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COMPOUND NUCLEUS FORMATION AND DE-EXCITATION WITH NEUTRONS

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A fost realizat un cod de calcul pentru calculul transmisiilor neutronilor prin bariere centrifugale. Au fost utilizate două modele: cel bazat pe interacţia tare şi cel care utilizează un potenţial complex. Sunt descrise aceste modele, iar greşelile apărute in publicaţii anterioare au fost eliminate. Rezultatele sunt obţinute pentru izotopii de Uraniu.

A computer code is realized in order to calculate the neutron transmissions through centrifugal barriers. Two models are used: the strong interaction model and the cloudy ball one. These models are described and errors appearing in previous publications are eliminated. Results are obtained for U isotopes.

1. Introduction

In order to design the new generations of nuclear reactors [1], a good knowledge of neutron induced fission cross sections is required. The evaluation of nuclear data is often realized within empirical models, the heights of the fission barriers being deduced from experimental data [2]. To reproduce the resonant structures in the cross section [3], transition states are introduced "by hand". In the last years, we proposed a new model to investigate the resonance structure of the fission cross sections. We showed that the rich resonant cross section structure in the threshold energy range can be due to single particle dynamical effects. A very important ingredient of the model [4] is the shell model that must supply the single particle levels schemes during the whole fission process. In this context, the rearrangements of the shells was obtained within a realistic version of the superasymmetric two center shell model. Another very important ingredient is a model for neutron transmissions. This work will focuses on this purpose.

2. Compound nucleus

Let consider a region of space where the target nucleus in any excited state is enclosed. This domain is known as internal region and the remaining space is the external region. The surface is well defined by some nuclear shape

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parametrization. A nucleon hits this region and a compound nucleus is formed. An unbound state is obtained and the nucleus disintegrates after a mean life time $\hbar/\Gamma$. The compound nucleus is created during a short period of time, enabling a redistribution of the kinetic energy and the momentum of the incoming particle among the entire system. Thus, the mathematical treatment of the nuclear system resembles in many ways to the treatment of classical thermodynamics systems. The expression of the cross-section for reactions X(a,b)Y which proceed through the intermediary compound state emerges naturally [5]

$$\sigma_{ab} = \sigma_a^c P_b \tag{1}$$

Where σ_{ab} is the cross section for the (a,b) reaction, σ_a^c denotes the cross section for the formation of a compound nucleus (X+a) and P_b is the probability that the compound nucleus disintegrates in the channel b. It is considered that the P_b for reactions X(a,b)Y which proceed through the intermediary compound state emerges naturally [6]:

$$\Psi = \Psi(R)\Phi_X\Phi_a \tag{2}$$

Where $\Psi(R)$ describes the relative motion and satisfies the Schrodinger equation

$$-\frac{\hbar^2}{2M_R}\Delta_R\Psi(R) + V(R)\Psi(R) = \epsilon\Psi(R) \tag{3}$$

and Φ_i (i=a, X) are related to the internal states of the two initial particles. Φ_i are not R dependent and, for convenience, will be omitted from now on. M_R is the reduced mass of the system. $V(R)$ is the potential between the two particles. A suitable choice for wave function of the relative motion (in the external region) has the form

$$\Psi(R) \propto \frac{u_l(R)}{R}Y_{lm}(\Omega) \tag{4}$$

where Y_{lm} are the spherical harmonic polynomials, and $u_l(R)$ is the radial function for orbital momentum l. Inserting the solution (4) in the Eq. (3) we obtain the radial equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{l(l+1)}{R^2} - \frac{2M_R}{\hbar^2}(V(R)-E_l)\right]u_l(R) = 0 \tag{5}$$

For positive energy channels, two linearly independent solutions to Eq. (5) occur, one of them is the so–called incoming wave function $u^{(-)}$, and the other is the out coming one $u^{(+)}$. The asymptotic behavior of each of these solutions are given by the forms:

$$u_l^{(-)}(R) \propto \exp(-ikR - \frac{1}{2}l\pi) \tag{6}$$

$$u_l^{(+)}(R) \propto \exp(ikR - \frac{1}{2}l\pi) \tag{7}$$

where $k = (2M_R E/\hbar^2)^{1/2}$ is the wave number. If the incident particle is a neutron, the reduced mass $M_R \approx m_n$ the neutron mass. Using the values $\hbar c$=197.32891 MeV fm (c being the velocity of light), $m_n c^2$=939.55 MeV the value of k can be obtained in fm^{-1} units ($k = [2m_n c^2 E/(\hbar c)^2]^{1/2}$). It is customary to work with real solutions which are regular ($F_l(R)$) and irregular ($G_l(R)$) at the origin and with asymptotic forms

$$F_l(R) \propto \sin(kR - \frac{1}{2}l\pi) \tag{8}$$

$$G_l(R) \propto \cos(kR - \frac{1}{2}l\pi) \tag{9}$$

In the case of a neutron as the particle a, the $V(R)$ potential vanishes, the Schrödinger equation becomes

$$\frac{\mathrm{d}^2 u_l(R)}{\mathrm{d}r^2} + \left[k^2 - \frac{l(l+1)}{R^2}\right] u_l(R) = 0 \tag{10}$$

The independent solutions are

$$F_l(R) = \left(\frac{\pi k R}{2}\right)^{1/2} J_{l+1/2}(kR) \tag{11}$$

$$G_l(R) = -\left(\frac{\pi k R}{2}\right)^{1/2} N_{l+1/2}(kR) \tag{12}$$

leading to

$$u_l^{(+)}(R) = G_l(R) + iF_l(R) \tag{13}$$

$$u_l^{(-)}(R) = F_l(R) - iF_l(R) \tag{14}$$

Finally, the external radial function is a combination of the two solutions:

$$u_l(R) = u_l^{(-)}(R) + \eta_l u_l^{(+)}(R) \tag{15}$$

where η is the fraction reflected of the ingoing wave.

Here $J_{l+1/2}(kR)$ is Bessel function and $N_{l+1/2}(kR)$ is the Neumann one. The expansions used in this work for the special functions are described in the Appendix 1.

Several approximations can be used to describe the wave function in the internal region. In the following, two approximations are used. The first one considers in that the internal wave function can be expressed as a dumping exponential function.

$$u_l(R) = C_l \exp(-iKR), \qquad R \leq R_s \tag{16}$$

Here K can be approximated with $[2M_R(E_n+V)/\hbar^2]^{1/2}$, V denoting the depth of the potential of the absorbing nucleus. This approximation is known as the strong interaction model.

The second way assumes a complex potential for the neutron-target interaction, and the associated model is called the cloudy crystal-ball. The interaction potential is

$$V(R) = V_0(1 - i\chi), \qquad 0 < \chi < 1 \tag{17}$$

In the actinide region, $V_0 \approx$ 42 MeV and $\chi \approx$ 0.1 [7]. Solving the Schödinger equations for a square well the interior solutions to the radial wave are found to be

$$u_l(R) = C_l k R j_l(KR) \tag{18}$$

where j_l is the Bessel function, C_l is a orthonormalization constant and

$$K = [2M_R(E_n - V)/\hbar^2]^{1/2} = [k^2 + (2M_R V_0/\hbar^2)(1 - i\chi)]^{1/2} \tag{19}$$

where $k = \sqrt{2M_R E_n/\hbar^2}$. The square root of a complex number is realized easily if this number is put on the form $a + ib = (\sqrt{a^2+b^2})\exp[i\arctan(b/a)]$.

Our aim is to deduce the fraction reflected of the external wave function denoted η_l, sometimes called the U-collision matrix. This parameter is obtained by solving the equation for the continuity of the logarithmic derivative on the surface, that means at the distance R_s:

$$R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s} = \frac{\dot{u}_l^{(-)}(R) + \eta_l \dot{u}_l^{(+)}(R)}{u_l^{(-)}(R) + \eta_l u_l^{(+)}(R)}\Bigg|_{R=R_s} \tag{20}$$

$R_s = 1.40 A_0^{1/3}$, A_0 the mass number of the compound nucleus. The inverse of the logarithmic derivative has the significance of the R-matrix.

Solving the Eq. (20) the complex number η_l is obtained and the compound nucleus cross section can be written:

$$\sigma_c = \pi\lambda^2 \sum_{l=0}^{\infty}(2l+1)(1 - |\eta_l|^2) \tag{21}$$

The quantities

$$T_l = (2l+1)(1 - |\eta_l|^2) \tag{22}$$

can be considered as the transmission of the neutron for a particular l, angular momentum carried by the neutron. Moreover, the compound cross section can be developed as:

$$\sigma_c = \pi\lambda^2 \sum_{l=0}^{\infty} \sum_{j=|i-1/2|}^{i+1/2} \frac{1}{2j+1} \sum_{J=|j-l|}^{j+l} \sum_{\Omega=-J}^{J} < j\Omega l0|J\Omega >^2 \ (2l+1)(1-|\eta_l|^2) \tag{23}$$

so that the probability to obtain the nucleus with a particular spin J, that means σ_c^J, is realized by retaining only the elements corresponding to J in the previous sum. Here the 1/2 comes from the neutron spin, i is the spin of the target, l is the orbital momentum carried by the neutron while Ω is the projection spin on the z-axis of the compound nucleus. In the previous equation, the square of Clebsh-Gordon coefficients $< j\Omega l0|J\Omega >$ intervene. λ is the wavelength in the incident channel divided by 2π (unfortunately the font for λ-bar was not found). Using the relativistic formula is is easy to derive this quantity. By measuring the incident neutron energy E_n in $m_n c^2$=939.565 MeV units, the linear momentum is

$$p = \sqrt{E_n(E_n+2)} \tag{24}$$

so that in MeV/c

$$p = 939.565\sqrt{E_n/939.565(E_n/939.565+2)} \tag{25}$$

while $\hbar c$=197.32891 MeV fm and the value of $\lambda = \hbar/p$ is obtained in fm.

The total cross-section (elastic+compound) is given by relation

$$\sigma_t = \pi\lambda^2 \sum_l (2l+1)(|1-\eta_l|^2 + 1 - |\eta_l|^2) \tag{26}$$

To put the Eq. (20) in a more suitable form, the phase constant ζ is defined in the external region

$$\exp(2i\zeta) = \frac{u_l^{(-)}(R)}{u_{(+)}(R)} \tag{27}$$

It is convenient to work with the real quantities $S_l(R)$ and $P_l(R)$ called shift factor and penetration factor, respectively, which satisfy the equations in the external region

$$R\frac{\frac{\mathrm{d}u_l^{(+)}}{\mathrm{d}R}}{u_l^{(+)}(R)} = S_l(R) + iP_l(R) \quad \text{or} \quad R\frac{\frac{\mathrm{d}u_l^{(-)}}{\mathrm{d}R}}{u_l^{(-)}(R)} = S_l(R) - iP_l(R) \tag{28}$$

in order to obtain (from Eq. (20))

$$\eta_l = \left. \frac{R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right) - S_l + iP_l}{R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right) - S_l - iP_l} \exp(2i\zeta_l) \right|_{R=R_s} \tag{29}$$

η_l must be obtained now by using some models for the logarithmic derivative in the internal region. As mentioned previously, two models are employed in this work: the strong interaction model and the cloudy crystal ball. These two approximations are treated separately.

In the strong interaction model

$$R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s} = -iKR \tag{30}$$

and we obtain the result

$$(1 - |\eta_l|^2) = \left. \frac{4P_l(R)KR}{S_l^2(R) + (KR + P_l(R))^2} \right|_{R=R_s} \tag{31}$$

In the cloudy crystal-ball model,

$$R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s} = R\left.\frac{KkRj_l'(KR) + kj_l(KR)}{kRj_l(KR)}\right|_{R=R_s} \tag{32}$$

An iterative procedure can be used in order to obtain the logarithmic derivative for any value of l starting with l=0. For $l = 0$

$$R\left(\frac{\frac{\mathrm{d}u_0(R)}{\mathrm{d}R}}{u_0(R)}\right)\Bigg|_{R=R_s} = 1 + \left.\frac{KR\cos(KR) - (KR)^2\sin(KR)}{\sin(KR)}\right|_{R=R_s} = KR\cot(KR) \tag{33}$$

After some calculations

$$R\left(\frac{\frac{\mathrm{d}u_0(R)}{\mathrm{d}R}}{u_0(R)}\right)\Bigg|_{R=R_s} = [\Re(KR) + i\Im(KR)]\left.\frac{1 - i\tan[\Re(KR)]\tanh[\Im(KR)]}{\tan[\Re(KR)] + i\tanh[\Im(KR)]})\right|_{R=R_s} \tag{34}$$

(the real part being denoted $\Re$ while the imaginary one $\Im$)so that

$$\Re\left\{ R\left(\frac{\frac{\mathrm{d}u_0(R)}{\mathrm{d}R}}{u_0(R)}\right)\Bigg|_{R=R_s}\right\} = \left.\frac{\Re(KR)\cos[\Re(KR)]\sin[\Re(KR)] + \Im(KR)\cosh[\Im(KR)]\sinh[\Im(KR)]}{\cosh^2[\Im(KR)] - \cos^2[\Re(KR)]}\right|_{R=R_s}$$

$$\Im\left\{ R\left(\frac{\frac{\mathrm{d}u_0(R)}{\mathrm{d}R}}{u_0(R)}\right)\Bigg|_{R=R_s}\right\} = \left.\frac{\Im(KR)\cos[\Re(KR)]\sin[\Re(KR)] - \Re(KR)\cosh[\Im(KR)]\sinh[\Im(KR)]}{\cosh^2[\Im(KR)] - \cos^2[\Re(KR)]}\right|_{R=R_s} \quad (35)$$

as shown in Appendix 2. These formulas are published with a wrong expression in [5]. Using the recurrence relation for Bessel functions (given in the appendix), we obtain the recurrence relations between logarithmic derivatives

$$R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s} = \left.\frac{KRJ_{l-1}(KR)}{J_l(KR)}\right|_{R=R_s} - l \quad (36)$$

so that

$$R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s} = \left.\frac{(KR)^2}{l - R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)}\right|_{R=R_s} - l \quad (37)$$

and two equations for the real and imaginary parts for the recurrence relations:

$$\Re\left\{ R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s}\right\} = \left.\frac{\{[\Re(KR)]^2 - [\Im(KR)]^2\}\left\{l - \Re\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]\right\} - 2\Re(KR)\Im(KR)Im\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]}{\left\{l - \Re\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]\right\}^2 + \left\{\Im\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]\right\}^2}\right|_{R=R_s} - l$$

$$\Im\left\{ R\left(\frac{\frac{\mathrm{d}u_l(R)}{\mathrm{d}R}}{u_l(R)}\right)\Bigg|_{R=R_s}\right\} = \left.\frac{\{[\Re(KR])^2 - [\Im(KR)]^2\}\Im\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right] + 2\Re(KR)\Im(KR)\left\{l - \Re\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]\right\}}{\left\{l - \Re\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]\right\}^2 + \left\{\Im\left[R\left(\frac{\frac{\mathrm{d}u_{l-1}(R)}{\mathrm{d}R}}{u_{l-1}(R)}\right)\right]\right\}^2}\right|_{R=R_s} - l \quad (38)$$

This last equation is introduced in formula (29) to obtain the cross section. We use the parameter $R_s = 1.4A^{1/3}$ (fm) [7].

3. Results

The results obtained with the code are displayed in figs.1, 2, 3 and 4. The total cross section, the compound cross section, the partial compound-cross sections for different values of J, and the transmissions are displayed.

In fig.1 the total reaction cross section σ_T is obtained within the strong interaction model as function of the incident energy of the neutron E_n. The reaction n+^{238}U was considered. The σ_T-values are very large for thermal neutrons and decrease rapidly for E_n of several MeV. The section σ_c for the formation of the compound nucleus is much lower and is given by the sum of partial cross sections for the formation of compound nucleus in different states of spin J=1/2, 3/2, 5/2... At very low values of E_n, the major contributions in the total compound cross section is given only by the partial cross sections for J=1/2 and 3/2. For energies up to 3-4 MeV, of interest for fission studies in the threshold region, only the partial cross sections with J=9/2 are sufficient to be taken into account to obtain results accurately enough.

In fig. 2 the same quantities as in Fig.1 are plotted, calculated in the frame of the cloudy ball model. It can be observed that an oscillatory behavior is exhibited by the total cross section with an average width of about 2 MeV as experimentally observed. The partial contribution of the J=3/2 channel is much stronger as in the previous case (the strong interaction model) and the partial contributions for J= 5/2 can be neglected. In the framework of this model the low spin channels have a stronger contribution in the compound cross section as in the frame of the previous one.

In fig. 3 the transmission of the centrifugal barrier for different values of the neutron angular momentum l=0, 1, 2... are plotted as function of the neutron energies. The strong interaction model is used. As expected, all transmissions tends to 1 when the kinetic energy of the neutron is increased.

In Fig. 3 the transmission of the centrifugal barrier for different values of the neutron angular momentum l=0, 1, 2... are plotted as function of the neutron energies. The strong interaction model is used. As expected, all transmissions tends to 1 when the kinetic energy of the neutron is increased.

In fig. 4 the transmission of the centrifugal barrier for different values of the neutron angular momentum l=0, 1, 2... are displayed in the frame of the cloudy ball model. The transmissions show an oscillatory behavior.

In fig. 5 we plotted a comparison between theoretical (cloudy ball model) and experimental [8] values for ^{235}U. This time the partial cross sections are for integers value of J. This plot give an estimation of the accuracy of our calculations. Due to the fact that the target has the spin 7/2, the partial cross sections for l=3 and 4 are larger at low energy. Deviations of about 30% between experimental and theoretical values are obtained.

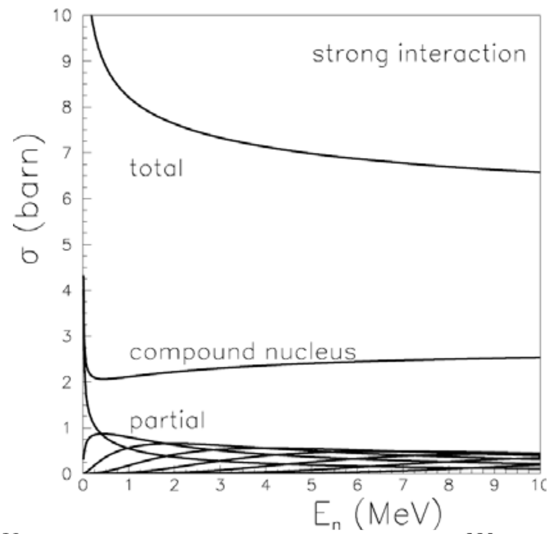

Fig.1 The reaction ^{238}U+n. The total cross section, the compound ^{239}U cross section and partial cross section for formation of the compound nucleus in different final spin J=0.5, 1.5…using the strong interaction model. E_n is the kinetic energy of neutrons.

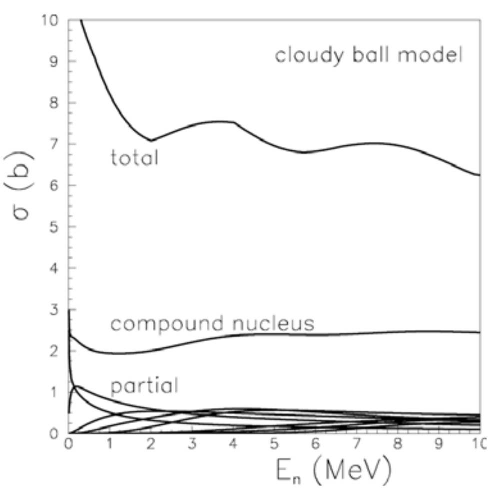

Fig.2 The reaction ^{238}U+n. The total cross section, the compound ^{239}U cross section and partial cross section for formation of the compound nucleus in different final spin J=0.5, 1.5…using the cloudy-ball (imaginary potential) model. E_n is the kinetic energy of neutrons.

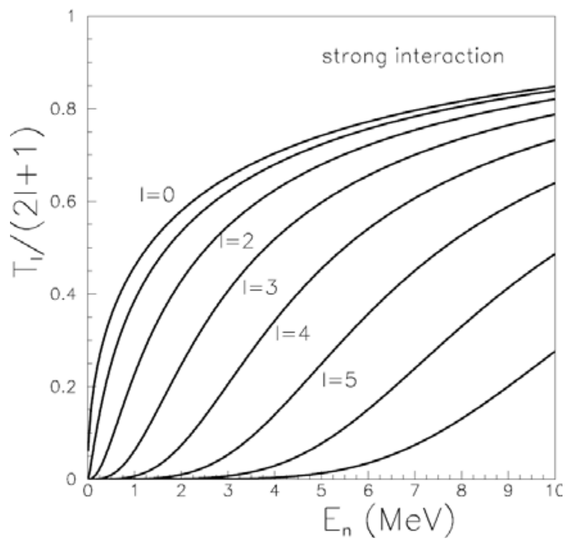

Fig.3 The reaction ^{238}U+n. The transmission coefficients of neutrons (Eq. 22) divided by $(2l+1)$ for l=0, 1…using strong interaction model. Asymptotically the value 1 is reached. E_n is the kinetic energy of neutrons.

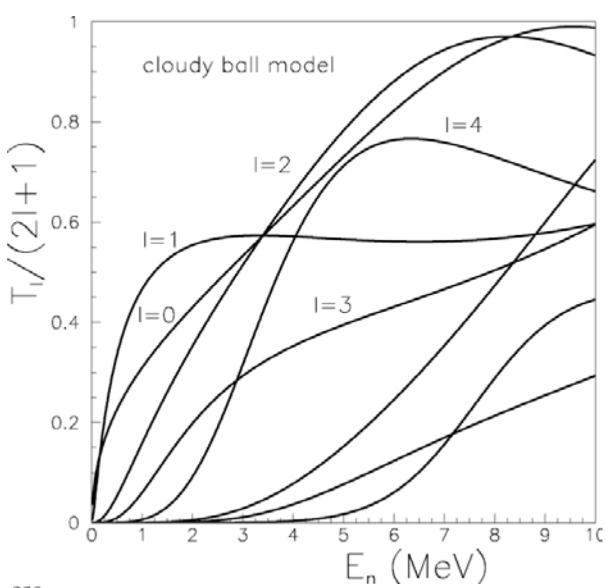

Fig. 4 The reaction ^{238}U+n. The transmission coefficients of neutrons (Eq. 22) divided by $(2l+1)$ for l=0, 1…using the cloudy-ball (imaginary potential) model. Asymptotically the value 1 is reached. E_n is the kinetic energy of neutrons

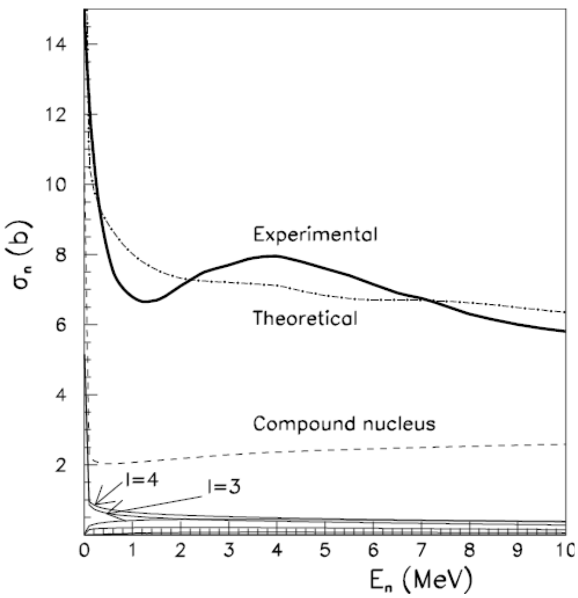

Fig. 5 Comparison between the total evaluated experimental cross section of ^{235}U (full line) [8] and the theoretical one (dot – dashed line) as function of the neutron incident energy. The compound nucleus cross section is plotted with dashed line. The partial cross sections for the formation of the compound nucleus with a given spin are plotted with full lines. Two spin are marked on the plot: *l*=3 and *l*=4.

4. Conclusions

In order to investigate the neutron induced fission, an essential ingredient is represented by the neutron transmission through centrifugal barriers. This quantity allows us to estimate the compound nucleus cross section and the probability that the compound nuclear system decays on a given channel. The transmission depends on the incident energy of the neutron, its orbital momentum, and the spin of the target. For this purpose, the theory concerning the neutron transmission was revisited and errors appearing in the literature were corrected. Two approaches were analysed: the strong interaction model and the cloudy ball one, the last model being based on a complex potential. The theory is presented in details. Numerical codes were realised for both versions. These codes are able to generate the compound nucleus cross sections in reactions induced by an incident neutron that interacts with a target nucleus, the total cross sections, and the spin dependent partial cross sections that depend on the states of the target nucleus. The theoretical simulations agree well with experimental data. In this context, the numerical codes realised in this work are well suited for the study of induced fission phenomena.

Appendix 1

Bessel and Neumann functions. In the following some useful series expansions and recurrence relations between special functions are displayed

$$J_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)} \tag{39}$$

$$N_\nu(z) = \frac{J_\nu \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{40}$$

$$J'_\nu(z) = \frac{1}{2}[J_{\nu-1}(z) - J_{\nu+1}(z)] \tag{41}$$

$$N'_\nu(z) = \frac{1}{2}[N_{\nu-1}(z) - N_{\nu+1}(z)] \tag{42}$$

If ν is close from an integer, then:

$$N_\nu(z) = -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!\Gamma(\nu+k+1)} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \ln\left(\frac{1}{2}z\right) J_n(z) - \frac{\left(\frac{1}{2}z\right)^n}{\pi} \sum_{k=0}^{\infty} \left\{\Psi(k+1) + \Psi(n+k+1)\right\} \frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!} \tag{43}$$

where the Digamma function is

$$\Psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}, \tag{44}$$

for $n \geq 2$ and

$$\Psi(1) = -\gamma \tag{45}$$

with

$$-\gamma = 2\ln 2 - 1.963510026021423 \tag{46}$$

Recurrence relations for Bessel functions

$$\frac{\mathrm{d}z^{m+1}J_m(z)}{\mathrm{d}z} = z^{m+1}J_{m-1}(z)$$
$$\frac{\mathrm{d}z^{m-1}J_m(z)}{\mathrm{d}z} = -z^{-m}J_{m+1}(z) \tag{47}$$

Appendix 2

Determination of the correct form for the wrong formulas of [5]

$$f_0 = (a+ib)\frac{\cos(a+ib)}{\sin(a+ib)} \tag{48}$$

$$\cos(a+ib) = \cos(a)\cos(ib) - \sin(a)\sin(ib)$$
$$\sin(a+ib) = \cos(a)\sin(ib) + \sin(a)\cos(ib) \tag{49}$$

$$f_0 = (a+ib)\frac{\cos(a)\cos(ib) - \sin(a)\sin(ib)}{\cos(a)\sin(ib) + \sin(a)\cos(ib)} =$$
$$(a+ib)\frac{\cos(a)[\exp(b)+\exp(-b)] - i\sin(a)[\exp(b)-\exp(-b)]}{i\cos(a)[\exp(b)-\exp(-b)] - \sin(a)[\exp(b)+\exp(-b)]}$$
$$(a+ib)\frac{1 - i\tan(a)\tanh(b)}{\tan(a) + i\tanh(b)} = (a+ib)\frac{[\tan(a) - \tan(a)\tanh^2(b)] - i[\tanh(b) + tanh(b)\tan^2(a)]}{\tan^2(a) + \tanh^2(b)} \tag{50}$$

$$\Re(f_0) = \frac{a[\tan(a) - \tan(a)\tanh^2(b)] + b[\tanh(b) + tanh(b)\tan^2(a)]}{\tan^2(a) + \tanh^2(b)} = \frac{\frac{a\tan(a)}{cosh^2(b)} + \frac{b\tanh(b)}{\cos^2(a)}}{\tan^2(a) + \tanh^2(b)} =$$
$$\frac{a\tan(a)\cos^2(a) + b\tanh(b)\cosh^2(b)}{[\tan^2(a) + \tanh^2(b)]\cosh^2(b)\cos^2(a)} = \frac{a\sin(a)\cos(a) + b\sinh(b)\cosh(b)}{\sin^2(a)\cosh^2(b) + \sinh^2(b)\cos^2(a)} =$$
$$\frac{a\sin(a)\cos(a) + b\sinh(b)\cosh(b)}{\cosh^2(b) - \cos^2(a)} \tag{51}$$

because

$$\sin^2(a)\cosh^2(b) + \sinh^2(b)\cos^2(a) = [1 - \cos^2(a)]\cosh^2(b) + \sinh^2(b)\cos^2(a) =$$
$$\cosh^2(b) - (\cosh^2(b) - \sin^2(b))\cos^2(a) = \cosh^2(b) - \cos^2(a) \tag{52}$$

R E F E R E N C E S

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