CHAOTIC BEHAVIOR OF IDEAL FOUR-LEVEL LASER WITH PERIODIC PUMP MODULATION: II

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In lucrare se realizează o investigare a predictabilității folosind algoritmul Grassberger – Proccaccia în cazul laserului ideal cu 4 nivele cu pompaj periodic, continuind astfel o cercetare anterioară. S-au calculat exponenții Lyapunov maximi, si s-a evidențiat prezenta haosului la aceeaşi valoare de prag a frecventei de pompaj prin doua metode, şi anume prin scaderea brusca a timpului de dublare a erorii în datele inițiale(calculat prin intermediul entropiei Kolmogorov), precum şi printr-o schimbare de semn al coeficientilori Lyapunov maximi. Rezultatele sunt în foarte bună concordantă cu cele obținute în literature clasică de specialitate.

A predictability investigation using the Grassberger-Proccaccia algorithm is performed for the ideal four - level laser in the case of periodic pump term, continuing a previous research. In addition, maximum Lyapunov exponents are calculated, and chaos is evidenced at the same threshold value of pump frequency by two ways, namely by a sudden decrease in the error–doubling time (computed via the Kolmogorov entropy) and by the change in sign of the maximum Lyapunov exponents. Results are in very good agreement with those obtained in classical literature.

Key words: predictability, chaos, error doubling time, Lyapunov exponents.

1. Introduction

The heredity of the last century for actual physics consists in three major unsolved problems: relativity, quantum mechanics, and chaos. The presence of chaos was reported in many areas of science [1,2], such as physics, chemistry,

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biology, economical sciences, etc. Therefore, the question of detecting and quantifying chaos has become an important one. The predictability of a system reveals the degree of confidence we may have in the knowledge of its temporal evolution. When modeling a chaotic system, it is very important to know the time after which the outputs of the model still have any meaning. One of the mathematical measure of predictability is the error - doubling time, which gives us information about the amplification of errors in the initial state of the system. On the other side, the spectrum of Lyapunov exponents has proven to be the most useful dynamical diagnostic for chaotic systems.

In a previous paper [3], we have show the presence of windows of predictability for the ideal four level laser with periodic pump modulation for frequencies ω belonging to the interval 0.010 - 0.100 (in γ^{-1} rescaled time units, with γ = the decay constant), starting from the well-known ODE system described in [4], numerically integrated using the Crank-Nicholson scheme [5], and applying the Grassberger-Proccaccia algorithm [6,7]. In **section 2**, we complete the analysis by evidencing the onset of chaos for ω belonging to 0.031 to 0.0140, in order to include the threshold value of ω = 0.0136 at which the system behavior becomes chaotic. In this scope, we follow the same way as in [3]. In **section 3,** the same phenomenon is revealed by computing the maximum Lyapunov exponents conformal to the method of Wolf&al [8]. **Section 4** concludes both methods yield the same quantitative results, in very good agreement with those obtained elsewhere.

2. Predictability estimates

We started with the well - known normalized two ODE describing the four-level laser with sinusoidal modulation [3,4]:

$$\begin{aligned} \dot{q} &= -q + nq + sn \\ \dot{n} &= p_0(1 + p_m \sin \omega t) - nq \end{aligned} \tag{1}$$

Here, p_m is the degree of modulation, p_m = 1, p_0 is the constant pump term, p_0 =6x10^{-4}, s is the spontaneous emission rate into the laser mode, $s = 10^{-7}$, and ω - the modulation frequency. q and n are the normalized photon number and population.

These two ODE were numerically integrated using the iterated Crank-Nicholson method with two iterations. The number of iterations was chosen in order to have numerical stability. Ten runs were made, corresponding to ω belonging to the interval [0.0131 – 0.0140] with an increment of 0.0001. The normalized population difference was found to be very suggestive in evidencing well-behaved or erratic patterns. Each run contained a number of 11000 data, from which the first 1000 transiental were removed, and the Grassberger–Procaccia algorithm was applied to the next 5000. When applying the algorithm, the autocorrelation function *r* was computed for choosing the lag time τ. The minimum values of *r* for reasonable values of τ were about 0.5, greater than *1/e*, which is the most common used threshold for determining τ.

Results of numerical integrations of the two ODE are shown in figures 1a and 2a (in caption). We see that erratic patterns (chaotic behavior) occur for $\omega < 0.0136$, and regular behavior is present for $\omega \geq 0.0136$. The phase portraits from figures 1b and 2b present limit cycles for regular behavior, and nondefinite aspect for chaos. Likewise, the correlation dimension, which approximates the attractor's dimension, is about two for regular patterns, and greater than two for irregular patterns (fig. 3). Moreover, the error doubling time decreases suddenly from approximatively 35 for $\omega \geq 0.0136$ to above 5 for $\omega < 0.0136$ (fig. 4).

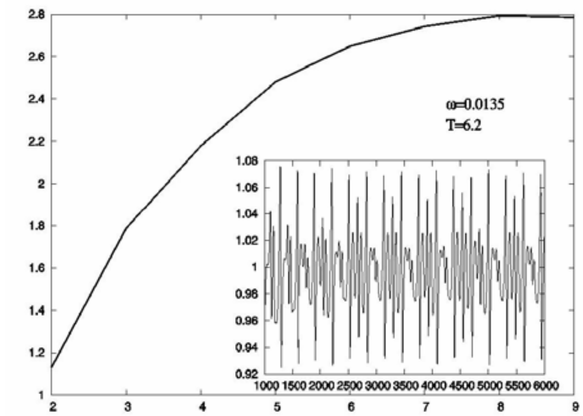

Fig. 1a: Correlation integral vs. embedding dimension for ω =0.0135;
Caption: normalized population difference vs time (arbitrary units)

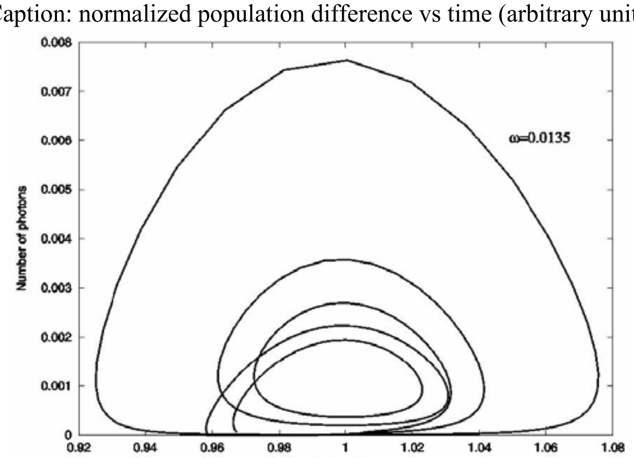

Fig.1b: Phase portrait for ω = 0.0135

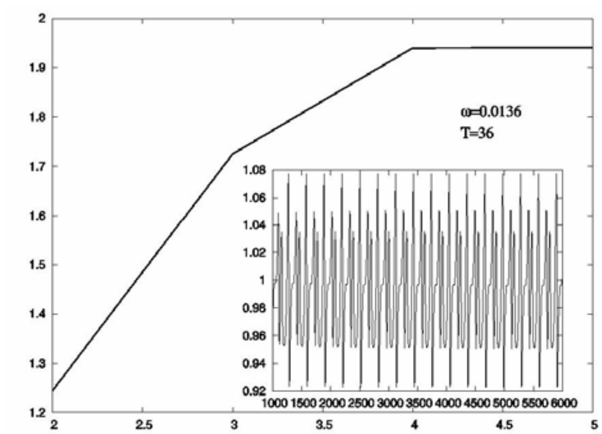

Fig. 2a: Correlation integral vs. embedding dimension for ω =0.0136;
Caption: normalized population difference vs time (arbitrary units)

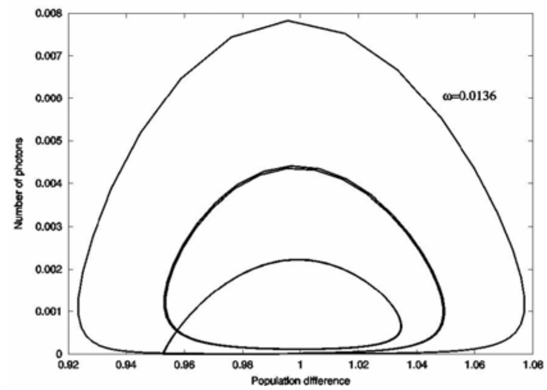

Fig.2b: Phase portrait for ω = 0.0136

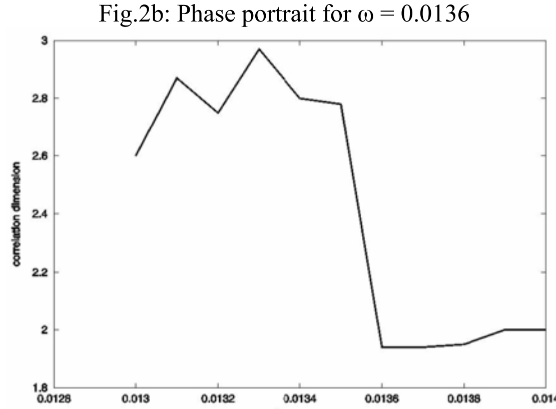

Fig.3: Correlation dimension vs. pump frequency

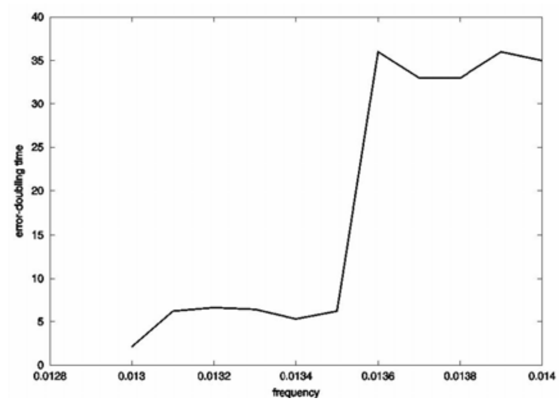

Fig. 4 Error-doubling time (arbitrary units) vs. pump frequency

3. Maximum Lyapunov exponents

The spectrum of Lyapunov exponents has proven to be the most useful dynamical diagnostic for chaotic systems. Wolf &al, in a reference paper [6], developed a technique for computing the maximum Lyapunov exponent from experimental (numerical) data series. Their paper describes the method in detail and is completed by a Fortran code. We present the results obtained for each ωbelonging to the interval 0.0131 - 0.0140, where transition to chaos occurs. When this happens, the maximum Lyapunov exponent must change sign, from negative to positive. Chaotic behavior is known to occur for $\omega < 0.0136$, so for these values the maximum Lyapunov exponent should be positive, whereas for $\omega \geq 0.0136$ (regular pattern) it must be negative. Results shown that this method is able to detect the transition from regular to chaotic behavior, as can be seen from the figure below.

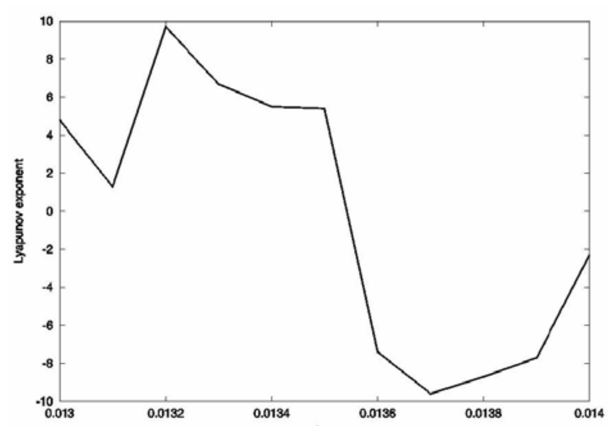

Fig.5 Maximum Lyapunov exponents vs. pump frequency

For ω greater then 0.0136 negative maximum Lyapunov exponents were obtained, whereas for ω less then 0.0136 they are positive, indicating the presence of chaos. These results are in good agreement with those of section 2, and with those from [2]. We have thus obtained the same result by two different methods. The first one starts with the calculation of the correlation integral in the reconstructed phase space and characterizes the attractor's dimension, whereas the second one involves monitoring the evolution of two nearby initial states in the same phase space.

4. Conclusions

Predictability estimates on four - level laser system with periodic pump term were performed by error -doubling time calculations in order to evidence the onset of chaos. The same scope was achieved by computing the maximum Lyapunov exponents. Both methods yield the same results. Results can be compared with experimental ones, in order to obtain information about the chaotic behaviour of lasers in the laboratory. For the moment, we obtained numerical results in very good agreement with those reported for example in [2]. Another method for computing the spectrum of all Lyapunov exponents is by using the original ODE system together with two liniarized versions; this method will be applied too, and the results will be published elsewhere.

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