TRAIN CONTROL PROBLEM

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This paper deals with the optimization of the railway transport system from the traction point of view. Optimizing the traction segment in a railway company means especially respecting the timetable and of course, the lowest fuel/electric power consumption. Our aim is three-fold: (1) to review and detail the optimal control theory of train movement compared with the presentations in the papers [1, 2 6 - 16]; (2) to determine the continuous transition from one phases to another for a globally optimal strategy on a track; (3) to formulate and to solve the problem of stochastic optimal control of train movement. It is reconfirmed that the optimal driving strategy for a train takes the form of a power-speed hold-coast-brake strategy, unless the track contains steep grades.

Keywords: optimal control involving ODEs, train optimal control, optimal stochastic control, bang-bang control.

1. Mathematical and physical ingredients

The actual requirements of the dynamic market economy are forcing the railway system to transform into a reliable alternative to the road and air traffic. From this perspective, the railways have to fulfil two key elements: (1) economical efficiency and reliability; (2) to offer what the potential customer needs. In particular, the railway system has to fulfil the following specific conditions: (1) freight service must be safe, cheap, fast and accessible (taking into account the complete service pack to be offered to customers situated far from the railway line); (2) long distance passengers service must be fast, highly comfortable (representing a true alternative to the airways) and to allow conditions for leisure, rest and entertainment; (3) short distance passenger service (including the metropolitan railways) must ensure fast links from the centers of the cities to the suburbs at low prices, compared to the bus services.

These are the main requirements demanded by the potential customers desiring prompt, safe and affordable services. It is important to know that their perception of the quality level of the transport service changes continuously.

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Railway operators have many analysis elements which might be influencing their economic efficiency. One of the main elements is the respect of the timetables or the decrease of the running times. The running time is the main referential, especially when it is related to the fuel or power consumption. The optimization of the running times and the fuel/electric power consumption is strictly related to the safety and modern signalling system.

The specialists in the train control problems used the following data: T is the time allowed for the journey, x is the distance between two stations, $u(t)$ is the accelerations applied to the train, $v(t)$ is the speed of the train, and $-r(v(t))$ is the resistive acceleration due to the friction. The movement of the train is governed by the Newton law

$$\ddot{x}(t) = u(t) - r\big(v(t)\big), \tag{1}$$

where $r(v)$, $v \in [0,\infty)$ is strictly increasing and convex function and the acceleration $u(t)$ (control variable) is limited by the relation $|u(t)| \leq 1$. The theory (see energy consumption) involves also the positive part of $u(t)$, defined by

$$u_+(t) = \frac{1}{2}(u(t) + |u(t)|) \tag{2}$$

The increasing and convex function $r(v)$ is exemplified by the formula

$$r(v) = a + bv + cv^2, v \in [0,\infty), \tag{3}$$

where a, b, c are known real numbers subject to $a > 0$, $b > 0$, $c > 0$. For simulations, it is used

$$r(v) = 0.015 + 0.00003v + 0.000006v^2 \tag{4}$$

2. Train control problem

The problem of finding the best way to drive to the next destination can be formulated as an optimal control problem (local energy minimization principle). That is, we wish to find the sequence of control settings that will get the train to the next destination on time, and with minimal energy consumption.

Mathematical assumptions (i) $U=L^\infty([0,T])$ is the set of measurable and bounded functions on the interval $[0,T]$, endowed with the norm

$$\|u\|_\infty = sup|u(t)|,\ t \in [0,T] \tag{5}$$

(ii) $v=C^{0,1}([0,T])$ is the set of Lipschitz functions on the interval [0, T], endowed with the norm

$$\|v\| = \|v\|_\infty + \|\dot{v}\|_\infty. \tag{6}$$

A feasible pair $(u,v)\epsilon F = U \times V$ must satisfies $\|u\|_\infty \leq 1$ and $v(0) = v(T) = 0$.

In the following problem, x and v are state variables and u is the *control variable*.

Deterministic Problem *Minimize the mechanical energy consumption*

$$J(u(\cdot) = \int_0^T u_+(t)v(t)dt \tag{7}$$

subject to

(i) *the ODE constraints*

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = u(t) - r\big(v(t)\big), \quad v(0) = v(T) = 0, \tag{8}$$

(ii) *the isoperimetric constraint*

$$\int_0^T v(t)dt = X, \tag{9}$$

(iii) *the control inequality constraint*

$$|u(t)| \leq 1. \tag{10}$$

Solution. We shall look to apply the Pontryagin maximum principle. For that we use the Hamiltonian

$$H(x,u,v) = -u_+ v + p_1 v + p_2\big(u - r(v)\big), \tag{11}$$

where $p_1= p_1(t)$ and $p_2=p_2(t)$ are the Lagrange multipliers. The Hamiltonian can be rewritten as a piecewise function of degree at most one with respect to u, namely

$$H(x,v,u) = \begin{cases} p_2 u + p_1 v - p_2 r(v), & \text{for} -1 \leq \text{u} < 0 \\ u(p_2 - v) + p_1 v - p_2 r(v), & \text{for } 0 \leq \text{u} \leq 1 \end{cases} \tag{12}$$

The adjoint ODEs

$$\dot{p_1}(t) = -\frac{\delta H}{\delta x}, \quad \dot{p_2}(t) = -\frac{\delta H}{\delta v} \tag{13}$$

become $\dot{p_1}(t) = 0 \quad \dot{p}_2(t) = u_+(t) - p_1 + p_2(t)\frac{\delta r}{\delta v}(t)$ (14)

Consequently, $p_1(t)=p_1$ (constant). For $p_2(t)$, we have explicitly

$$\dot{p_2}(t) = u(t) - p_1 + p_2(t)\frac{\delta r}{\delta v}(t) \qquad \text{for } \ 0 \le u \le 1 \tag{15}$$

$$\dot{p_2}(t) = -p_1 + p_2(t)\frac{\delta r}{\delta v}(t) \qquad \text{for } \ -1 \le u < 0. \tag{16}$$

If the Hamiltonian is linear in the control variables and the control variables have simple bounds then the optimal control is a combination of bang-bang control and singular arcs.

The Hamiltonian is piecewise linear (function of degree at most one) in the control, the control variable have simple bounds, and the switching functions are $p_2(t)$ and $p_2(t)–v(t)$, respectively. Therefore the optimal control is a combination of *bang-bang control and singular arcs*. The optimal control $u^*(t)$ is discontinuous: it jumps from a minimum to a maximum and viceversa in response to each change in the sign of switching function.

(i) The optimal control as determined by the switching function $p_2(t)$ is

$$u^*(t) = \begin{cases} 0, \ \text{for } p_2(t) > 0 \text{ bang} - \text{bang control} \\ -1, \ \text{for } p_2(t) < 0 \text{ bang} - \text{bang control} \\ \text{undetermined}, \ \text{for } p_2(t) = 0 \end{cases} \tag{17}$$

Suppose $t=t_s$ is the switching time, i.e., a solution of the equation $p_2(t)=0$. Then the optimal control is rewritten.

$$u^*(t) = \begin{cases} \text{either } 0 \text{ or } -1, \ \text{for } t \in [0,T] \text{ and } -1 \ \text{ for } \ t \in [t_s, T] \\ 0, \ \text{for } t \in [0, t_s) \text{and } \ 0 \quad \text{ for } \ t \in [t_s, T] \\ -1, \ \text{for } t \in [0, t_s) \end{cases} \tag{18}$$

The most interesting case is those of finite number (or countable set) of switching times.

(ii) The optimal control as determined by the switching function $p_2(t)–v(t)$ is

$$u^*(t) = \begin{cases} 1, \text{for } \ p_2(t) > v(t) \text{bang} - \text{bang control} \\ 0, \ \text{for } p_2(t) < v(t) \text{bang} - \text{bang control} \\ \text{undetermined}, \ \text{for } p_2(t) = v(t) \end{cases} \tag{19}$$

Suppose $t=t_s$ is the switching time, i.e., a solution of the equation $p_2(t)=v(t)$. Then the optimal control is rewritten

$$u^*(t) = \begin{cases} either\ 1\ or\ 0, & \text{for } t \in [0,\mathrm{T}] \\ 1, & for\ t \in [0,t_s) \text{ and 0 for } t \in [t_s,T] \\ 0, & for\ t \in [0,t_s) \text{ and 1 for } t \in [t_s,T] \end{cases} \tag{20}$$

The most interesting case is those of finite number (or countable set) of switching times.

3. Maximum acceleration, coast and maximum brake

There are three cases which correspond respectively to *maximum acceleration*, *coast and maximum brake*. These all occur in a typical optimal control strategyof the train, but their presence is only piecewise, splitting the interval [0,T] into subintervals, i.e., $T=T_1+...+T_n$.

Case 1 (phase 1: maximum acceleration):

$$p_2(t) > v(t) \Longrightarrow u^*(t) = 1. \tag{21}$$

The time t is in a first subinterval $[0,T_1]$ of the interval $[0,T]$. This case include the optimal adjoint evolution

$$p_1(t) = p_1,\ \dot{p_2}(t) = 1 - p_1 + p_2(t)\frac{\mathrm{d}r}{\mathrm{d}v}(t) \tag{22}$$

and the optimal initial evolution

$$\dot{x}(t) = v(t),\ \ \dot{v}(t) = 1 - r\big(v(t)\big). \tag{23}$$

For details, we shall use the expression (1) for $r(v)$. Then the second initial ODE become

$$\frac{dv}{cv^2+bv+a-1} = -\mathrm{d}v \tag{24}$$

Let $\Delta_1=b^2-4c(a-1)$ be the discriminant of the polynominal $cv^2+bv+a-1$. If $\Delta_1\geq 0$, then we find the roots

$$\alpha = \frac{-b-\sqrt{\Delta_1}}{2c}\ ,\beta = \frac{-b+\sqrt{\Delta_1}}{2c}. \tag{25}$$

We remark that $\alpha < 0$ and $\beta < 0$ for $\alpha > 1$, and $\beta > 0$ for $\alpha < 1$. In our theory, we need $\beta > 0$ and $v \in [0,\beta)$ or $v \in [0,\infty)$. Taking into consideration the conditions $v(0)=0$, $v(T)=0$, we must work on the interval $[0,\beta)$ only. Then we find

$$\int \frac{dv}{cv^2+bv+a-1} = \begin{cases} \frac{1}{c(\alpha-\beta)} \ln \frac{v-\alpha}{\beta-v} & \text{for } \Delta_1 > 0 \\ -\frac{1}{c(v-\alpha)} & \text{for } \Delta_1 = 0 \\ \frac{2}{\sqrt{-\Delta_1}} \operatorname{atan} \frac{2cv+b}{\sqrt{-\Delta_1}} & \text{for } \Delta_1 < 0. \end{cases} \tag{26}$$

Conclusions:

(i) For $\Delta_1 > 0$, the optimal evolution is

$$t = C_1 - \frac{1}{c(\alpha-\beta)} \ln \frac{v-\alpha}{\beta-v}, \quad v(t) = \frac{\alpha+\beta e^{c(\alpha-\beta)(C_1-t)}}{1+e^{c(\alpha-\beta)(C_1-t}}, \tag{27}$$

$$x(t) = \alpha(t - C_1) + \frac{1}{c} \ln\left(e^{c(\alpha-\beta)(C_1-t)} + 1\right) + C_2. \tag{28}$$

Imposing the condition $v(0)=0$, we find

$$C_1 = \frac{1}{c(\alpha-\beta)} \ln \frac{-\alpha}{\beta}. \tag{29}$$

Similary, the condition $x(0)=0$ produces

$$C_2 = \alpha C_1 - \frac{1}{c} \ln\left(e^{c(\alpha-\beta)C_1} + 1\right). \tag{30}$$

Eliminating the parameter t, we obtain $x=x(v)$. By parametric pilot, we find $v=v(x)$.

Let us consider the ODE $\dot{v}(t) = 1 - (cv^2(t) + bv(t) + a)$. If $a < 1$, then its equilibrium (critical) point is the positive solution β of the equation $0=1-(cv^2+bv+a)$. The solution β is a supremum of the function $v(t)$ since in its left the function $v(t)$ is increasing ($\lim_{t\to\infty} v(t)=\beta$) and in the right is decreasing to $\lim_{t\to-\infty} v(t)=\alpha < 0$. The train movement suppose $v(t)$ is bounded and increasing.

(ii) For $\Delta_1 \leq 0$, the ODE $\dot{v}(t) = 1 - (cv^2(t) + bv(t) + a)$ shows that thefunction $v(t)$ is decreasing. Hence this case is not convenient for a strating phase (acceleration).

Case 2 (phase 2: coast):

$$p_2(t) < v(t) \Rightarrow u^*(t) = 0. \tag{31}$$

The time t is in a second subinterval [T_1,T] of the interval [0,T]. This case include the optimal adjoint evolution

$$p_1(t) = p_1,\ \dot{p_2}(t) = -p_1 + p_2(t)\frac{dr}{dv}(t) \tag{32}$$

and the optimal initial evolution

$$\dot{x}(t) = v(t),\ \dot{v}(t) = -r\big(v(t)\big). \tag{33}$$

Obviously, the speed $v(t) \geq 0$ is decreasing (see the second ODE), as requires the coast phase. Introducing the expression (1) for $r(v)$, we obtain the details.

Case 3 (phase 3: maximum brake):

$$p_2(t) < 0 \Longrightarrow u^*(t) = -1. \tag{34}$$

The time t is in a third subinterval [T_3,T_4] of the interval [0,T]. This case include the optimal adjoint evolution

$$p_1(t) = p_1,\ \dot{p_2}(t) = -p_1 + p_2(t)\frac{dr}{dv}(t) \tag{35}$$

and the optimal initial evolution

$$\dot{x}(t) = v(t),\ \dot{v}(t) = -1 - r\big(v(t)\big). \tag{36}$$

Obviously, the speed $v(t) \geq 0$ is decreasing (see the second ODE), as requires the maximum brake phase. Introducing the expression (1) for $r(v)$, we obtain the details.

4. Singular control

If the switching function vanishes identically for some time interval, the control u has no influence on the Hamiltonian function H, i.e., the maximum principle fails. This is referred to *singular control*.

Case 1 (intermediary phase: velocity hold):

$$p_2(t) = v(t) \Longrightarrow u^*(t) \in [0,1]. \tag{37}$$

The time t is in a subinterval $[T_2,T_3]$ of the interval $[0,T]$. Since the equality $p_2(t)=v(t)$ must be maintained on a non-trivial interval, it follows $\dot{p}_2(t) = \dot{v}(t)$. From the evolution ODE and from the second adjoint ODE, we find the conditions

$$\frac{d}{dv}\left(vr(v)\right) = p_1. \tag{38}$$

Hence $v(t)r(v(t))=p_1v(t)$, i.e., $r(v)=p_1$ with the solution $v(t)=V$ (velocity hold) on our interval, since $r(v)$ is strictly increasing and convex. The optimal control is $u^*(t)=r(V)$. Hence $v(t)$, $(t)\in[T_1,T_2]$ and the continuity of phases give the condition $V=v(T_2)$. The function $\varnothing(v)=vr(v)$ is strictly increasing and convex.

Now $d\varnothing/dv(t)\geq r(0)=a$ and this situation can only occur if $p_1 \geq a > 0$.

Case 2 (partial brake):

This case requires that $p_2(t)=0$ is true on a non-trivial interval $[0,t_0)\subset[0,T]$ and corresponds to partial braking. Since, $u_+^*(t)=0$, the second adjoint equation implies $0=p_1$, and consequently on this interval the initial ODEs and the adjoint ODEs are inactive. Hence $H^*=0$. Since, $v(0)=0$, an optimal strategy can only start with $u^*(t)=1$, for $p_2(t) > v(t) > 0$, and switch to $u^*(t)=0$ at the moment t_1, for $0 < p_2(t) < v(t)$. Set $\eta(t)=p_2(t)/v(t)$. Then $\eta(t_1)=1$. On the other hand, for $u^*(t)=0$, $t\in[t_0,t_1]\cup[t_1,T]$ the second initial ODE and the second adjoint ODE become

$$\dot{v}(t) = -r\big(v(t)\big),\ \ \dot{p_2}(t) = p_2(t)\frac{\mathrm{d}}{\mathrm{d}v}r\big(v(t)\big). \tag{39}$$

Consequently,

$$\dot{\eta}(t) = \frac{1}{v(t)}\frac{\mathrm{d}}{\mathrm{d}v}r\big(v(t)\big) > 0. \tag{40}$$

This gives $\eta(t) > 1$ for $t > t_1$ wich contradicts the double inequality $0<\eta(t)< 1$. Consequently, we cannot switch from $u^*(t)=1$ to $u^*(t)=0$ and this case cannot occur (it rests $p_1\neq 0$).

5. Energy-efficient speed profile

The book [9] suggested that an energy-efficient speed profile should contain at least three or four phases coupled by continuity: (i) *maximum acceleration, coast and maximum brake*; (ii) *maximum acceleration, hold speed,* *coast and maximum brake*. All the experiments confirmed that these strategies are indeed efficient.

Accelerate-brake strategy The set $\mathcal{F}$ is non-empty. Indeed the initial condition problem

$$\dot{v}(t) = 1 - r\big(v(t)\big), \quad v(0) = 0 \tag{41}$$

has a unique solution $v_1(t)$, $t \geq 0$, and the final condition problem

$$\dot{v}(t) = -1 - r\big(v(t)\big), \quad v(T) = 0 \tag{42}$$

has a unique solution $v_2(t)$, $t \leq T$. Further there exists a unique point $t=T_1$ where $v_1(T_1)=v_2(T_1)$ (the two phases are joined by continuity). The pair of piecewise functions

$$u(t) = \begin{cases} 1 & \text{for } \; t \in (0, T_1) \\ -1 & \text{for } \; t \in (T_1, T) \end{cases} \tag{43}$$

and $$v(t) = \begin{cases} v_1(t) & \text{for } \; t \in (0, T_1) \\ v_2(t) & \text{for } \; t \in (T_1, T) \end{cases} \tag{44}$$

satisfies the conditions and represents an *accelerate-brake strategy*.

Accelerate- coast-brake strategy Let us look for more feasible pairs. We choose $T_2 \in [0, T_1]$ and find the unique solution $v_3(t)$, $t \geq T_2$ of the problem

$$\dot{v}(t) = -r\big(v(t)\big), \quad v(T_2) = v_1(T_2). \tag{45}$$

In the condition $v_3(T) \geq 0$, there exists a unique point $T_3 \in [T_1, T]$, with $v_3(T_3)=v_2(T_3)$ it follows that the pair

$$u(t) = \begin{cases} 1 & \text{for } \; t \in (0, T_2) \\ 0 & \text{for } \; t \in [T_2, T_3] \\ -1 & \text{for } \; t \in (T_3, T) \end{cases}$$

(46)

and $$v(t) = \begin{cases} v_1(t) & \text{for } \; t \in (0, T_2) \\ v_3(t) & \text{for } \; t \in [T_2, T_3] \\ v_2(t) & \text{for } \; t \in (T_3, T) \end{cases} \tag{47}$$

represents an *accelerate-coast-brake strategy*.

Mathematical reasoning made above confirms again the theory, and they can be summarized by

Theorem 1. *An efficient speed profile consists in at least four steps coupled by continuity:*

(1) *The condition* $v(0)=0$ *imposes that the first phase must be "maximum acceleraion", on the interval* $[0,T_1]$, *solution of the inequation* $p_2(t)>v(t)$. *It follows* $v(T_1)<\beta$.

(2) *Then,* $p_2(t)=v(t)$, *for* $t\in[T_1,T_2]$,, *when* $v(t)=v(T_1)$, *i.e., hold speed.*

(3) *Further,* $0 < p_2(t) < v(t)$, *for* $t\in[T_2,T_3)$, *and* $p_2(T_3)=0$ *i.e., coast case.*

(4) *Finally,* $p_2(t)<0$, *for* $t\in[T_3,T]$, *and* $v(T)=0$ *i.e., total brake.*

6. Numerical simulation

The numerical simulation scenarios in [9] reveals the optimal control of the train movement. Our numerical simulations shows that the speed (acceleration) profile for „accelerate-hold-cost-brake" strategy is represented by the shape in the Fig. 1 (Fig. 2).

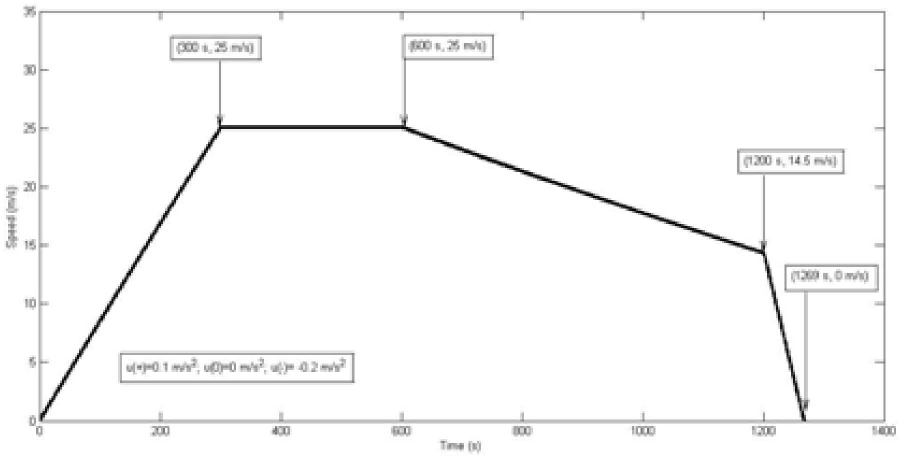

Fig. 1. Speed profile

7. Discrete train control problem

The movement process of the train can be conceptualized also in discrete time. This is perhaps the greatest source of confusion among practitioners, both in terms of implementation and psyhical interpretation. In general, the train movement occurs in continuous time but we observe it at fixed discrete-time intervals. Thus, continuous time in conceptually and theoretically appealing, but in practice it is perhaps more intuitive to interpret movement in discrete intervals.

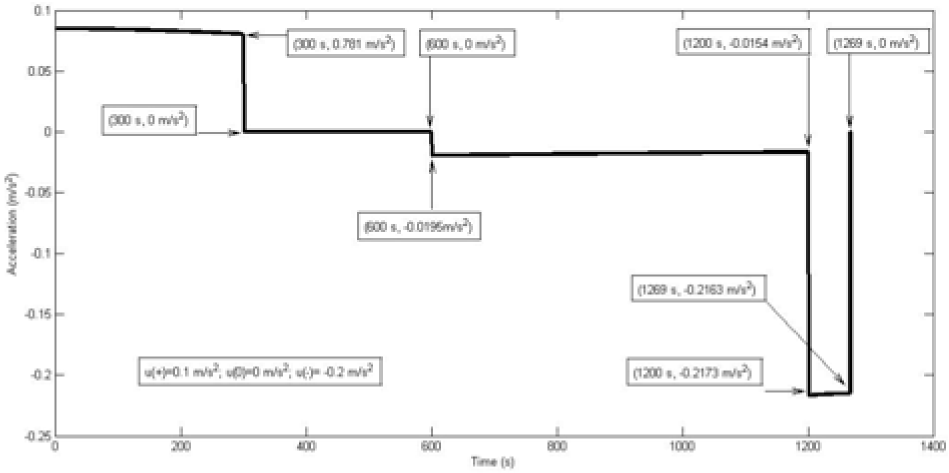

Fig. 2. Acceleration profile

Minimize the mechanical energy consumption

$$J\big(u_k(\cdot)\big) = \sum_{k=1}^{N-1} u_{+k} v_k \tag{48}$$

subject to

(i) The Ode constraints

$$x_{k+1} - x_k = v_k, v_{k+1} - v_k = u_k - r(v_k), \\ 1 \le k \le N-1, v(0) = v(N) = 0, \tag{49}$$

(ii) The isoperimetric constraint

$$\sum_{k=1}^{N-1} v_k = X, \tag{50}$$

(iii) The control inequality constraint

$$|u_k| \le 1. \tag{51}$$

Here x_k, u_k, are the state variables. The maximum principle of Pontryagin shows that if we denote by p_k and q_k the costate or adjoint variables, and denoting

$$H_k = -u_{+k}v_k + p_k v_k + q_k\big(u_k - r(v_k)\big), \tag{52}$$

Then $p_k - p_{k-1} = -\frac{\delta H_k}{\delta x_k}, \quad q_k - q_{k-1} = -\frac{\delta H_k}{\delta v_k}, \;\; 2 \le k \le N-1$ (53)

and $H_k^* = max_{uk} H_k, \;\; 1 \le k \le N.$ (54)

8. Optimal stochastic movement of the train

Notably, the speed of movement in intrinsically linked in current continuous-time random walk formulations, and this can have important implications when interpreting train behavior (see [3-5]).

Let $t \in [0,T]$. Let $x(t)$ be the stochastic position variable, $v(t)$ be the stochastic speed variable, $(x(t),v(t))$ *be a stochastic or diffusion process*, $W(t)$ be a Wiener process, σ be a diffusion coefficient and $u(t)$ be the *random control variable*. The stochastic process $(x(t),v(t))$ is usually a Markov process.

Stochastic problem (*train stochastic optimal control problem*) Find

$$max_u I\big(u(\cdot)\big) = E\left\{-\int_0^T u_+(t)v(t)dt\right\} \tag{55}$$

constrained by

$$\mathrm{d}x(t) = v(t)dt, \;\; \mathrm{d}v(t) = \Big(u_+(t) - r\big(v(t)\big)\Big)\,\mathrm{d}t + \sigma \mathrm{dW(t)}, \tag{56}$$
$$\mathrm{x}(0) = 0, \;\; v(0) = 0.$$

Solution In our context, we use a *control Hamiltonian stochastic 1-form*

$$\mathcal{H}(t,x,u,p) = \big(-u_+v + p_1 v + p_2\big(u_+ - r(v)\big)\big)dt. \tag{57}$$

and its pullback. The adjoint linear stochastic differential system

$$\mathrm{d}p_1(t) = -\frac{\delta\mathcal{H}}{\delta x}, \; \mathrm{d}p_1(t) = -\frac{\delta\mathcal{H}}{\delta v} \tag{58}$$

is $\mathrm{d}p_1(t) = 0, \;\; \mathrm{d}p_2(t) = \left(u_+(t) - p_1 + p_2(t)\frac{dr}{dv}(t)\right)\mathrm{d}t.$ (59)

Consequently, $p_1(t)=p_1$ (constant). For $p_2(t)$, we can write explicitly

$$\mathrm{d}p_2(t) = \left(u(t) - p_1 + p_2(t)\frac{\mathrm{d}r}{\mathrm{d}v}(t)\right)\mathrm{d}t \qquad \text{for} \qquad 0 \le u \le 1 \qquad (60)$$

$$\mathrm{d}p_2(t) = \left(-p_1 + p_2(t)\frac{\mathrm{d}r}{\mathrm{d}v}(t)\right)\mathrm{d}t \qquad \text{for} \qquad -1 \le u \le 0 \qquad (61)$$

9. Conclusions

Circumstances which make train control a pressing problem at the present time are very well known. However, automatic control can not be done without knowledge of the mathematical theory of optimal control. That is why, in our paper we clarify the idea of cost functional, ODE constraints, isoperimetric constraint, Pontryagin maximum principalefor a train control problem (see [1, 2, 6-16]).

This article is addressed not only to mathematicians wanting to know more about mathematical issues associated with concrete applications, but also to engineers already acquainted with classical techniques of optimal control, wishing to get more familiar with the more modern approaches of geometric control and other mathematical notions that have demonstrated significant enhancements in classical train problem, or to discipline to nontrivial examples in transport problems.

The article presents a scholarly research application, with mathematical solutions for strategy development of optimization of energy consumption with implications in the field of railway transportation. It is a first step to achieve an intelligent railway vehicle. Control system of the vehicle drive regime for optimization of energy consumption is an emerging technology for railway traction. Optimization energies resource consumption is for railway system a high priority both in terms of the efficiency of the system and in terms of protecting the environment. Acceptance and usage of new technologies for drive regime of traction railway vehicles will determine new technologie applications for railway infrastructure and new strategies for vehicle driving and traffic management for a high economic efficency of railway system.

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