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S-ACTS WITH FINITE LENGTH ON CONGRUENCES

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An S-act A over a semigroup S is called strongly noetherian if it satisfies the ascending chain condition on its congruences. This is equivalent to being finitely generated of each congruence on A. We provide some fundamental facts about strongly noetherian acts. Another notion concerning chain conditions studied here is the property of being A of finite length on congruences. It is proved that every strongly noetherian as well as strongly artinian S-acts has finite length.

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1. Introduction and preliminaries

A finiteness condition for a class of algebraic systems is any property possessed by all finite members of that class. Such conditions are in fact a classical approach in the study of algebraic systems of different types and often formulated in terms of some notions concerning ordered sets, most importantly the maximal and the minimal conditions which are equivalent to the ascending and the descending chain conditions, respectively. A noetherian (artinian) algebraic system is the one which satisfies the ascending (descending) chain condition on its "substructures". Noetherian and artinian rings and modules have been widely studied in the literature. Unlike to the case of rings and modules, there are two different approaches for chain conditions on S-acts over a semigroup S: one is via their subacts, and the other via their congruences, and the notions of noetherian (artinian) and strongly noetherian (strongly artinian) are used for S-acts with ascending (descending) chain conditions on their subacts and congruences, respectively. The study of right noetherian semigroups and noetherian S-acts were initiated by Hotzel [5] and Normak [14], respectively. Further studies on these notions or their connections with other algebraic properties can be found in a number of papers, for example, see [3, 4, 6, 8, 9, 11, 12, 15, 16, 17]. It is well-known that a module M has finite length if and only if it is both noetherian and artinian, where the length of M is defined to be the length of the longest chain of submodules of M. Here we aim to introduce and study the length of acts over semigroups and those ones with finite lengths. Using the notion of a saturated chain of congruences of S-acts, we define the length of an S-act as the shortest length of its saturated chains. It is shown that, in contrast to the case of modules, two saturated chains of congruences for an S-act A do not generally have the same length and also being A of finite length is not necessarily equivalent to being it strongly noetherian

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as well as strongly artinian. Let us first recall some definitions and ingredients concerning S-acts needed in the sequel. For more information and the notions not mentioned here, see [7, 9].

Throughout the paper S stands for a semigroup with or without identity unless otherwise stated. Also, we set $S^1 = S \dot{\cup} \{1\}$ where 1 is an identity adjoined to S provided that S has no identity element, and otherwise, $S^1 = S$. A non-empty set A is said to be a (*right*) *S-act* if there is a, so called, *action* $\mu : A \times S \to A$ such that, denoting $\mu(a, s) := as$, $a(st) = (as)t$ and if S is a monoid with 1, $a1 = a$, for all $a \in A$ and $s, t \in S$. Each semigroup S can be considered as an S-act with the action given by its operation. An element $\theta \in A$ is said to be a *fixed* (*zero*) *element* if $\theta s = \theta$ for all $s \in S$. The S-act $A \cup \{\theta\}$ with a fixed element θ adjoined to A is denoted by A^θ. A non-empty subset B of A is called a *subact* of A if $bs \in B$, for every $s \in S$ and $b \in B$. A non-empty subset X of A is a *generating set* for A if $A = XS$, where $XS = \{xs \mid x \in X, s \in S\}$ for the case where S is a monoid, and if S is not a monoid, $A = XS \cup \{x\}$. By a *cyclic* act we mean an S-act with a singleton generating set. Any non-empty set A can be made into an S-act by setting $as = a$ for all $a \in A, s \in S$, namely *trivial action*. A *simple* S-act is an S-act with no proper subacts. Let A and B be two S-acts. A mapping $f : A \to B$ is called a *homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. The *product* of a non-empty family $\{A_i \mid i \in I\}$ of S-acts is their Cartesian product $\prod_{i\in I} A_i$ with the componentwise action, and the *coproduct* $\coprod_{i\in I} A_i$ of this family is their disjoint union with the action $(a, i)s = (as, i)$ for every $s \in S$ and $a \in A_i, i \in I$. An (*act*) *congruence* on an S-act A is an equivalence relation ρ on A for which $a\rho a'$ implies that $(as)\rho(a's)$ for any $a, a' \in A$ and $s \in S$. For $H \subseteq A \times A$, the *congruence generated by* H, that is, the smallest congruence on A containing H, is denoted by $\rho(H)$. For $a, b \in A$, one has $a\rho(H)b$ if and only if either $a = b$ or there exist $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n \in A, s_1, s_2, \dots, s_n \in S^1$ where $(p_i, q_i) \in H \cup H^{-1}$ for $i = 1, \dots, n$, such that $a = p_1s_1, q_1s_1 = p_2s_2, q_2s_2 = p_3s_3, \dots, q_ns_n = b$, where $H^{-1} = \{(y, x) \mid (x, y) \in H\}$. The above sequence is called an H-sequence of length n. We also denote the set of all congruences on A by $\mathrm{Con}(A)$. For a subact B of an s-act A, the Rees congruence ρ_B is defined as follow, $a\rho_B b$ if and only if $a = b$ or $\{a, b\} \subseteq B$. The factor act A/ρ_B may be denoted by A/B.

2. Strongly noetherian S-act

In this section we provide some fundamental facts about strongly noetherian S-acts. We investigate the behaviour of the property of being strongly noetherian under some act-theoretic constructions.

For an S-act A, the congruences $\Delta_A = \{(a, a) \mid a \in A\}$ and $\nabla_A = A \times A$ are called *diagonal congruence* and *universal congruence*, respectively. We say that a congruence is *trivial* if it is diagonal or universal. Otherwise, it is called *non-trivial*. By a *non-diagonal* (*non-universal*) congruence, we mean a congruence θ on A which $\theta \neq \Delta_A$ ($\theta \neq \nabla_A$).

Definition 2.1. *A congruence ρ_2 on an S-act A is called a cover of a congruence ρ_1 and denoted by $\rho_1 \sqsubset \rho_2$ if $\rho_1 \subset \rho_2$ and there is no congruence strictly between ρ_1 and ρ_2. Also ρ_2 is called a principal extension of ρ_1 provided that there exists $(a, b) \in A \times A$ such that $\rho_2 = \rho(\rho_1 \cup \{(a, b)\})$.*

Note. It is clear that each cover is a principal extension, but the converse is not generally true. For instance, consider a semilattice $S = (L, \wedge)$ as an act over itself with top and bottom elements $\top$ and $\bot$, respectively. Then ∇_S is a principal extension of Δ_S of the form $\nabla_S = \rho(\Delta_S \cup \{(\bot, \top)\})$ whereas S may have non-trivial congruences.

Recall that an S-act A is said to be *noetherian* (*artinian*) if every ascending (descending) chain of subacts of A is eventually stationary. Considering chain of congruences instead of chain of subacts in these definitions, we have the following:

Definition 2.2 ([15])**.** *An S-act A is said to be strongly noetherian (strongly artinian) if every ascending (descending) chain of congruences of A is eventually stationary.*

Remark 2.1. (i) *If A is a strongly noetherian (strongly artinian) S-act, then it is noetherian (artinian). These follow from the fact that for any subacts A_i, A_j of A, if $A_i \subset A_j$, then $\rho_{A_i} \subset \rho_{A_j}$. But the converse is not valid in general. For this, consider a group S as an S-act not finitely generated as a group. Clearly, S is simple (i.e. it has no proper nontrivial normal subgroups) and so noetherian and artinian. Since S is not finitely generated, it is not strongly noetherian (see Lemma* **3**.*3). In Example* **2**.*1(i), we present an S-act that is artinian but not strongly artinian.*

(ii) *Each subact of a strongly noetherian A is finitely generated. In particular, A is finitely generated. But finitely generated acts are not necessarily strongly noetherian (see Example* **2**.*1(ii)).*

Lemma 2.1. *Let A be an S-act. The following are equivalent:*

(i) *A is strongly noetherian.*

(ii) *Every $\theta \in \mathrm{Con}(A)$ is finitely generated.*

(iii) *For each non-diagonal $\theta \in \mathrm{Con}(A)$, there is a finite chain $\Delta_A = \theta_0 \subset \theta_1 \subset \cdots \subset \theta_n = \theta$ of congruences on A in which θ_i is a principal extension of θ_{i-1}, for each $1 \leq i \leq n$.*

(iv) *Every set of congruences on A has a maximal element.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iv) are clear.

(ii) $\Rightarrow$ (iii) Let $\theta \in \mathrm{Con}(A)$. Using (ii), $\theta = \rho(\{(a_1, b_1), \ldots, (a_n, b_n)\})$ for some $a_i, b_i \in A, i = 1, \ldots, n$. Taking $\theta_0 = \Delta$ and $\theta_i = \rho(\{(a_1, b_1), \ldots, (a_i, b_i)\})$ for every $1 \leq i \leq n$, each θ_i is a principal extension of θ_{i-1}.

(iii) $\Rightarrow$ (ii) Let $\theta \in \mathrm{Con}(A)$. It follows from the assumption that there is a finite chain $\Delta = \theta_0 \subset \theta_1 \subset \cdots \subset \theta_n = \theta$ of congruences on A in which θ_i is a principal extension of θ_{i-1}, for each $1 \leq i \leq n$. Then for each $1 \leq i \leq n$, $\theta_i = \rho(\theta_{i-1} \cup \{(a_i, b_i)\})$, for some $a_i, b_i \in A$. This clearly gives that $\theta = \rho(\{(a_1, b_1), \ldots, (a_n, b_n)\})$, which means that θ is finitely generated. □

Using Rees congruences on A, it is not difficult to check that if every $\theta \in \mathrm{Con}(A)$ is finitely generated, then every subact of A is finitely generated. So we have the following:

Corollary 2.1. *If A is a strongly noetherian S-act, then every subact of A is finitely generated.*

Corollary 2.2. *If A is a strongly noetherian S-act with zero, then there exists a finite set $T \subseteq A \times A$ of the form $\{0\} \times X$ such that for any $a, b \in A$, there is a T-sequence from a to b of length at most 2.*

Proof. By Remark **2**.1(ii), A is generated by a finite set X. Consider $T = \{0\} \times X$. So for any $a, b \in A$ there exist $x_1, x_2 \in X$ and $s, t \in S$ such that $a = x_1 s$ and $b = x_2 t$. Now we have the T-sequence $a = x_1 s, 0s = 0t, x_2 t = b$ from a to b of length at most 2. □

Here we give some examples used in the sequel.

Example 2.1. (i) *Consider the semigroup $S = (\mathbb{N}, \min)$. The following strict chain shows that S is not a strongly noetherian S-act:*

$$\Delta_S \subset \rho_{\{1,2\}} \subset \rho_{\{1,2,3\}} \subset \rho_{\{1,2,3,4\}} \subset \cdots$$

Also it is clear that for each $n \in \mathbb{N}$, $\theta_n = \Delta_S \cup \{(a,b) \mid a,b \geq n+1\}$ *is a congruence in which* $\nabla_S \supset \theta_1 \supset \theta_2 \supset \cdots$ *is a strict chain. So* S *is not strongly artinian as an* S*-act. Note that this is an example of an artinian* S*-act which is not strongly artinian.*

(ii) *Similarly to* (i), *the semigroup* $S = (\mathbb{N}, \max)$ *is not strongly noetherian nor strongly artinian as an* S*-act.*

(iii) *Let* $S = (\mathbb{N}, +)$. *The strict chain*

$$\nabla_S \supset \rho_{\{2,3,\ldots\}} \supset \rho_{\{3,4,\ldots\}} \supset \rho_{\{4,5,\ldots\}} \supset \cdots$$

shows that S *is not strongly artinian as an* S*-act. But* S *is a strongly noetherian* S*-act. For this, take a congruence* ρ *on* S. *Using Lemma* **2**.*1, it must be shown that* ρ *is finitely generated. Let* n_0 *be the smallest natural number for which there exists* $m \in \mathbb{N}$ *with* $n_0 < m$ *and* $n_0 \rho m$. *Assume also that* m_0 *is the smallest natural number such that* $n_0 < m_0$ *and* $n_0 \rho m_0$. *Then* $m_0 = n_0 + t_0$ *for some* $t_0 \in \mathbb{N}$. *Since* $(n_0+t)\rho(m_0+t) = (n_0+t)+t_0$ *for each* $t \in \mathbb{N}$, $n\rho(n+t_0)$ *for each* $n \geq n_0$ *so that* $n\rho(n+kt_0)$ *for all* $k \in \mathbb{N}$. *Let* $x, y \geq n_0$ *and* $x \equiv y \pmod{t_0}$. *So* $y - x = k_0 t_0$ *and hence* $x\rho y$. *Now suppose that* $x, y \geq n_0$ *and* $x \not\equiv y \pmod{t_0}$. *Then there exist* $r_1, r_2 \in \mathbb{N}$ *such that* $x\rho(n_0+r_1)$ *and* $y\rho(n_0+r_2)$. *Without loss of generality, assume that* $0 \leq r_1 < r_2 \leq t_0$. *If* $(n_0+r_1)\rho(n_0+r_2)$, *then* $(n_0+r_1+t_0-r_2)\rho(n_0+r_2+t_0-r_2)$ *and hence* $(n_0+r_1+t_0-r_2)\rho m_0 \rho n_0$ *in which* $r_1+t_0-r_2 < t_0$ *which is impossible. Thus* $(x,y) \notin \rho$ *which gives that all equivalence classes of* ρ *are* $[n_0]_\rho, [n_0+1]_\rho, [n_0+2]_\rho, \ldots, [n_0+(t_0-1)]_\rho$. *Hence,* ρ *is generated by the finite set* $\{(n_0,m_0),(n_0+1,m_0+1),\ldots,(n_0+t_0-1,m_0+t_0-1)\}$, *as required.*

As usual, a non-universal (non-diagonal) congruence θ on an S-act A is said to be maximal (minimal), if there is no congruence ρ on A lying strictly between θ and ∇_S (Δ_S). We say that an S-act A satisfies the maximal (minimal) condition for congruences if each non-universal (non-diagonal) congruence is contained into (contains) a maximal (a minimal) congruence.

Remark 2.2. *Every strongly noetherian (strongly artinian)* S*-act satisfies the maximal (minimal) condition for congruences. The converses of these facts are not generally true. For this, take any infinite set* $A = \{a_i\}_{i=1}^{\infty}$ *with trivial action of a monoid* S *on* A. *Note that in this case every equivalence relation is a congruence. Clearly, each equivalence relation with two equivalence classes is a maximal congruence and each Rees relation* $\rho_{\{a_i,a_j\}}$ *is a minimal congruence on* A. *It is easily seen that any non-universal congruence* ρ *on* A *is contained into an equivalence relation* θ *with two equivalence classes which is maximal. Now let* ρ *be a non-diagonal congruence and* $(a_i,a_j) \in \rho$. *Then* ρ *contains* $\rho_{\{a_i,a_j\}}$ *which is minimal. However,* A *is neither strongly noetherian nor strongly artinian. Indeed, there are the following infinite strict chains of congruences:*

$$\Delta \subset \rho_{\{a_1,a_2\}} \subset \rho_{\{a_1,a_2,a_3\}} \subset \cdots \quad \text{and} \quad \nabla \supset \rho_{A\setminus\{a_1\}} \supset \rho_{A\setminus\{a_1,a_2\}} \supset \cdots$$

Lemma 2.2. *If* A *is a strongly noetherian* S*-act, then every subact and every homomorphic image of* A *are also strongly noetherian.*

Proof. Consider a subact B of A and $\theta \in \mathrm{Con}(B)$. Using [9, Theorem 2.1(i)], $\bar{\theta} = \theta \cup \Delta_A$ is a congruence on A which is finitely generated by Lemma **2**.1. So θ is finitely generated and hence B is strongly noetherian. Let $\rho \in \mathrm{Con}(A)$. Since there is a lattice isomorphism between the interval $[\rho, \nabla_A] = \{\theta \in \mathrm{Con}(A) \mid \rho \subseteq \theta\}$ of the lattice $\mathrm{Con}(A)$ and $\mathrm{Con}(A/\rho)$ (see [9, Theorem 2.1(ii)] and [2, Theorem II.6.20]), every ascending chain of congruences of the S-act A/ρ has a correspondence ascending chain of congruences in the interval $[\rho, \nabla_A]$, which is eventually stationary. More precisely, here the lattice isomorphism $\alpha : [\rho, \nabla_A] \to \mathrm{Con}(A/\rho)$

is given by $\alpha(\theta) = \theta/\rho$ for any $\theta \in [\rho, \nabla_A]$, where $\theta/\rho = \{([a]_\rho, [b]_\rho) \in (A/\rho)^2 \mid (a,b) \in \theta\}$. Thus, using the fact that each lattice isomorphism is also an order-embedding, i.e. $\alpha(\theta_i) \subseteq \alpha(\theta_{i+1})$ implies $\theta_i \subseteq \theta_{i+1}$, the second assertion also holds. □

Corollary 2.3. *Let $\{A_i \mid i \in I\}$ be a non-empty family of S-acts. If $\prod_{i\in I} A_i$ is strongly noetherian, then so is each A_i.*

We need the following theorem in the sequel proved in [8, Lemma 4.1].

Theorem 2.1. *Let A be a subact of an S-act B. Then A and B/A are strongly noetherian if and only if B is strongly noetherian.*

Now we are going to study the behaviour of being strongly noetherian with respect to coproducts.

Proposition 2.1. *Let B be a subact of an S-act A for which $A \setminus B$ is a finite set. Then A is strongly noetherian if and only if so is B.*

Proof. This follows immediately from Theorem **2**.1, since A/B is finite and hence obviously strongly noetherian. □

Corollary 2.4. *An S-act A is strongly noetherian if and only if so is A^θ.*

Using Theorem **2**.1, Corollary **2**.4 and the fact that $\dfrac{A \sqcup B}{A} \simeq B^\theta$ for any S-acts A and B, we have the following:

Corollary 2.5. (i) *Let A and B be two S-acts. Then $A \sqcup B$ is strongly noetherian if and only if A and B are strongly noetherian.*

(ii) *Let $\{A_i \mid i \in I\}$ be a non-empty family of S-acts. If $\coprod_{i\in I} A_i$ is strongly noetherian, then so is each $A_i, i \in I$.*

Let A be a T-act and $\alpha : S \to T$ be a semigroup epimorphism. Then A can be made into an S-act by setting $as = a\alpha(s)$ for each $a \in A$ and $s \in S$. Also, if $\theta \in \mathrm{Con}(A_T)$, then $\theta \in \mathrm{Con}(A_S)$ so that if A is strongly noetherian as the S-act, then it is strongly noetherian as the T-act.

Proposition 2.2. *Let S be strongly noetherian as an act over itself. Then an S-act A is strongly noetherian if and only if it is finitely generated.*

Proof. If A is strongly noetherian, then it is finitely generated by Corollary **2**.1. For the converse, let A be generated by $\{x_1, x_2, \ldots, x_n\}$. So A is a homomorphic image of $\coprod_{i=1}^n S$. Using Corollary **2**.5 and Theorem **2**.1, the result follows. □

Remark 2.3. *It is easily seen that each congruence on an S-act A is a subact of the S-act $A \times A$. So if $A \times A$ is noetherian, then A is strongly noetherian. But the converse is not generally true. Indeed, the semigroup $S = (\mathbb{N}, +)$ explained in Example **2**.1(iii) is a strongly noetherian S-act. Clearly, $(a,b)S = \{(a+k, b+k) \mid k \in \mathbb{N}\}$. Thus*

$$(1,1)S \subset (1,1)S \cup (2,1)S \subset (1,1)S \cup (2,1)S \cup (3,1)S \subset \cdots$$

is a strict ascending chain of subacts of $S \times S$, which means that $S \times S$ is not noetherian and hence not a strongly noetherian S-act.

Proposition 2.3. *Let A be an S-act. If $A \times A$ is a noetherian S-act, then A has finitely many minimal congruences.*

Proof. Assume that $\Gamma = \{\theta_i\}_{i\in I}$ is a set of distinct minimal congruences of A where I is infinite. Let $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \cdots$ be a strict chain in $\mathcal{P}(\Gamma)$, where $\mathcal{P}(\Gamma)$ denotes the power set of Γ. Consider the chain $\bigcup_{i\in\Gamma_1}\theta_i \subset \bigcup_{i\in\Gamma_2}\theta_i \subset \cdots$. Since each $\bigcup_{i\in\Gamma_j}\theta_i$ is a subact of $A\times A$ and $A\times A$ is noetherian, the chain is eventually stationary. So there is $N\in\mathbb{N}$ such that for every $j \geq N$, $\bigcup_{i\in\Gamma_j}\theta_i = \bigcup_{i\in\Gamma_N}\theta_i$. Thus for each $i\in\Gamma_j$, $\theta_i \subseteq \bigcup_{i\in\Gamma_N}\theta_i$. Since θ_i is minimal, it is monogenic as $\theta_i = \rho(a,b), a, b \in A$. Then there is $\theta_k \in \Gamma_N$ such that $(a,b)\in\theta_k$ and hence $\theta_i = \theta_k$, by minimality of θ_k, which is a contradiction. □

Similarly to the proof of Proposition **2**.3, one can show that a noetherian S-act for which every congruence is a Rees congruence has finitely many minimal congruences.

Remark 2.4. *Let A be a strongly artinian or a strongly noetherian S-act. Then it has only finitely many congruences $\Sigma = \{\theta_1, \theta_2, \ldots, \theta_m\}$ satisfying $\theta_j \subseteq \rho(\bigcup_{j\neq i=1}^{m}\theta_i)$ for each $\theta_j\in\Sigma$. To this end, let $\{\theta_i \mid i\in\mathbb{N}\}$ be an infinite set of congruences of A. For the case where A is strongly artinian, consider the chain $\rho(\bigcup_{i=1}^{\infty}\theta_i) \supseteq \rho(\bigcup_{i=2}^{\infty}\theta_i) \supseteq \cdots$ of congruences of A. So there exists $N\in\mathbb{N}$ such that $\rho(\bigcup_{i=N}^{\infty}\theta_i) = \rho(\bigcup_{i=N+1}^{\infty}\theta_i)$. Thus $\theta_N \subseteq \rho(\bigcup_{i=N+1}^{\infty}\theta_i)$. If A is strongly noetherian, then the chain $\theta_1 \subseteq \rho(\theta_1\cup\theta_2) \subseteq \rho(\theta_1\cup\theta_2\cup\theta_3) \subseteq \cdots$ of congruences of A gives that there exists $N\in\mathbb{N}$ such that $\rho(\bigcup_{i=1}^{N}\theta_i) = \rho(\bigcup_{i=1}^{N+1}\theta_i)$ and hence $\theta_{N+1} \subseteq \rho(\bigcup_{i=1}^{N}\theta_i)$.*

3. S-Acts with finite length

This section is devoted to introduce the notion of length of an S-act by means of finiteness conditions on its congruences. We investigate whether the property of being of finite length is inherited by being strongly noetherian and strongly artinian and vice versa. In the case that S has finite length as an S-act, it is shown that each S-act has finite length if and only if it is finitely generated. Some results are also obtained when S is a group.

Let A be an S-act and $\rho, \theta \in \mathrm{Con}(A)$. If $\rho\subset\theta$ and no congruence of A lies strictly between ρ and θ, we say that θ is a *cover* for ρ and use the notation $\rho \sqsubset \theta$ in which "$\sqsubset$" denotes the *cover relation* on $\mathrm{Con}(A)$. Likewise, the cover relation on the set of subacts of an S-act can also be defined.

Definition 3.1. *Let A be an S-act. Any chain of the form*

$$\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \nabla_A, n\in\mathbb{N},$$

of congruences of A is called a saturated chain (of congruences of length n) for A. The length of A, represented as $l_c(A)$, is the shortest length of saturated chains of congruences of A. If A has no saturated chain, then we define $l_c(A)=\infty$. More generally, let $\theta\in\mathrm{Con}(A)$. Every chain of the form

$$\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \theta, n\in\mathbb{N},$$

of congruences of A is called a saturated chain of (congruences of length n) for θ. The length of θ, represented as $l(\theta)$, is the shortest length of saturated chains of θ. If θ has no saturated chain, then we define $l(\theta)=\infty$. So $l(\nabla_A) = l_c(A)$. Moreover, any saturated chain of the form

$$\theta \sqsubset \theta_1 \sqsubset \theta_2 \sqsubset \cdots \sqsubset \theta_n = \nabla_A$$

is said to be a saturated chain containing θ.

Proposition 3.1. *For an S-act A, we have the following assertions:*

(i) *If A is strongly artinian, then each non-universal congruence of A has a cover. In particular, A has a minimal congruence. Also every set of congruences of A has a minimal element.*

(ii) *If A is strongly noetherian, then each non-diagonal congruence of A is a cover of a congruence of A. In particular, A has a maximal congruence. Also every set of congruences of A has a maximal element.*

Proof. (i) Let ρ be a non-universal congruence of A. If $\rho \sqsubset \nabla$, then we are done. Otherwise, there is a chain $\rho \subset \theta_1 \subset \nabla$. If $\rho \sqsubset \theta_1$, there is nothing to prove. Otherwise, there is a chain $\rho \subset \theta_2 \subset \theta_1 \subset \nabla$. Continuing the similar argument, if each congruence θ_i is not a cover of ρ, then we have an infinite strict chain of congruences, which is a contradiction.

(ii) The proof is similar to (i). □

Lemma 3.1. *If A is strongly noetherian as well as strongly artinian, then for all congruences $\rho \subset \theta$ there is a saturated chain of congruences $\rho \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \theta$.*

Proof. If $\rho \sqsubset \theta$, then we are done. Otherwise, since A is strongly artinian, by Proposition **3**.1, the set $\{\tau \in \mathrm{Con}(A) \mid \rho \subset \tau \subseteq \theta\}$ has a minimal element, say ρ_1. So $\rho \sqsubset \rho_1 \subset \theta$. If $\rho_1 \sqsubset \theta$, then there is nothing to prove. Otherwise, there exists $\rho_2 \in \mathrm{Con}(A)$ such that $\rho_1 \sqsubset \rho_2 \subset \theta$. Continuing the similar process, if each ρ_i is a proper subset of θ, then we get the strict infinite chain of congruences which contradicts being strongly noetherian of A. Therefore, $\rho_n = \theta$ for some $n \in \mathbb{N}$. Hence, $\rho \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \theta$ is a saturated chain of congruences. □

Remark 3.1. *As a corollary of Lemma* **3**.*1, it is concluded that if A is both strongly noetherian and strongly artinian, then A has finite length. But the following example shows that the converse of this fact is not generally true. Let $S = Z_2 = \{1, x\}$, and $A = \{a_i, b_i \mid i \in \mathbb{N}\}$ be the S-act with action given by $a_i x = b_i$ and $b_i x = a_i$. Consider $B_0 = \emptyset$ and for each $i \in \mathbb{N}$, $B_i = B_{i-1} \cup \{a_i, b_i\}$. So the saturated chain of Rees congruences $\Delta_A \sqsubset \rho_{B_1} \sqsubset \rho_{B_2} \sqsubset \cdots$, shows that A is not necessary strongly noetherian. But A has finite lenght, for, let θ be an equivalence relation on A induced by the classes $\{a_i \mid i \in \mathbb{N}\}$ and $\{b_i \mid i \in \mathbb{N}\}$. Clearly, θ is a congruence on A and the saturated chain $\Delta_A \sqsubset \theta \sqsubset \nabla_A$, shows that A has finite lenght.*

In contrast to the case of modules that two saturated chains of submodules over a ring have the same lengths and if N is a submodule of M, then $l(N) \leq l(M)$, the following example shows that two saturated chains of congruences for an S-act A with finite length do not have necessarily the same lengths and also there exists a congruences σ and θ on A for which $\sigma \subset \theta$ but $l(\theta) < l(\sigma)$.

Example 3.1. *Let $S = Z_2 = \{1, x\}$, and let $A = \{a, b, c, d, e\}$ be the S-act with action given as follows:*

$$ax = b, bx = a, cx = d, dx = c, ex = e.$$

Let $B_1 = \{a, b\}, B_2 = \{c, d\}, C = \{a, b, e\}$. Clearly, A is the disjoint union of its subacts C and $A \backslash C$. Let σ be the congruence with classes C and $A \backslash C$. Then we have a saturated chain of congruences $\Delta_A \sqsubset \rho_B \sqsubset \rho_C \sqsubset \sigma \sqsubset \nabla_A$ of length 4. Now let τ be the congruence on A with classes $\{a, c, e\}, \{b, d, f\}$. Clearly, there is no congruence θ on A such that $\Delta_A \subset \theta \subset \tau$. Thus $\Delta_A \subset \tau \subset \nabla_A$ is a saturated chain of congruences of length 2. So there are two saturated chains on congruences of A with different lengths. Also σ has a saturated chain as $\Delta_A \sqsubset \rho_B \sqsubset \rho_C \sqsubset \sigma$, but it has no saturated chain of length 2. So $l(\sigma) = 3 > 2 = l(\nabla)$.

A finite S-act is clearly strongly noetherian as well as strongly artinian (and so has finite length). But the converse does not hold in general (see [10]). In continue up to

Proposition **3**.2, we discuss about the converse of this fact in the case being S a group. First, consider the following example.

Example 3.2. *Let A be an S-act with trivial action. Then we have the saturated chain of Rees congruences $\Delta_A \sqsubset \rho_{\{a_1\}} \sqsubset \rho_{\{a_1,a_2\}} \sqsubset \cdots \sqsubset \nabla_A$, where $a_i \in A$. Hence, A is finite if and only if $l_c(A)$ is finite.*

Clearly, for a semigroup S, act congruences on the S-act S are precisely the right congruences on S. For a group S, note the following:

Lemma 3.2. *Let S be a group.*

(i) *There exists a lattice isomorphism between act congruences on S and subgroups of S.*

(ii) *There exists a lattice isomorphism between group congruences on S and normal subgroups of S.*

Proof. (i) Let T be a subgroup of S. We construct an act congruence on S as follows: $x\rho_T y$ if and only if $xy^{-1} \in T$ (and hence $T = [1]_{\rho_T}$). Conversely, consider an act congruence ρ on S. Then $T = [1]_\rho$ is clearly a subgroup of S. It is also easily seen that this correspondence is one to one.

(ii) The proof is similar to that of part (i). □

Corollary 3.1. *Let a group S have finite length as an S-act. Then S has a finite length on its group congruences.*

Using Lemma **3**.2(i), we have

Corollary 3.2. *Let S be a group. Then the following are equivalent:*

(i) $l_c(S) = n$.

(ii) *There is a saturated chain $\{1\} \sqsubset S_1 \sqsubset S_2 \sqsubset \cdots \sqsubset S_n = S$ of length n of subgroups of S.*

Lemma 3.3. *Let S be a group. Then*

(i) *If S is a cyclic group which is strongly artinian as an S-act, then it is finite.*

(ii) *If S is strongly noetherian as an S-act, then every subgroup of S is a finitely generated group.*

Proof. (i) Let $S = \langle s \rangle$. Then $S = \langle s \rangle \supseteq \langle s^2 \rangle \supseteq \langle s^3 \rangle \supseteq \cdots$ is a descending chain of subgroups. Using Lemma **3**.2, there exists $n \in \mathbb{N}$ such that $\langle s^n \rangle = \langle s^{n+1} \rangle$. Then $s^n = s^{k(n+1)}$, for some $k \in \mathbb{N}$ and hence $s^{k(n+1)-n} = 1$, which implies that S is finite.

(ii) Let T be a non-finitely generated subgroup of S generating by an infinite set X. Let $\{s_i\}_{i=1}^{\infty} \subseteq X$. Consider the subgroups $T_1 = \langle s_1 \rangle$, $T_2 = \langle s_1, s_2 \rangle, \ldots$ of S and the S-act congruences ρ_i on S constructed as follow: $x\rho_i y$ if and only if $xy^{-1} \in T_i$ (and so $T_i = [1]_{\rho_i}$). So the infinite strict chain $\rho_1 \subset \rho_2 \subset \rho_3 \subset \cdots$ contradicts the hypothesis. □

Proposition 3.2. *Let an abelian group S have finite length as an S-act. Then the following assertions hold:*

(i) *S is finite.*

(ii) *Each finitely generated S-act is finite.*

Proof. (i) First we prove the assertion for an abelian group S with finite length as an S-act. By Remark **3**.1 and Lemma **3**.3(ii), S is a finitely generated group. Let S be generated by the set $X = \{x_1, x_2, \ldots, x_k\}$. Take any $x_i \in X$. We claim that the subgroup $T_i = \langle x_i \rangle$ has finite length as a T_i-act. Otherwise, T_i is not a strongly noetherian S-act or not a strongly artinian T_i-act. Thus T_i has an ascending or descending strict infinite chain of congruences. By Lemma **3**.2, T_i has an ascending or descending strict infinite chain of subgroups, which is in fact an ascending or descending strict infinite chain of subgroups of S. Using Lemma **3**.2, S has infinite length, which is a contradiction. Now Remark **3**.1 and Lemma **3**.3(i) give that x_i has finite order and hence S is finite.

(ii) Follows from (i). □

A congruence θ on an S-act A is said to be *prime* if $\rho_1 \cap \rho_2 \subseteq \theta$ implies $\rho_1 \subseteq \theta$ or $\rho_2 \subseteq \theta$, for $\rho_1, \rho_2 \in \mathrm{Con}(A)$.

Theorem 3.1. *If A has finite length and every maximal congruence of A is prime, then the number of maximal congruences of A is finite.*

Proof. Let $\Sigma = \{\bigcap_{i=1}^{n} \rho_i \mid n \in \mathbb{N}, \rho_1, \dots, \rho_n$ are maximal congruences of $A\}$. Using Proposition **3**.1 and Remark **3**.1, Σ is non-empty and has a minimal element $\bigcap_{i=1}^{t} \rho_i$, say. Now let ρ be a maximal congruence of A. So $\rho_1 \cap \cdots \cap \rho_t \cap \rho \in \Sigma$ which implies $\rho_1 \cap \cdots \cap \rho_t \subseteq \rho$. Since ρ is prime by the assumption, $\rho_i \subseteq \rho$ for some $1 \le i \le t$. So $\rho_i = \rho$ by maximality of ρ_i. Hence, A has a finitely many maximal congruences. □

Remark 3.2. (i) *It is clear that any S-act A is simple if and only if $l_c(A) = 1$.*

(ii) *If S-acts A and B are isomorphic, then $l_c(A) = l_c(B)$. But the converse fails in general. For this, take the monoid $S = \{1, s\}$ where $s^2 = 1$. Consider two S-acts $A = \{a, b, c\}$ with trivial action and $B = \{a, b, c, d\}$ with a and b are fixed elements and $c1 = ds = c$ and $d1 = cs = d$. The non-trivial congruences of A are Rees congruences $\rho_{\{a,b\}}, \rho_{\{a,c\}}$ and $\rho_{\{b,c\}}$, and that of B are Rees congruences $\rho_{\{a,b\}}, \rho_{\{a,c,d\}}, \rho_{\{c,d\}}$ and $\rho_{\{b,c,d\}}$. So $l_c(A) = l_c(B) = 2$ whereas A and B are non-isomorphic.*

(iii) *For each S-act A, $l_c(A) \le |A|$. Indeed, if A is infinite, then we are done. Let $|A| = n$. Then A has finite length as $l_c(A) = m$. Let $\rho_0 = \Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_m = \nabla_A$ be a saturated chain for A. We know each ρ_i is a principal extension of ρ_{i-1}. So for all $1 \le i \le m$ there are distinct elements $(a_i, b_i) \in \rho_i \setminus \rho_{i-1}$. Consider $L = \{(a_1, b_1), \dots, (a_m, b_m) \mid (a_i, b_i) \in \rho_i \setminus \rho_{i-1},\ i = 1, 2, \dots, m\}$ and $L_i = \{a_1, \dots, a_i, b_1, \dots, b_i \mid (a_j, b_j) \in L, 1 \le j \le i\}$ for $1 \le i \le m$. By induction on $1 \le k \le m$, we show that $k \le |L_k|$. For $k = 1$ we are done. Let $k - 1 \le |L_{k-1}|$. Let $(a_k, b_k) \in L$ and then $(a_k, b_k) \in \rho_k \setminus \rho_{k-1}$. If $a_k, b_k \in L_{k-1}$, then $\rho_k = \rho_{k-1}$, which is a contradiction. So one of a_k or b_k does not belong to L_{k-1}. Then $|L_k| \ge |L_{k-1}| + 1 \ge (k-1) + 1 = k$. Therefore, $m \le |L_m| \le |A|$.*

Lemma 3.4. *Let A and B be S-acts and $\rho_1, \rho_2 \in \mathrm{Con}(A)$ and $\theta \in \mathrm{Con}(B)$.*

(i) *If $\rho_1 \sqsubset \rho_2$, then $\rho_1 \sqcup \theta \sqsubset \rho_2 \sqcup \theta$ in $\mathrm{Con}(A \sqcup B)$.*

(ii) $\max\{l_c(A), l_c(B)\} \le l_c(A \sqcup B) \le l_c(A) + l_c(B) + 1$.

Proof. (i) Clearly, $\rho_i \sqcup \theta \in \mathrm{Con}(A \sqcup B), i = 1, 2$. Let $\rho_1 \cup \theta \subset \psi \subset \rho_2 \cup \theta$ for $\psi \in \mathrm{Con}(A \sqcup B)$. Then $\rho_1 \subset \psi|_A \subset \rho_2$ and $\psi|_B = \theta$. So $\psi|_A = \rho_1$ or $\psi|_A = \rho_2$ and hence $\psi = \rho_1 \cup \theta$ or $\psi = \rho_2 \cup \theta$.

(ii) The first inequality follows from the fact that the restriction of any congruence of $A \sqcup B$ to A (or B) is the congruence of A (or B). For the second inequality, if $l_c(A) = \infty$ or $l_c(B) = \infty$, then there is nothing to prove. Let $l_c(A) = n$ and $l_c(B) = m$. Then there are saturated chains $\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \nabla_A$ and $\Delta_B \sqsubset \theta_1 \sqsubset \theta_2 \sqsubset \cdots \sqsubset \theta_m = \nabla_B$ of congruences for A and B, respectively. Then, using (i), $\Delta_{A \sqcup B} = \Delta_A \sqcup \Delta_B \sqsubset \rho_1 \sqcup \Delta_B \sqsubset \rho_1 \sqcup \theta_1 \sqsubset \rho_2 \sqcup \theta_1 \sqsubset \cdots \sqsubset \rho_n \sqcup \theta_1 = \nabla_A \sqcup \theta_1 \sqsubset \nabla_A \sqcup \theta_2 \sqsubset \cdots \sqsubset \nabla_A \sqcup \theta_m = \nabla_A \sqcup \nabla_B \sqsubset \nabla_{A \sqcup B}$. Hence, $l_c(A \sqcup B) \le n + m + 1 = l_c(A) + l_c(B) + 1$. □

The following example shows that the above inequality can be proper or sharp.

Example 3.3. (i) *Let* $A = \{a, b\}$ *and* $B = \{c\}$ *be* S*-acts with trivial actions. Then* $\Delta_A \sqsubset \nabla_A$, $\Delta_B = \nabla_B$ *and* $\Delta_{A \sqcup B} \sqsubset \Delta_{A\sqcup B} \cup \{(a, b), (b, a)\} \sqsubset \nabla_{A\sqcup B}$. *So* $l_c(A) = 1$, $l_c(B) = 0$ *and* $l_c(A \sqcup B) = 2$.

(ii) *Consider the* S*-act given in Example* **3**.*1. Then* $A = B_1 \sqcup B_2$ *in which* $B_1 = \{a, b\}$ *and* $B_2 = \{c, d\}$. *Clearly,* $l_c(B_1) = l_c(B_2) = 1$ *and hence* $l_c(A) = 2 < l_c(B_1) + l_c(B_2) + 1$.

Corollary 3.3. (i) *Any* S*-acts* A *and* B *are of finite lengths if and only if so is* $A \sqcup B$.

(ii) *An* S*-act* A *has finite length if and only if the length of* A^θ *is finite.*

Proof. (i) We get the result by appling Lemma **3**.4(ii).

(ii) This is a direct consequence of part (i). □

Corollary 3.4. *Let* B *be a subact of an* S*-act* A*. Then* A *has finite length if and only if* B *and* A/B *have also finite length.*

Proof. Follows from Corollary **3**.3 and the fact that $B \sqcup (A/B) \cong A^\theta$. □

Proposition 3.3. *Let* S *have finite length as an* S*-act. Then an* S*-act* A *is finitely generated if and only if it has finite length.*

Proof. Let A be generated by $\{x_1, x_2, \ldots, x_n\}$. Thus A is a homomorphic image of $\coprod_{i=1}^n S$. Using the assumption and Corollary **3**.3, $\coprod_{i=1}^n S$ has finite length so that A has also finite length. For the converse, let $\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \nabla_A$ be a saturated chain of congruences of A. By Note after Definition **2**.1, each cover is a principal extension. So there is a set $H = \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\} \subseteq \nabla_A$ which $\nabla_A = \rho(H)$. Now for each $a \in A$, choose an element $a \neq b_a \in A$. Since $(a, b_a) \in \nabla_A$, there is a $p \in \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and $s \in S^1$ such that $a = ps$. Hence, $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ is a generating subset of A. □

Let B be a subact of A. Every $\rho \in \mathrm{Con}(B)$ can be extended to a congruence $\overline{\rho} = \rho \cup \Delta_{A\setminus B} \in \mathrm{Con}(A)$. Thus there is a one to one correspondence between the sets $\mathrm{Con}(B)$ and $\{\theta \in \mathrm{Con}(A) \mid \theta \subseteq \rho_B\}$.

Theorem 3.2. *Let* $\rho \in \mathrm{Con}(A)$ *and* B *be a subact of* A*. Then the following assertions hold:*

(i) $l_c(A) \leq l(\rho) + l_c(A/\rho)$.

(ii) $l_c(A) \leq l_c(B) + l_c(A/B)$.

Proof. If $l_c(A/\rho) = \infty$ or $l_c(A) = \infty$, so we are done. Let $l(\rho) = n$ and $l_c(A/\rho) = m$. Then there are saturated chains $\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \rho$ and $\Delta_{A/\rho} \sqsubset \theta_1/\rho \sqsubset \theta_2/\rho \sqsubset \cdots \sqsubset \theta_m/\rho = \nabla_{A/\rho}$ of congruences for ρ and A/ρ, respectively. So A has the saturated chain $\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \rho \sqsubset \theta_1 \sqsubset \theta_2 \sqsubset \cdots \sqsubset \theta_m = \nabla_A$ of congruences. Thus $l_c(A) \leq n + m = l(\rho) + l_c(A/\rho)$.

(ii) It is not difficult to check that $l_c(B) = l(\rho_B)$ by using the fact that there is a one to one correspondence between the sets $\mathrm{Con}(B)$ and $\{\theta \in \mathrm{Con}(A) \mid \theta \subseteq \rho_B\}$. So the assertion follows from (i). □

The following example shows that the inequality in the previous theorem can be sharp or proper.

Example 3.4. (i) *Let* S *be an arbitrary semigroup. Consider the* S*-act* $A = \{a_0, a_1, a_2, a_3\}$ *with trivial action and the subact* $B = \{a_0, a_1\}$ *of* A*. It is not difficult to check that* $l_c(A) = 3$, $l_c(B) = 1$ *and* $l_c(A/B) = 2$.

(ii) *In Example* **3**.*1, it has been shown that* $l_c(A) = 2$ *and* $l_c(B) = 1$*. Also, clearly the* S*-act* A/B *is isomorphic to the* S*-act* $D = \{a_0, c, d\}$ *in which* a_0 *is a fixed element,* $cx = d$ *and* $dx = c$ *and* $\Delta_D \subset \rho_{\{c,d\}} \subset \nabla_D$. *Thus* $l_c(D) = 2$.

Proposition 3.4. *Let A be an S-act and $\Delta_A \sqsubset \rho_1 \sqsubset \rho_2 \sqsubset \cdots \sqsubset \rho_n = \nabla_A$ be a saturated chain of congruences of length $n \geq 1$ for A. Then the following assertions hold:*

(i) *Each $\rho_m, 1 \leq m \leq n$, is finitely generated by m elements.*

(ii) *If $l_c(\rho) = k$ where $\rho \in \mathrm{Con}(A)$, then ρ is generated by k elements.*

(iii) *A is finitely generated with $2n$ generators.*

Proof. (i) For each $1 \leq i \leq m$, choose and fix $(a_i, b_i) \in \rho_i \setminus \rho_{i-1}$. So $\rho_i = \langle \rho_{i-1} \cup (a_i, b_i)\rangle$. Consider $H = \{(a_i, b_i) \mid 1 \leq i \leq m\}$. Then $H \subseteq \rho_m$ and hence $\rho(H) \subseteq \rho_m$. On the other hand, $\rho_1 \subseteq \rho(H), \rho_2 = \langle \rho_1 \cup \{(a_2, b_2)\}\rangle \subseteq \rho(H), \rho_3 = \langle \rho_2 \cup \{(a_3, b_3)\}\rangle \subseteq \rho(H), \ldots, \rho_m = \langle \rho_{m-1} \cup \{(a_m, b_m)\}\rangle \subseteq \rho(H)$. So $\rho_m \subseteq \rho(H)$, which means that ρ_m is finitely generated with m generators.

(ii) We are done by Lemma **3**.1 and a similar argument as in part (i).

(iii) By part (i), ∇_A is generated by the set $H = \{(a_1, b_1), \ldots, (a_n, b_n)\}$. So, for any two distinct elements $x, y \in A$, we have

$$x = p_1 s_1, q_1 s_1 = p_2 s_2, \ldots, q_m s_m = y.$$

Then each $x \in A$ is of the form $x = ps$ where $p \in \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. Thus A is generated by $2n$ elements. □

Theorem 3.3. *Let A be an S-act. If each subact of $A \times A$ is cyclic, then the following assertions hold:*

(i) *A is strongly noetherian.*

(ii) *The lattice* $\mathrm{Con}(A)$ *forms a chain.*

Proof. (i) Let $\theta_1 \subseteq \theta_2 \subseteq \cdots$ be an ascending chain in $\mathrm{Con}(A)$. Consider $\theta = \bigcup_{i=1}^{\infty} \theta_i$. Then there exist $a, b \in A$ such that $\theta = \rho(a, b)$. So there is $N \in \mathbb{N}$ such that $(a, b) \in \theta_N$, which implies $\theta = \theta_N$ and hence A is strongly noetherian.

(ii) It follows from the assumption that each congruence on A is monogenic. Let $\rho(a_1, b_1)$ and $\rho(a_2, b_2)$ be two congruences on A. Since $\rho(a_1, b_1)$ and $\rho(a_2, b_2)$ are subacts of $A \times A$, there exists $(a, b) \in A \times A$ such that $\rho(a_1, b_1) \cup \rho(a_2, b_2) = (a, b)S$. Thus $(a, b) \in \rho(a_1, b_1)$, say. Then $\rho(a_2, b_2) \subseteq (a, b)S \subseteq \rho(a, b) \subseteq \rho(a_1, b_1)$, which means that $\mathrm{Con}(A)$ is a chain. □

The following is a straightforward implication of Theorem **3**.3, which gives a condition for establishing the converse of Remark **3**.1.

Corollary 3.5. *If each subact of $A \times A$ is cyclic, then the following are equivalent:*

(i) *A has finite length.*

(ii) $\mathrm{Con}(A)$ *is a finite chain.*

(iii) *A is strongly noetherian as well as strongly artinian.*

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