrm{d}\nu$$
$$+\sum_{i=1}^{m}\int_{\tau_0}^{\tau}\frac{-\lambda e^{-\lambda\nu}(e^{-\lambda\tau}-e^{-\lambda\nu})^{\beta_i-1}}{\Gamma(\beta_i)}\mathrm{g}_i(\nu,x(\nu))\mathrm{d}\nu,\ \tau\in\Delta.$$

Finally, we testified the above results through different examples.

Example 4.1. *Consider the following NQFIE:*

$$\begin{aligned} x(\tau) &= \int_0^\tau \frac{(\tau-\nu)^{\frac{1}{5}-1}}{5\Gamma_5(1)} \frac{1}{2(e^\nu+1)}(x(\nu)+\sqrt{1+x^2(\nu)})\mathrm{d}\nu \\ &\times \int_0^\tau \frac{(\tau-\nu)^{\frac{2}{5}-1}}{5\Gamma_5(2)}(\frac{1+\sin(x(\nu))}{\nu+2})d\nu \\ &+\sum_{i=1}^{3}\int_0^\tau \frac{(\tau-\nu)^{\frac{\beta_i}{\mathrm{k}_i^\star}-1}}{\mathrm{k}_i^\star\Gamma_{\mathrm{k}_i^\star}(\beta_i)}\frac{\nu}{50i}(1+\frac{x(\nu)}{1+x(\nu)})\mathrm{d}\nu, \end{aligned}$$

which is a special case of our problem (**1**.1) *with* $\psi(\tau)=\phi(\tau)=\psi_i^\star(\tau)=\tau; i=1,\ldots,3, \alpha_1=1; \alpha_2=2; \beta_1=1; \beta_2=3; \beta_3=4; \mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_i^\star=5; i=1,\ldots,3, \tau_0=0, \tau_f=1, \Delta=[0;1]$,

$$\mathrm{f}_1(\tau,x)=\frac{1}{2(e^\tau+1)}(x+\sqrt{1+x^2});$$

$$\mathrm{f}_2(\tau,x)=\frac{1+\sin x}{\tau+2};$$

$$\mathrm{g}_i(\tau,x)=\frac{\tau}{50i}\left(1+\frac{|x|}{1+|x|}\right); i=1,\ldots,3.$$

Evidently, the functions $\mathrm{f}_1, \mathrm{f}_2$, *and* g_i; $i=1,\ldots,3$. *are continuous. Additionally, observe that the conditions (H2) and (H3) are fulfilled. Since, for any* $x, y \in \mathbb{R}$ *and* $\tau \in [0;1]$ *we have*

$$|\mathrm{f}_1(\tau,x)-\mathrm{f}_1(\tau,y)| \le \frac{1}{2}|x-y|,$$

$$|\mathrm{g}_i(\tau,x)-\mathrm{g}_i(\tau,y)| \le \frac{1}{50i}|x-y|,\ i=1,\ldots,3,$$

and

$$|\mathrm{f}_2(\tau;x)| \le \frac{1}{\tau+2}(1+|x|),$$

with $\mathbb{L}_{\mathrm{f}_1}=\frac{1}{2}$, $\mathbb{L}_{\mathrm{g}_i}=\frac{1}{50i}, \mu(\tau)=\frac{1}{\tau+2}$, *and* $\omega(|x|)=1+|x|$. *It remains to show that (H4) holds. To achieve this task, one may set*

$$\mathrm{f}_1^\star := \sup_{\tau\in\Delta}|\mathrm{f}_1(\tau,0)| = \frac{1}{4},\ \mathrm{g}_i^\star := \sup_{\tau\in\Delta}|\mathrm{g}_i(\tau,0)| = \frac{1}{50i}, i=1,\ldots,3;$$

$$\mu^\star := \sup_{\tau\in\Delta}|\mu(\tau)| = \frac{1}{2},\ \mathbb{K}_{\psi,\alpha_1,\mathrm{k}_1} := \frac{(\psi(\tau_f)-\psi(\tau_0))^{\frac{\alpha_1}{\mathrm{k}_1}}}{\Gamma_{\mathrm{k}_1}(\alpha_1+\mathrm{k}_1)} = \frac{1}{\Gamma_5(1)},$$

$$\mathbb{K}_{\phi,\alpha_2,\mathrm{k}_2} := \frac{(\phi(\tau_f)-\phi(\tau_0))^{\frac{\alpha_2}{\mathrm{k}_2}}}{\Gamma_{\mathrm{k}_2}(\alpha_2+\mathrm{k}_2)} = \frac{1}{2\Gamma_5(2)},\ \mathbb{K}_{\psi_1,\beta_1,\mathrm{k}^\star} = \frac{1}{\Gamma_5(1)},$$

$$\mathbb{K}_{\psi_2,\beta_2,\mathrm{k}_2{}^\star} := \frac{(\psi_2(\tau_f)-\psi_2(\tau_f))^{\frac{\beta_2}{\mathrm{k}_2^\star}}}{\Gamma_{\mathrm{k}_2^\star}(\beta_2+\mathrm{k}_2^\star)} = \frac{1}{3\Gamma_5(3)},$$

$$\mathbb{K}_{\psi_3,\beta_3,\mathrm{k}_3{}^\star} := \frac{(\psi_3(\tau_f)-\psi_3(\tau_0))^{\frac{\beta_3}{\mathrm{k}_3^\star}}}{\Gamma_{\mathrm{k}_3^\star}(\beta_3+\mathrm{k}_3^\star)} = \frac{1}{4\Gamma_5(4)}.$$

From above information, we can choose $\xi \in [0.03576, 7.0885]$. *Accordingly, all the assumptions of Theorem* **3**.*1 are fulfilled. One could directly detect that the aforementioned problem has at least one solution* $x \in C([0,1],\mathbb{R})$.

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