WINTGEN INEQUALITIES ALONG RIEMANNIAN SUBMERSIONS

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In this paper, a Wintgen inequality is obtained depending on O'Neill's tensor field $\mathcal{T}$ along a Riemannian submersion from a real space form to a Riemannian manifold and the geometric meaning of the equality case is provided. Then, a Wintgen inequality is derived along a Riemannian submersion from a complex space form to a Riemannian manifold, and a geometric result is provided in the case of equality. In addition, a Wintgen inequality is obtained using concepts based on O'Neill's tensor field $\mathcal{A}$, and it is shown that the condition for equality is essentially equivalent to the integrability of the horizontal distribution. This condition is also investigated in the case of a complex space form.

Keywords: Wintgen inequality, Riemannian manifold, Riemannian submersion, Anti-invariant Riemannian submersion, Invariant Riemannian submersion, Complex space form

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1. Introduction

In geometry, inequalities establish a connection between intrinsic invariants and extrinsic invariants and play an important role in the characterization of the geometric object. For a surface $\mathfrak{M}$ in the Euclidean space E^3, the Euler inequality

$$\rho \leq ||H||^2$$

is fulfilled, where ρ is the (intrinsic) Gauss curvature of $\mathfrak{M}$ and $||H||^2$ is the (extrinsic) squared mean curvature of $\mathfrak{M}$. Furthermore, $\rho = ||H||^2$ everywhere on $\mathfrak{M}$ if and only if $\mathfrak{M}$ is totally umbilical.

Wintgen [22] provided a basic relationship among the Gauss curvature ρ, the normal curvature $\rho^{\perp}$ and the squared mean curvature $||H||^2$ of the surface $\mathfrak{M}$ in $\mathbb{E}^4$ as follows:

$$\parallel H \parallel^2 \geq \rho + \rho^{\perp}. \tag{1}$$

The equality in the Wintgen inequality holds if and only if the curvature ellipse of $\mathfrak{M}$ is a circle. This inequality is called *the Wintgen inequality*. A surface $\mathfrak{M}$ in $\mathbb{E}^4$ is called an ideal Wintgen surface if it satisfies the equality case of the Wintgen inequality (1) identically. Chen [5] classified ideal Wintgen surfaces in the 4-dimensional Euclidean space.

In a real space form, Guadalupe and Rodriguez provided the Wintgen inequality for a surface [10]. The Wintgen inequality for a submanifold with codimension 2 of a real space form $N^{n+m}(c)$ with constant sectional curvature c was obtained by De Smet, Dillen, Verstraelen, and Vrancken [7]:

$$\parallel H \parallel^2 \geq \rho + \rho^{\perp} - c, \tag{2}$$

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where ρ is the normal scalar curvature, $\rho^{\perp}$ is the normalized normal scalar curvature and $||H||^2$ is the squared mean curvature of $\mathfrak{M}$. Moreover, the authors claimed as a conjecture that this inequality is valid for a submanifold with arbitrary codimension. We note that Chen [6] proved this inequality (2) for normally flat submanifolds earlier in 1996. Xie [21] classified the equality case for codimension ≥ 3, depending on the constancy of the mean curvature, the scalar curvature and the normal curvature. This the conjecture was independently proved by Ge-Tang [12] and Lu [13]. Then, Dillen, Fastenakels, and Veken [8] and Mihai [14] investigated the Wintgen inequality for submanifolds of a complex space form. After these studies, this subject has been studied very actively, [1], [2], [8], [14, 15, 16]. This Wintgen inequality is also obtained for submanifolds of statistical manifolds [3] and [17]. For a list of publications on this topic and more results on submanifolds, refer to the survey paper [4].

The aim of this paper is to derive the Wintgen inequality for Riemannian submersions where the domain is a real space form and complex space form and to provide a geometric interpretation for the equality case.

The paper is organized as follows. In section 2, the basic notions and formulas required for the paper are reminded. In section 3, a lemma is provided that allows for the definition of the notion of normal curvature for a Riemannian submersion. Then, Wintgen inequality is provided using O'Neill's tensor field $\mathcal{T}$ for a Riemannian submersion. In section 4, Wintgen inequality is obtained by tensor field $\mathcal{T}$ again for a Riemannian submersion from a complex space form to a Riemannian manifold. In section 5, the Wintgen inequality is obtained by O'Neill's tensor field $\mathcal{A}$ for a Riemannian submersion. Here, it is shown that the condition for equality is equivalent to the integrability of the horizontal distribution. Finally, the Wintgen inequality is obtained using notions defined by O'Neill's tensor field $\mathcal{A}$ in the case where the total manifold of the Riemannian submersion is a complex space form.

2. **Preliminaries**

Let Ψ be a Riemannian submersion from a real space form $(\mathfrak{M}^m(c), \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ). Let $\chi^{\diamond}(\mathfrak{M}) = \Gamma((\ker\Psi_*)^{\perp})$ and $(\chi^{\natural}(\mathfrak{M}) = \Gamma(\ker\Psi_*)$ be the spaces of horizontal and vertical fields, respectively. Then we have [9]

$$\nabla_{\varsigma_1}\varsigma_2 = \mathcal{T}_{\varsigma_1}\varsigma_2 + \widehat{\nabla}_{\varsigma_1}\varsigma_2, \tag{3}$$

$$\nabla_{\varsigma_1}\zeta_1 = h\nabla_{\varsigma_1}\zeta_1 + \mathcal{T}_{\varsigma_1}\zeta_1, \tag{4}$$

$$\nabla_{\zeta_1}\varsigma_1 = \mathcal{A}_{\zeta_1}\varsigma_1 + v\nabla_{\zeta_1}\varsigma_1, \tag{5}$$

$$\nabla_{\zeta_1}\zeta_2 = h\nabla_{\zeta_1}\zeta_2 + \mathcal{A}_{\zeta_1}\zeta_2. \tag{6}$$

for horizontal vector fields ζ_1 and ζ_2 and vertical vector fields ς_1 and ς_2, where ∇ is the Levi-Civita connection of $\mathfrak{M}$, $\mathcal{T}$ and $\mathcal{A}$ are the O'Neill tensor fields. We note that the tensor $\mathcal{A}$ measures the integrability of the horizontal distribution. Denote by $\mathbf{R}^{\mathfrak{M}}$ and $\widehat{\mathbf{R}}$ the Riemannian curvature tensor of $\mathfrak{M}$ and the vertical distribution, respectively. Then, we have

$$g(\mathbf{R}^{\mathfrak{M}}(U,\varsigma_1)\varsigma_2,\varsigma_3) = g(\widehat{\mathbf{R}}(U,\varsigma_1)\varsigma_2,\varsigma_3)) - \hbar_{\mathfrak{M}}(\mathcal{T}(U,\varsigma_3),\mathcal{T}(\varsigma_1,\varsigma_2)) + \hbar_{\mathfrak{M}}(\mathcal{T}(\varsigma_1,\varsigma_3),\mathcal{T}(U,\varsigma_2)) \tag{7}$$

for $U,\varsigma_1,\varsigma_2,\varsigma_3 \in \chi^{\natural}(\mathfrak{M})$. For more details, see [20]. Let $(\mathfrak{M}, g)$ and $\left(B, g^{'}\right)$ be Riemannian manifolds, and $\pi : (\mathfrak{M}, g) \to (B, g^{'})$ a Riemannian submersion. Then the following identities are satisfied

$$\begin{aligned} g((\nabla_U\mathcal{T})_{\varsigma_1}\zeta_1,\zeta_2) &= g(\mathcal{T}_U\zeta_1,\mathcal{T}_{\varsigma_1}\zeta_2) - g(\mathcal{T}_U\zeta_2,\mathcal{T}_{\varsigma_1}\zeta_1),\\ g((\nabla_{\zeta_1}\mathcal{A})_{\zeta_2}U,\varsigma_1) &= g(\mathcal{A}_{\zeta_1}U,\mathcal{A}_{\zeta_2}\varsigma_1) - g(\mathcal{A}_{\zeta_1}\varsigma_1,\mathcal{A}_{\zeta_2}U). \end{aligned} \tag{8}$$

for horizontal vector fields $\zeta_1, \zeta_2, \zeta_3$ and vertical vector fields $U, \varsigma_1, \varsigma_2$ [20].

3. Wintgen inequalities along Riemannian submersions on Vertical distributions

Let $\mathfrak{M}^m(c)$ be a real space form of constant sectional curvature c. Then the Riemannian curvature tensor $\mathbf{R}^{\mathfrak{M}}$ takes the following expression:

$$\hbar_{\mathfrak{M}}(\mathbf{R}^{\mathfrak{M}}(\zeta_1,\zeta_2)\zeta_3,\varsigma_2) = c\left\{\hbar_{\mathfrak{M}}(\zeta_1,\varsigma_2)g_{\mathfrak{M}}(\zeta_2,\zeta_3) - \hbar_{\mathfrak{M}}(\zeta_1,\zeta_3)\hbar_{\mathfrak{M}}(\zeta_2,\varsigma_2)\right\} \tag{9}$$

for vector fields $\zeta_1,\zeta_2,\zeta_3,\varsigma_2$ tangent to $\mathfrak{M}^m(c)$.

We also recall the following two important results that we will use in our theorems.

Theorem 3.1. [12] *Let $B_1,...,B_m$ be $(n\times n)$ real symmetric matrices. Then*

$$\sum_{r,s=1}^{m} ||[B_r,B_s]||^2 \leq \left(\sum_{r=1}^{m}||B_r||^2\right)^2,$$

where the equality holds if and only if under some rotation[1] *all B_r's are zero except two matrices which can be written as PH_1P^t and PH_2P^t, where P is an $(n\times n)$ orthogonal matrix, and*

$$H_1 = diag(\mu,-\mu,0,...),\quad H_2 = diag(\begin{pmatrix} 0 & \mu \\ -\mu & 0\end{pmatrix},0,...).$$

Corollary 3.1. [12] *Let $f:\mathfrak{M}^n \longrightarrow N^{n+m}(c)$ be an isometric immersion. Then*

$$\rho+\rho^{\perp} \leq |H|^2 + c,$$

where the equality holds at some point $p\in\mathfrak{M}$ if and only if there exist an orthonormal frame $\{\upsilon_1,\dots,\upsilon_r\}$ of $T_p\mathfrak{M}$ and an orthonormal frame $\{\sigma_1,\dots,\sigma_m\}$ of $T_p^{\perp}\mathfrak{M}$, such that

$$A_{\sigma_1} = diag(\lambda_1+\mu,\lambda_1-\mu,\lambda_1,...,\lambda_1),\quad A_{\sigma_2} = diag(\begin{pmatrix}\lambda_2 & \mu \\ \mu & \lambda_2\end{pmatrix},\lambda_2,...,\lambda_2),$$

and all other shape operators are $A_{\sigma_r} = \lambda_r I_n$, where $\mu,\lambda_1,...,\lambda_m$ are real numbers.

We begin by providing the expression for curvature that we will use in the process of constructing Wintgen inequalities.

Lemma 3.1. *Let F be a Riemannian submersion from a real space form $(\mathfrak{M}^m(c),\hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n,ℓ). $\mathbf{R}^{\mathfrak{M}}$ and $\mathbf{R}^{\natural^{\perp}}$ the Riemannian curvature tensor of $\mathfrak{M}$ and the vertical distribution, respectively. We have*

$$\hbar_{\mathfrak{M}}(\mathbf{R}^{\natural^{\perp}}(U,\varsigma_1)\zeta_1,\zeta_2) = 3\hbar_{\mathfrak{M}}(\mathcal{T}_{\varsigma_1}\zeta_1,\mathcal{T}_U\zeta_2) - 3\hbar_{\mathfrak{M}}(\mathcal{T}_U\zeta_1,\mathcal{T}_{\varsigma_1}\zeta_2) \tag{10}$$

for $U,\varsigma_1\in\chi^{\natural}(\mathfrak{M})$ and $\zeta_1,\zeta_2\in\chi^{\diamond}(\mathfrak{M})$.

Proof. Let $U,\varsigma_1\in\chi^{\natural}(\mathfrak{M})$ and $\zeta_1,\zeta_2\in\chi^{\diamond}(\mathfrak{M})$. Using equations (3) and (4), we obtain

$$\begin{aligned}\hbar_{\mathfrak{M}}(\mathbf{R}^{\natural^{\perp}}(U,\varsigma_1)\zeta_1,\zeta_2) &= \hbar_{\mathfrak{M}}((\nabla_{\varsigma_1}\mathcal{T})_U\zeta_1,\zeta_2) - \hbar_{\mathfrak{M}}((\nabla_U\mathcal{T})_{\varsigma_1}\zeta_1,\zeta_2)\\ &\quad + \hbar_{\mathfrak{M}}(\mathcal{T}_U\zeta_2,\mathcal{T}_{\varsigma_1}\zeta_1) - \hbar_{\mathfrak{M}}(\mathcal{T}_{\varsigma_1}\zeta_2,\mathcal{T}_U\zeta_1).\end{aligned} \tag{11}$$

From (11) and the first equation of (8), we obtain (10). □

For $p\in\mathfrak{M}$, let $\{\upsilon_1,...,\upsilon_r,\sigma_{r+1},...,\sigma_n\}$ be an orthonormal frame of $T_p\mathfrak{M}$ such that $\{\upsilon_1,...,\upsilon_r\}$ is an orthonormal frame of $\natural_p = \ker F_{*p}$ and $\{\sigma_{r+1},...,\sigma_n\}$ is an orthonormal frame for $\diamond_p = (\ker F_{*p})^{\perp}$. In this section, unless otherwise stated, the base of the total manifold will be considered as above. The normalized scalar curvature $\rho^{\natural}$ on $\natural_p$ is expressed as

$$\rho^{\natural} = \frac{2}{r(r-1)}\sum_{1\leq i<j\leq r}\hbar_{\mathfrak{M}}(\widehat{\mathbf{R}}(\upsilon_i,\upsilon_j)\upsilon_j,\upsilon_i). \tag{12}$$

A new notion of the normal scalar curvature $\rho^{\natural^{\perp}}$, is defined for every point $p \in \mathfrak{M}$ as follows,

$$\rho^{\natural^{\perp}} = \frac{2}{r(r-1)} \left(\sum_{1\le i<j}^{r} \sum_{r+1\le k<l\le n} \hbar_{\mathfrak{M}}(\mathbf{R}^{\natural^{\perp}}(\upsilon_i,\upsilon_j)\sigma_k,\sigma_l)^2 \right)^{1/2}, \tag{13}$$

where $\mathbf{R}^{\natural^{\perp}}$ is the normal curvature tensor field of $\natural_p$.
From (10), the normal scalar curvature on the vertical distribution can be expressed as

$$\rho^{\natural^{\perp}} = \frac{2}{r(r-1)}\sqrt{\frac{9}{2}\sum_{k,l=r+1}^{n}\sum_{i,j=1}^{r}\hbar_{\mathfrak{M}}\left([\mathfrak{T}^k,\mathfrak{T}^l]\upsilon_i,\upsilon_j\right)^2} = \frac{3\sqrt{2}}{r(r-1)}\sqrt{\sum_{k,l=r+1}^{n}\|[\mathfrak{T}^k,\mathfrak{T}^l]\|^2},$$

where $\|[\mathfrak{T}^k,\mathfrak{T}^l]\|^2 = \sum_{k,l=r+1}^{n}\sum_{i,j=1}^{r}\hbar_{\mathfrak{M}}\left([\mathfrak{T}^k,\mathfrak{T}^l]\upsilon_i,\upsilon_j\right)^2$.
We put

$$\begin{aligned} \mathfrak{T}_{ij}^{\alpha} &= \hbar_{\mathfrak{M}}(\mathfrak{T}_{\upsilon_i}\upsilon_j,\sigma_\alpha),\ i,j=1,\cdots,r,\ \alpha=r+1,\ldots n, \\ \|\mathfrak{T}\|^2 &= \sum_{i,j=1}^{r}\hbar_{\mathfrak{M}}(\mathfrak{T}_{\upsilon_i}\upsilon_j,T_{\upsilon_i}\upsilon_j), \quad \text{trace}\mathfrak{T} = \sum_{i=1}^{r}\mathfrak{T}_{\upsilon_i}\upsilon_i, \\ \|\text{trace}\mathfrak{T}\|^2 &= \hbar_{\mathfrak{M}}(\text{trace}\mathfrak{T},\text{trace}\mathfrak{T}). \end{aligned} \tag{14}$$

In the sequel we are going to state and prove the Wintgen inequality for Riemannian submersions.

Theorem 3.2. *Let Ψ be a Riemannian submersion from a real space form $(\mathfrak{M}^m(c),\hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n,ℓ) with rank$\natural = r \ge 3$. Then, we have*

$$\frac{1}{r(r-1)}\,||trace\mathfrak{T}||^2 \ge \rho^{\natural} + \frac{\sqrt{2}}{6}\rho^{\natural^{\perp}} - c. \tag{15}$$

The equality holds at point $p \in \mathfrak{M}$ if and only if there exist an orthonormal frame $\{\upsilon_1,\ldots,\upsilon_r\}$ on $\natural_p$ and an orthonormal frame $\{\sigma_{r+1},\ldots,\sigma_n\}$ on $\diamond_p$, such that

$$\mathfrak{T}_1 = diag(\mu,-\mu,0,...), \quad \mathfrak{T}_2 = diag(\begin{pmatrix} 0 & \mu \\ -\mu & 0\end{pmatrix},0,...)$$

and all other tensor fields $\mathfrak{T}_i$ vanish for $i = 3,...,r$, where μ is real number.

Proof. Combining (7) with (9), one obtains

$$\tau^{\natural} = r(r-1)c + \sum_{\alpha=r+1}^{n}\sum_{1\le i<j\le r}\Big\{\hbar_{\mathfrak{M}}(\mathfrak{T}^{\alpha}(\upsilon_i,\upsilon_i),\mathfrak{T}^{\alpha}(\upsilon_j,\upsilon_j)) - \hbar_{\mathfrak{M}}(\mathfrak{T}^{\alpha}(\upsilon_i,\upsilon_j),\mathfrak{T}^{\alpha}(\upsilon_j,\upsilon_i))\Big\}. \tag{16}$$

From (12) and (16), we get

$$\rho^{\natural} - c = \frac{2}{r(r-1)}\sum_{\alpha=r+1}^{n}\sum_{i<j}^{r}\{\mathfrak{T}_{ii}^{\alpha}\mathfrak{T}_{jj}^{\alpha} - (\mathfrak{T}_{ij}^{\alpha})^2\}, \tag{17}$$

where we put $\mathfrak{T}_{ij}^{\alpha} = \hbar_{\mathfrak{M}}(\mathfrak{T}(\upsilon_i,\upsilon_j),\sigma_\alpha)$. By direct calculation, we have

$$\begin{aligned} ||trace\mathfrak{T}||^2 &= \sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathfrak{T}_{ii}^{\alpha}\right)^2 \\ &= \frac{1}{r-1}\sum_{\alpha=r+1}^{n}\sum_{1=i<j}^{r}\Big((\mathfrak{T}_{ii}^{\alpha}-\mathfrak{T}_{jj}^{\alpha})^2 + 2r\mathfrak{T}_{ii}^{\alpha}\mathfrak{T}_{jj}^{\alpha}\Big). \end{aligned} \tag{18}$$

Using (18) and (17), we obtain

$$\begin{aligned}(r-1)\,||trace\mathcal{T}||^2 &= \sum_{\alpha=r+1}^{n}\sum_{1=i<j}^{r}\left((\mathcal{T}_{ii}^{\alpha}-\mathcal{T}_{jj}^{\alpha})^2+2r\mathcal{T}_{ii}^{\alpha}\mathcal{T}_{jj}^{\alpha}-2r\left(\mathcal{T}_{ij}^{\alpha}\right)^2+2r\left(\mathcal{T}_{ij}^{\alpha}\right)^2\right)\\ &= r^2(r-1)\left(\rho^{\natural}-c\right)+\sum_{\alpha=r+1}^{n}\sum_{1=i<j}^{r}\left((\mathcal{T}_{ii}^{\alpha}-\mathcal{T}_{jj}^{\alpha})^2+2r\left(\mathcal{T}_{ij}^{\alpha}\right)^2\right) \qquad (19)\\ &= r^2(r-1)\left(\rho^{\natural}-c\right)+r\sum_{\alpha=r+1}^{n}||\mathcal{T}^{\alpha}||^2-\sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}_{ii}^{\alpha}\right)^2.\end{aligned}$$

Using Theorem 3.1, from (19) and (13), we arrive at

$$\begin{aligned}(r-1)\,||trace\mathcal{T}||^2 &\geq r^2(r-1)\left(\rho^{\natural}-c\right)+r\left(\sum_{k,l=r+1}^{n}\left|\left|[\mathcal{T}^k,\mathcal{T}^l]\right|\right|^2\right)^{\frac{1}{2}}-\sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}_{ii}^{\alpha}\right)^2\\ &= r^2(r-1)\left(\rho^{\natural}-c\right)+\frac{\sqrt{2}}{6}r^2(r-1)\rho^{\natural^{\perp}}-\sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}_{ii}^{\alpha}\right)^2,\end{aligned}$$

which gives the inequality case. The equality case comes from Corollary 3.1. □

An example of the equality condition is given below.

Example 3.1. *The standard Riemannian* $\pi:\mathbb{S}^{15}\to\mathbb{S}^{8}(\frac{1}{2})$ *with totally geodesic fibers satisfy the equality cases of inequality in* (15).

4. Wintgen Inequalities along Riemannian submersions defined on a complex space form

In this section, the Wintgen inequality is provided for Riemannian submersions whose total manifold is a complex space form. First, we recall the context of a complex space form. Let $\mathfrak{M}$ be an almost Hermitian manifold with the almost Hermitian structure $(J,\hbar_{\mathfrak{M}})$. Then, $\mathfrak{M}$ becomes a Kaehler manifold if $\nabla^{\mathfrak{M}}J=0$, where $\nabla^{\mathfrak{M}}$ is the Riemannian connection of the Riemannian metric $\hbar_{\mathfrak{M}}$. A Kaehler manifold with constant holomorphic sectional curvature c is said to be a complex space form and its Riemannian curvature tensor $\mathbf{R}^{\mathfrak{M}}$ is given by [20]

$$\begin{aligned}\hbar_{\mathfrak{M}}(\mathbf{R}^{\mathfrak{M}}(\zeta_1,\zeta_2)\zeta_3,\varsigma_2) &= \frac{c}{4}\left\{\hbar_{\mathfrak{M}}(\zeta_1,\varsigma_2)\hbar_{\mathfrak{M}}(\zeta_2,\zeta_3)-\hbar_{\mathfrak{M}}(\zeta_1,\zeta_3)\hbar_{\mathfrak{M}}(\zeta_2,\varsigma_2)\right\}\\ &+\frac{c}{4}\left\{\hbar_{\mathfrak{M}}(\zeta_1,J\zeta_3)\hbar_{\mathfrak{M}}(J\zeta_2,\varsigma_2)-\hbar_{\mathfrak{M}}(\zeta_2,J\zeta_3)\hbar_{\mathfrak{M}}(J\zeta_1,\varsigma_2)\right\} \qquad (20)\\ &+\frac{2c}{4}\hbar_{\mathfrak{M}}(\zeta_1,J\zeta_2)\hbar_{\mathfrak{M}}(J\zeta_3,\varsigma_2)\end{aligned}$$

for all $\zeta_1,\zeta_2,\zeta_3,\varsigma_2\in\Gamma(T\mathfrak{M})$.

Let a Riemannian submersion Ψ be a submersion from a complex space form $(\mathfrak{M}^m(c),J,\hbar_{\mathfrak{M}})$ of complex dimension m to a Riemannian manifold (N^n,ℓ) of real dimension n. For $\zeta_1\in\Gamma(ker\Psi_*)$, we can write

$$J\zeta_1=\phi\zeta_1+\omega\zeta_1, \qquad (21)$$

where $\phi\zeta_1$ denotes the vertical component of $J\zeta_1$ and $\omega\zeta_1$ stands for the horizontal component of $J\zeta_1$. Next, we put

$$||\phi||^2=\sum_{i,j=1}^{r}\hbar_{\mathfrak{M}}^2(\phi\upsilon_i,\upsilon_j). \qquad (22)$$

The following theorem gives the Wintgen inequality in the case where the total manifold of a Riemannian submersion is a complex space form.

Theorem 4.1. *Let Ψ be a Riemannian submersion from a complex space form $(\mathfrak{M}^m(c), \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ) with rank$\natural = r \geq 3$. Then, we have*

$$\frac{1}{r(r-1)}\,||trace\mathcal{T}||^2 \geq \rho^{\natural} + \frac{\sqrt{2}}{6}\rho^{\natural^{\perp}} - \frac{c}{4} - \frac{3c}{4r(r-1)}\,||\phi||^2, \tag{23}$$

the equality holds at point $p \in \mathfrak{M}$ if and only if there exist an orthonormal frame $\{\upsilon_1, \dots, \upsilon_r\}$ on $(\natural_p)$ and an orthonormal frame $\{\sigma_{r+1}, \dots, \sigma_n\}$ on $(\diamond_p)$, such that

$$\mathcal{T}_1 = diag(\mu, -\mu, 0, ...), \quad \mathcal{T}_2 = diag(\begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, 0, ...),$$

and all other tensor fields $\mathcal{T}_i$ vanish for $i = 3, ..., r$, where μ is real number.

Proof. Let $\{\upsilon_1, \dots, \upsilon_r\}$ be an orthonormal frame on the $\natural_p$ and $\{\sigma_{r+1}, \dots, \sigma_n\}$ be on the horizontal space $\diamond_p$ at $p \in \mathfrak{M}$. Combining (7) with (20), one obtains

$$\begin{aligned} 2\tau^{\natural} = {} & \frac{r(r-1)c}{4} + \frac{3c}{4}\sum_{i,j=1}^{r}\hbar^2_{\mathfrak{M}}(\phi\upsilon_i, \upsilon_j) + 2\sum_{\alpha=r+1}^{n}\sum_{1\leq i<j\leq r}\hbar_{\mathfrak{M}}(\mathcal{T}^{h^\alpha}(\upsilon_i,\upsilon_i), \mathcal{T}^{h^\alpha}(\upsilon_j,\upsilon_j)) \\ & - \hbar_{\mathfrak{M}}(\mathcal{T}^{h^\alpha}(\upsilon_i,\upsilon_j), \mathcal{T}^{h^\alpha}(\upsilon_j,\upsilon_i)). \end{aligned} \tag{24}$$

From (12) and (24), we have

$$\rho^{\natural} - \frac{c}{4} - \frac{3c}{4r(r-1)}\,||\phi||^2 = \frac{2}{r(r-1)}\sum_{\alpha=r+1}^{n}\sum_{i<j}^{r}\left\{\mathcal{T}^{h^\alpha}_{ii}\mathcal{T}^{h^\alpha}_{jj} - (\mathcal{T}^{h^\alpha}_{ij})^2\right\} \tag{25}$$

where we put $\mathcal{T}^{h^\alpha}_{ij} = \hbar_{\mathfrak{M}}(\mathcal{T}^h(\upsilon_i,\upsilon_j), \sigma_\alpha)$. The length of the tensor field $\mathcal{T}$ is obtained as

$$\begin{aligned} ||trace\mathcal{T}||^2 &= \sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}^{\alpha}_{ii}\right)^2 \\ &= \frac{1}{r-1}\sum_{\alpha=r+1}^{n}\sum_{1=i<j}^{r}\left((\mathcal{T}^{\alpha}_{ii} - \mathcal{T}^{\alpha}_{jj})^2 + 2r\mathcal{T}^{\alpha}_{ii}\mathcal{T}^{\alpha}_{jj}\right). \end{aligned} \tag{26}$$

From (25) and (26), we derive

$$\begin{aligned} (r-1)\,||trace\mathcal{T}||^2 &= \sum_{\alpha=r+1}^{n}\sum_{1=i<j}^{r}\left((\mathcal{T}^{\alpha}_{ii} - \mathcal{T}^{\alpha}_{jj})^2 + 2r\mathcal{T}^{\alpha}_{ii}\mathcal{T}^{\alpha}_{jj} - 2r\left(\mathcal{T}^{\alpha}_{ij}\right)^2 + 2r\left(\mathcal{T}^{\alpha}_{ij}\right)^2\right) \\ &= r^2(r-1)\left(\rho^{\natural} - \frac{c}{4} - \frac{3c}{4r(r-1)}\,||\phi||^2\right) + \sum_{\alpha=r+1}^{n}\sum_{1=i<j}^{r}\left((\mathcal{T}^{\alpha}_{ii} - \mathcal{T}^{\alpha}_{jj})^2 + 2r\left(\mathcal{T}^{\alpha}_{ij}\right)^2\right) \\ &= r^2(r-1)\left(\rho^{\natural} - \frac{c}{4} - \frac{3c}{4r(r-1)}\,||\phi||^2\right) + r\sum_{\alpha=r+1}^{n}||\mathcal{T}^{\alpha}||^2 - \sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}^{\alpha}_{ii}\right)^2. \end{aligned}$$

From (13) and Theorem 3.1, we get

$$\begin{aligned} (r-1)\,||trace\mathcal{T}||^2 \geq {} & r^2(r-1)\left(\rho^{\natural} - \frac{c}{4} - \frac{3c}{4r(r-1)}\,||\phi||^2\right) + r\left(\sum_{k,l=r+1}^{n}\left|\left|[\mathcal{T}^k, \mathcal{T}^l]\right|\right|^2\right)^{\frac{1}{2}} \\ & - \sum_{\alpha=r+1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}^{\alpha}_{ii}\right)^2 \\ = {} & r^2(r-1)\left(\rho^{\natural} - \frac{c}{4} - \frac{3c}{4r(r-1)}\,||\phi||^2\right) + \frac{\sqrt{2}}{6}r^2(r-1)\rho^{\natural^{\perp}} - ||trace\mathcal{T}||^2. \end{aligned}$$

Thus the inequality case is completed. The equality case comes from Corollary 3.1. □

A Riemannian submersion Ψ from an almost Hermitian manifold $(\mathfrak{M}^m, J, \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ) is said to be an anti-invariant Riemannian submersion if $ker\Psi_*$ is anti- invariant with respect to the almost complex structure J, i.e., $J(ker\Psi_*) \subseteq (ker\Psi_*)^\perp$ in [19]. In this case, clearly (23) implies $||\phi||^2 = 0$ and from Theorem 4.1, we conclude the following result.

Corollary 4.1. *Let Ψ be an anti-invariant Riemannian submersion from a complex space form $(\mathfrak{M}^m(c), \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ) with $rank\natural = r \geq 3$. Then, we have*

$$\frac{1}{r(r-1)} ||trace\mathcal{T}||^2 \geq \rho^\natural + \frac{\sqrt{2}}{6}\rho^{\natural^\perp} - \frac{c}{4}, \tag{27}$$

the equality holds at point $p \in \mathfrak{M}$ if and only if there exist an orthonormal frame $\{\upsilon_1, \ldots, \upsilon_r\}$ on $(\natural_p)$ and an orthonormal frame $\{\sigma_{r+1}, \ldots, \sigma_n\}$ on $\diamond_p$, such that

$$\mathcal{T}_1 = diag(\mu, -\mu, 0, ...), \quad \mathcal{T}_2 = diag(\begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, 0, ...),$$

and all other tensor fields $\mathcal{T}_i$ vanish for $i = 3, ..., r$, where μ is real number.

5. Wintgen inequality along Riemannian submersions on a horizontal distribution

In this section, we will express a Wintgen's inequality in terms of horizontal distributions. We will see that the condition for equality is equivalent to the integrability of a horizontal distribution. Let Ψ be a Riemannian submersion from a real space form $(\mathfrak{M}^m(c), \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ). Let $\chi^\diamond(\mathfrak{M}) = \Gamma((\ker\Psi_*)^\perp)$ and $\chi^\natural(\mathfrak{M}) = \Gamma(\ker\Psi_*)$ be the spaces of horizontal and vertical vector fields over $\mathfrak{M}$, respectively. Denote by $\mathbf{R}^{\mathfrak{M}}$ and $\mathbf{R}^*$ the Riemannian curvature tensor of $\mathfrak{M}$ and the horizontal distribution, respectively. Then, we have the Gauss type equation:

$$\begin{aligned}\mathbf{R}^{\mathfrak{M}}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = {}& \mathbf{R}^*(\zeta_1, \zeta_2, \zeta_3, \zeta_4) + 2\hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_1}\zeta_2, \mathcal{A}_{\zeta_3}\zeta_4) - \hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_2}\zeta_3, \mathcal{A}_{\zeta_1}\zeta_4) \\ &+ \hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_1}\zeta_3, \mathcal{A}_{\zeta_2}\zeta_4)\end{aligned} \tag{28}$$

for $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \chi^\diamond(\mathfrak{M})$, see [20] for more details.

Lemma 5.1. *Let F be a Riemannian submersion from a real space form $(\mathfrak{M}^m(c), \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ). Then we have*

$$g(\mathbf{R}^{\diamond^\perp}(\zeta_1, \zeta_2)\varsigma_1, \varsigma_2) = 3\hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_2}\varsigma_1, \mathcal{A}_{\zeta_1}\varsigma_2) - 3\hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_1}\varsigma_1, \mathcal{A}_{\zeta_2}\varsigma_2) \tag{29}$$

for $\zeta_1, \zeta_2 \in \chi^\diamond(\mathfrak{M})$, $\varsigma_1, \varsigma_2 \in \chi^\natural(\mathfrak{M})$.

Proof. Let $\zeta_1, \zeta_2 \in \chi^\diamond(\mathfrak{M})$ and $\varsigma_1, \varsigma_2 \in \chi^\natural(\mathfrak{M})$. Using equations (5) and (6), we obtain:

$$\begin{aligned}g(\mathbf{R}^{\diamond^\perp}(\zeta_1, \zeta_2)\varsigma_1, \varsigma_2) = {}& \hbar_{\mathfrak{M}}((\nabla_{\zeta_1}\mathcal{A})_{\zeta_2}\varsigma_1, \varsigma_2) - \hbar_{\mathfrak{M}}((\nabla_{\zeta_2}\mathcal{A})_{\zeta_1}\varsigma_1, \varsigma_2) \\ &+ \hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_1}\varsigma_2, \mathcal{A}_{\zeta_2}\varsigma_1) - \hbar_{\mathfrak{M}}(\mathcal{A}_{\zeta_2}\varsigma_2, \mathcal{A}_{\zeta_1}\varsigma_1)\end{aligned} \tag{30}$$

for $\zeta_1, \zeta_2 \in \chi^\diamond(\mathfrak{M})$ and $\varsigma_1, \varsigma_2 \in \chi^\natural(\mathfrak{M})$. From (30) and the second equation of (8), we obtain (29).

□

Let $\{\upsilon_1, \upsilon_2, ..., \upsilon_r\}$ be an orthonormal frame of the horizontal space $\diamond_p$, for $p \in \mathfrak{M}$ and $\{\sigma_{r+1}, ..., \sigma_n\}$ be an orthonormal frame of the vertical space $\natural_p$. In this section, unless otherwise stated, the base of the total manifold will be considered as above. We define the scalar curvature $\tau^\diamond$ on $\diamond_p$ by

$$\tau^\diamond = \sum_{1 \leq i < j \leq r} \hbar_{\mathfrak{M}}(\mathbf{R}^*(\upsilon_i, \upsilon_j)\upsilon_j, \upsilon_i) \tag{31}$$

and the normalized scalar curvature $\rho^{\diamond}$ of $\diamond_p$ as

$$\rho^{\diamond} = \frac{2\tau^{\diamond}}{r(r-1)}. \tag{32}$$

A new notion of normal scalar curvature $\mathbf{R}^{\diamond^{\perp}}$, is defined for every point $p \in \mathfrak{M}$ as follows,

$$\rho^{\diamond^{\perp}} = \frac{2}{r(r-1)} \left(\sum_{1 \le i < j}^{r} \sum_{r+1 \le k < l \le n} \hbar_{\mathfrak{M}}(\mathbf{R}^{\diamond^{\perp}}(\upsilon_i, \upsilon_j)\sigma_k, \sigma_l)^2 \right)^{1/2}. \tag{33}$$

We now put

$$\mathcal{A}_{ij}^{\alpha} = \sum_{i,j=1}^{r} \sum_{\alpha=r+1}^{n} \hbar_{\mathfrak{M}}(\mathcal{A}_{\upsilon_i}\upsilon_j, \sigma_\alpha), \ \|\mathcal{A}\|^2 = \sum_{i,j=1}^{r} \hbar_{\mathfrak{M}}(\mathcal{A}_{\upsilon_i}\upsilon_j, \mathcal{A}_{\upsilon_i}\upsilon_j)$$

$$\text{trace}\mathcal{A} = \sum_{i=1}^{r} \mathcal{A}_{\upsilon_i}\upsilon_i, \ \|\text{trace}\mathcal{A}\|^2 = \hbar_{\mathfrak{M}}(\text{trace}\mathcal{A}, \text{trace}\mathcal{A}).$$

We have the following theorem.

Theorem 5.1. *Let Ψ be a Riemannian submersion from a real space form $(\mathfrak{M}^m(c), \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ) with rank$\diamond = r \ge 3$. Then at every point p, we have*

$$\rho^{\diamond} - c \ge 0. \tag{34}$$

Moreover, the equality case holds in the above inequality at a point $p \in \mathfrak{M}$ if and only if the horizontal distribution is integrable.

Proof. Combining (9) and (28), and using (32) one obtains

$$\rho^{\diamond} - c = \frac{6}{r(r-1)} \sum_{\alpha=r+1}^{n} \sum_{i<j=1}^{r} (\mathcal{A}_{ij}^{\alpha})^2, \tag{35}$$

where we have put $\mathcal{A}_{ij}^{\alpha} = \hbar_{\mathfrak{M}}(\mathcal{A}(\upsilon_i, \upsilon_j), \sigma_\alpha)$.
From Lemma 5.1, taking $\{\zeta_1 = \upsilon_i, \zeta_2 = \upsilon_j\} \in \chi^{\diamond}(\mathfrak{M})$ and $\{\varsigma_1 = \sigma_k, \varsigma_2 = \sigma_l\} \in \chi^{\natural}(\mathfrak{M})$, $i, j = 1, ..., r$, $k, l = r+1, ..., n$, we obtain

$$\hbar_{\mathfrak{M}}(\mathbf{R}^{\diamond^{\perp}}(\upsilon_i, \upsilon_j)\sigma_k, \sigma_l) = 0. \tag{36}$$

which implies $\rho^{\diamond^{\perp}} = 0$. From (35), we obtain

$$\rho^{\diamond} - c = \frac{6}{r(r-1)} \sum_{\alpha=r+1}^{n} \sum_{i<j=1}^{r} (\mathcal{A}_{ij}^{\alpha})^2 \ge 0. \tag{37}$$

Hence, the inequality in Theorem 5.1 is satisfied. We can easily verify that equality holds in the above inequality if and only if $\mathcal{A}_{ij}^{\alpha} = 0$, Moreover, since $\mathcal{A}_{ii}^{\alpha} = 0$ and $\mathcal{A}_{jj}^{\alpha} = 0$, the necessary and sufficient condition for equality in the inequality (37) is that the horizontal distribution is integrable. □

We now define

$$||\omega||^2 = \sum_{i<j=1}^{r} \hbar_{\mathfrak{M}}^2(\omega\upsilon_i, \upsilon_j).$$

We now have the following theorem.

Theorem 5.2. *Let Ψ be a Riemannian submersion from a complex space form $(\mathfrak{M}^m(c), J, \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ) with rank$\diamond = r \ge 3$. Then at every point p, we have*

$$\rho^{\diamond} - \frac{c}{4} - \frac{3c}{4r(r-1)} ||\omega||^2 \ge 0. \tag{38}$$

Moreover, the equality case holds in any of the above inequality at a point $p \in \mathfrak{M}$ if and only if the horizontal distribution is integrable.

Since the proof of the Theorem 5.2 is similar to the previous Theorem 5.1, we omit it.

If the Riemannian submersion is an invariant Riemannian submersion, we have the following result.

Corollary 5.1. *Let Ψ be an invariant Riemannian submersion from a complex space form $(\mathfrak{M}^m(c), J, \hbar_{\mathfrak{M}})$ to a Riemannian manifold (N^n, ℓ) with rank$\diamond = r \geq 3$. Then at every point p, we have*

$$\rho^{\diamond} - \frac{c}{4} \geq 0. \tag{39}$$

Moreover, the equality case holds in any of the above inequality at a point $p \in \mathfrak{M}$ if and only if the horizontal distribution is integrable.

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