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ON THE LAGRANGE COMPLEX INTERPOLATION

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In lucrare prezint unele rezultate legate de interpolarea Lagrange în domeniul complex (cor. prop. 1şi prop. 2). Formula (6) este o extindere a formulei lui Shannon de eşantionare (7) , pentru cazul momentelor echidistante de eşantionare . In §2 am adăugat un rezultat privind eşantionarea in domeniul frecvenţelor. In §3 am dat o extindere multidimensională a formulei de eşantionare , care nu foloseşte interpolarea Lagrange, ci o abordare distribuţională .

In this work, I present some results regarding the Lagrange interpolation in the complex domain (cor. prop. 1 and prop. 2). Formula (6) is an extension of the well-known Shannon's sampling formula (7) , for sampling equidistant moments. In §2 I present a simple result regarding the sampling in frequency. In §3 I give a multidimensional extension , which does not use the Lagrange interpolation but a distributional approach.

Key words: Lagrange interpolation in the complex domain, sampling theorem, multidimensional sampling theorem.

1. Lagrange interpolation in complex domain

Recall that a function $f:\mathbb{C}\to\mathbb{C}$ is called entire if it is holomorphic ($\equiv$analytical) in all the complex plane. If (a_n), $n\geq 1$ is a given sequence of complex numbers and $z_n\to\infty$, the Lagrange interpolation problem consists of finding an entire function f so that $f(z_n)=a_n, \forall n\geq 1$.
This problem has solutions only if we impose some supplementary conditions regarding the sequence (a_n) .

PROPOSITION 1. Fix a real number T>0. Let $z_n = n\mathrm{T}$, with $n\in\mathbb{Z}$ and the function $\varphi:\mathbb{C}\to\mathbb{C}$, $\varphi(z)=\sin\frac{\pi z}{\mathrm{T}}$. If a sequence (a_n) , $n\in\mathbb{Z}$ has the property that $\forall\varepsilon>0$, $\exists\mathrm{M}>0$ and $\mathrm{N}(\varepsilon)$ so that $|a_n|<\frac{\mathrm{M}}{|n|^{1+\varepsilon}}$ for $n\geq\mathrm{N}(\varepsilon)$, then the series

$$f(z)=\sum_{n=1}^{\infty}\frac{a_n\varphi(z)}{\varphi'(z_n)(z-z_n)}, \qquad (1)$$

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is uniformly convergent on any compact in $\mathbb{C}$.

Proof : The series (1) becomes $\sum_{n\in Z} a_n \sin\frac{\pi z}{T}\frac{1}{\frac{\pi}{T}(z-nT)\cos n\pi}=$ $\sum_{n\in Z} a_n sa\frac{\pi}{T}(z-nT)$, where we have denoted $sa(u)=\frac{\sin u}{u}$ ($u\neq 0$) and $sa(0)=1$.

If $R>0$ and $|z|\leq R$, then for every $|n|$ large enough, we have

$|a_n sa(\frac{\pi}{T}(z-nT))|\leq\frac{M\cdot S(R)}{|n|^{1+\varepsilon}}$, where $S(R)=\sup\{|sa(u)|;|u|\leq R\}$.

So, for any N fixed, $\left|\sum_{|n|>N} a_n\cdot sa(\frac{\pi}{T}(z-nT))\right|\leq 2M\cdot S(R)\cdot\sum_{n=N}^{\infty}\frac{1}{n^{1+\varepsilon}}$.

Accordingly, the rest of the considered series of functions is uniformly convergent towards zero for $N\to\infty$

Applying the fact that the sum of a series of entire functions, uniformly convergent on any compact, is an entire function, we get the following:

COROLLARY : Under the hypothesis of prop. 1, the function $f:\mathbb{C}\to\mathbb{C}$, $f(z)=\sum_{n\in Z} a_n sa\frac{\pi}{T}(z-nT)$ is entire and, moreover $f(nT)=a_n$, for any $n\in\mathbb{Z}$ [1].

2. A generalization of the sampling Shannon formula

Fix $T>0$ and let $f:\mathbb{C}\to\mathbb{C}$ be an entire function. Denote $\rho_n=(n+\frac{1}{2})T$ and consider the complex integral

$$I_n(z)=\frac{1}{2\pi i}\int_{|z|=\rho_n}\frac{f(u)du}{(u-z)\sin\frac{\pi u}{T}},\ n\in\mathbb{Z} \tag{2}$$

The residue of the function under the integral in the simple pole kT is equal to $\frac{f(kT)}{(k\pi-\frac{\pi z}{T})\cos k\pi}$ and for $z\neq kT$, the residue in z is equal to $\frac{f(z)}{\sin\frac{\pi z}{T}}$. Then, acording to the residues theorem, it results that

$$I_n(z)=\sum_{|kT|\leq\rho_n}\frac{f(kT)}{(k\pi-\frac{\pi z}{T})\cos k\pi}+\frac{f(z)}{\sin\frac{\pi z}{T}} \tag{3}$$

On the other hand , we estimate the integral (2) direct by the parametrization of the path of integration ; put $u = \rho_n \cdot e^{it}$, $t \in [0,2\pi]$, hence

$$2\pi \, | \, \mathrm{I}_n(z) \, | = \left| \int_0^{2\pi} \frac{f(\rho_n e^{it}) \cdot \rho_n i e^{it}}{(\rho_n e^{i\theta} - z) \cdot \sin(\frac{\pi}{\mathrm{T}} \rho_n e^{it})} d\theta \right| \tag{4}$$

But for any $t \in [0,2\pi]$, $\exp(\frac{\pi}{\mathrm{T}} \rho_n \, | \sin t \, |) < | \sin \frac{\pi}{\mathrm{T}} \rho_n e^{it} \, |$.

Moreover , for any z fixed , $\lim_{n \to \infty} | \, e^{it} - \frac{z}{\rho_n} \, | = 1$ so $| \, e^{it} - \frac{z}{\rho_n} \, | > \frac{1}{2}$ for any n large enough . From (4) one can obtain the estimation

$$| \, \mathrm{I}_n(z) \, | \leq \frac{4}{\pi} \int_0^{2\pi} \frac{| \, f(\rho_n e^{it}) \, |}{\exp(\frac{\pi}{\mathrm{T}} \rho_n \, | \sin t \, |)} dt . \tag{5}$$

PROPOSITION 2. Let $\mathrm{T} > 0$ fixed and $f : \mathbb{C} \to \mathbb{C}$ be an entire function with the property that it there is $\mathrm{M} > 0$ and a real number $\beta \in (0, \frac{\pi}{T})$ so that $| \, f(x) \, | \leq \mathrm{M}$ for any $x \in \mathbb{R}$ and $| \, f(z) \, | \leq \mathrm{M} \cdot \exp(\beta r \, | \sin t \, |)$ for any $z = re^{it}$. Then

$$f(z) = \sum_{n \in Z} f(n\mathrm{T}) \cdot sa(\frac{\pi}{\mathrm{T}}(z - n\mathrm{T})) . \tag{6}$$

Proof : Thus, $| \, f(\rho_n e^{it}) \, | \leq \mathrm{M} \cdot \exp(\beta \rho_n \, | \sin t \, |)$; so, according to (5) ,

$| \, \mathrm{I}_n(z) \, | \leq \frac{4\mathrm{M}}{\pi} \int_0^{2\pi} \exp((\beta - \frac{\pi}{\mathrm{T}}) \cdot \rho_n \, | \sin t \, |) dt$. Because $\beta < \frac{\pi}{\mathrm{T}}$, $\mathrm{I}_n(z) \to 0$ for $n \to \infty$, uniformly on any compact, the relation (6) follows .

NOTE. If the values $f(n\mathrm{T})$, $n \in \mathbb{Z}$ are previously fixed , the entire function f is uniquely determinated (by applying the identity principle).

CORROLARY. If $f : \mathbb{R} \to \mathbb{C}$ belongs to $\mathrm{L}^1 \cap \mathrm{L}^2$ and supp$\hat{f} \subset [-b, b]$, then

$$f(t) = \sum_{n \in Z} f(\frac{n\pi}{b}) \cdot sa(bt - n\pi) , \text{ a.e. } t \in \mathbb{R}. \tag{7}$$

This is the classical formulation of the Shannon formula. The formula (6) is a generalization of this. Here is a simpler argument to get (7), by using distributions: if u means the unit step function then

$\hat{f}(\omega) = \sum_{n \in Z} (u(\omega + b) - u(\omega - b)) \cdot \hat{f}(\omega - 2nb)$, for any $\omega \in \mathbb{R}$ [indeed, if $| \, \omega \, | \geq b$, then $\hat{f}(\omega)=0$ and $u(\omega+b)-u(\omega-b)=0$; and if $|\omega|<b$, then $\omega-2nb\in(-b,b)$ only for n = 0 and for $n\neq 0$, $\hat{f}(\omega-2nb)=0$]. From the previous relation, one gets $\hat{f}(\omega)=(u(\omega+b)-u(\omega-b))\cdot\hat{f}(\omega)*\sum_{n\in Z}\delta(\omega-2nb)$ and it is enough to apply the Fourier inversion formula .

NOTE. 1) The function $\varphi(z)=sa(\frac{\pi z}{\mathrm{T}})$ is a good interpolator , in the sense that whenever $f(z)=\sum_n \varphi(z-n\mathrm{T})\cdot f(n\mathrm{T})$, it follows for any δ>0 , that $f(z)=\sum_n \varphi(z-n\mathrm{T}+\delta)\cdot f(n\mathrm{T}-\delta)$ [3].

2) The signals which do appear in nature are random and one can extend the relation (7) to random signals: namely if (ξ_t) is a stationary random signal with a null mean and a limited band of frequency , then $\xi_t=\sum_{n\in Z}\xi_{\frac{n\pi}{b}}\cdot sa(bt-n\pi)$ [2], [4].

The above corollary suggest a dual result , regarding the sampling in frequency of the spectral function $\hat{f}$ of f : qualitatively, the value of $\hat{f}$ in any frequency is well determined from the values in some discrete frequencies .

PROPOSITION 3. Suppose that $f\in \mathrm{L}^1\cap\mathrm{L}^2$ has a bounded duration (that is , $\exists\tau>0$ such that $f(t)=0$ for $|t|\geq\tau$). Then

$$\forall\omega\in\mathbb{R},\ \hat{f}(\omega)=\sum_{n\in Z}\hat{f}(\frac{n}{\tau})\cdot sa\pi(\tau\omega-n) \tag{8}$$

Proof. We have $f(t)=(u(t+\frac{\tau}{2})-u(t-\frac{\tau}{2}))\cdot(f(t)*\sum_{n\in Z}\delta(t-n\tau))$ and apply the Fourier transform : $\hat{f}(\omega)=\frac{\sin\pi\omega\tau}{\pi\omega}*[\hat{f}(\omega)\cdot\frac{1}{\tau}\sum_{n\in Z}\delta(\omega-\frac{n}{\tau})]=$

$=\sum_{n\in Z}\delta(\omega-\frac{n}{\tau})\cdot[\frac{\sin\pi\omega\tau}{\pi\omega\tau}*\hat{f}(\omega)]$, whence (8) .

3. A multidimensional extension of the Shannon formula

In the case of the multidimensional signals, the Lagrange type interpolation cannot be directly applied . Instead, in this case , one can use the previous argument , by making use the distributions.

For the multidimensional case , the Lagrange interpolation from §1, 2 is not directly possible, but it could be applied the previous argument , by making use the distributions. Namely, we prove:

PROPOSITION 4. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function from $L^2(\mathbb{R}^n)$ with supp $\hat{f}$ bounded. Then there are vectors $v_1,\ldots,v_n \in \mathbb{R}^n$ so that a.e. $\forall x \in \mathbb{R}^n$, $f(x)$ is well determined by the values of f in the set of values $\Omega = \left\{ \sum_{i=1}^{n} k_i v_i \,/\, k_1,\ldots,k_n \in Z \right\}$.

Proof. Choose $v_1,\ldots,v_n$ linear independent and let $u_1,\ldots,u_n$ their reciprocals (such that it takes place the following relation between euclidean inner products : $\langle u_i, v_j \rangle = 2\pi\delta_{ij}$ for every i,j) . We shall note $\underline{k} \cdot \underline{v} = \sum_{i=1}^{n} k_i v_i$ and we shall define a sampling function s , similar with "*sa*" , so that the following formula holds:

$$f(x) = \sum_{k \in Z^n} f(\underline{k} \cdot \underline{v}) \cdot s(x - \underline{k} \cdot \underline{v}) \text{ a.e. } x \in \mathbb{R}^n. \tag{9}$$

If such a function really exists, then

$$f(\underline{k} \cdot \underline{v}) \cdot s(x - \underline{k} \cdot \underline{v}) = \int_{R^n} f(y) \cdot s(x-y) \cdot \delta(y - \underline{k}\underline{v}) dy$$

But, $f(x) = \frac{1}{W} \sum_{k \in Z^n} f(y) e^{-i<y, \underline{k} \cdot \underline{u}>} \cdot s(x-y) dy$, where W is the volume of the parallelipiped built on the vectors $u_1,\ldots,u_n$. Applying the Fourier n-dimensional operator , it results $\hat{f}(\omega) = \frac{1}{W} \cdot \hat{s}(\omega) \cdot \sum_{k \in Z^n} \hat{f}(\omega + \underline{k} \cdot \underline{u})$ for any $\omega \in \mathbb{R}^n$. Then it is enough to choose vectors $v_1,\ldots,v_n$ so that the supports of $\hat{f}(\omega + \underline{k} \cdot \underline{u})$ are disjoint and consider a function s so that $\hat{s} = W$, constant on supp $\hat{f}$ and null for the values ω where $\hat{f}(\omega + \underline{k} \cdot \underline{u}) \neq 0$, for $\underline{k} \neq 0$. The formula (9) is then obtained by applying the Fourier inversion formula .

4. Conclusions

In this work , one presents some results regarding the Lagrange interpolation in the complex plane , which have close connections with the Shannon sampling theorem (propositions 1 , 2) . One can also apply the adopted method for the case of the random signals and that of nonequidistant moments . In the paragraph 2 I have added a formula (8) for the sampling in frequency. One asserts and proves an extension of the sampling theorem to the case of functions of several variables .

R E F E R E N C E S

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