AN ITERATIVE APPROACH FOR THE SOLUTION OF THE VARIATIONAL INCLUSION AND THE SPLIT FIXED POINT PROBLEM

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In this paper, we study the variational inclusion and the split fixed point problem in Hilbert spaces. We propose an iterative algorithm by using the resolvent operator technique for solving the variational inclusion and the split fixed point problem involved in two pseudocontractive operators. Strong convergence analysis of the suggested algorithm is demonstrated.

Keywords: Variational inclusion, split fixed point, monotone operator, pseudocontractive operator.

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1. Introduction

In this paper, we consider the following variational inclusion (shortly, VI) which consists in finding a point $u^\dagger \in H_1$ with the property

$$0 \in f(u^\dagger) + g(u^\dagger), \tag{1}$$

where $f : H_1 \to 2^{H_1}$ is a multi-valued operator of a real Hilbert space H_1 and $g : H_1 \to H_1$ is a single-valued operator. The solution set of the VI is denoted by Ω_1.

The VI includes several problems as special cases.

SI (i). For the case when $H_1 = R^n$, the VI reduces to the generalized equation investigated by Robinson [26].

SI (ii). If the operator g is identically null, then the VI reduces to the inclusion problem investigated by Rockafellar [27].

SI (iii). With the choice $f = \partial h$, the subdifferential of a proper and lower semicontinuous convex function h, the VI reduces to the following mixed quasi-variational inequality ([13, 23]) of finding $u^\dagger \in H_1$ such that

$$\langle g(u^\dagger), u - u^\dagger\rangle + h(u) - h(u^\dagger) \geq 0,\ \forall u \in H_1.$$

SI (iv). Setting $f = \partial(\delta_C)$, the subdifferential of the indicator function δ_C of a nonempty closed convex subset C, the VI reduces to the following variational inequality ([29]) of finding $u^\dagger \in C$ such that

$$\langle g(u^\dagger), u - u^\dagger\rangle \geq 0,\ \forall u \in C. \tag{2}$$

Variational inequality and variational inclusion problems have attracted so much attention, see, e.g., [1, 5, 6, 45, 46, 51]. The VI will continue to be one of the central problems in nonlinear analysis and optimization ([15, 28, 34, 47, 48, 53, 56]). We begin by recalling some notations as well as summarizing existing algorithms for solving the VI. Let H_1 be a

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real Hilbert space with corresponding inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A multi-valued operator $f : H_1 \to 2^{H_1}$ with domain $\mathrm{D}(f) := \{x \in H_1 : f(x) \neq \emptyset\}$ is called monotone if $\langle x-y, u-v\rangle$ for all $u \in f(x)$ and $v \in f(y)$. A monotone operator f is maximal if its graph, i.e., the set $\mathrm{Graph}(f) := \{(x,u) : x \in \mathrm{D}(f), u \in f(x)\}$ is not properly contained in the graph of any other monotone operator. It is well known that a monotone operator f is maximal if and only if for $(x,u) \in H_1 \times H_1, \langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in \mathrm{Graph}(f)$ implies that $u \in f(x)$. A single-valued operator $g : H_1 \to H_1$ is said to be ϖ-inverse strongly monotone if for all $x, x^\dagger \in H_1$, $\omega\|\varphi(x)-\varphi(x^\dagger)\|^2 \leq \langle\varphi(x)-\varphi(x^\dagger), x-x^\dagger\rangle$, where ω is a positive constant.

There are many iterative algorithms for solving the VI. Among them, a common method is to use resolvent operator technique introduced in [27]. Especially, Zhang, Lee and Chan [55] introduced an iterative scheme for finding a common element of the set of solutions to the VI and the set of fixed points of nonexpansive mappings in Hilbert spaces. Kheawborisut and Kangtunyakarn [18] proposed a modified subgradient extragradient method for solving a system of variational inclusion problem and finite family of variational inequalities problem. Zhang, Dong and Chen [54] presented a multi-step inertial proximal contraction algorithm for solving monotone variational inclusion problem. Cholamjiak, Suantai and Sunthrayuth [12] introduced an explicit parallel algorithm for solving variational inclusion problem and fixed point problem. Kazmi et. al. [17] investigated a hybrid iterative algorithm for solving monotone variational inclusion and hierarchical fixed point problems.

Recently, the split fixed point problem has been investigated extensively due to it is an extension of the split feasibility problem ([7]). In this paper, we are interested in the split fixed point problem for the class of pseudocontractive operators. This more general class, which properly includes the classes of nonexpansive operators, directed operators and demicontractive operators, is more desirable in fixed point methods [31, 32, 33, 38, 58]. To begin with, let us recall that the split feasibility problem (shortly, SFP) is to find a point $x^\dagger$ with the property

$$x^\dagger \in C \quad \text{such that} \quad Ax^\dagger \in Q, \tag{3}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator.

The SFP arises from phase retrievals and in medical image reconstruction ([3, 4, 8]) and has been studied extensively, see [35, 37, 49]. In the case where C and Q in the SFP are the fixed point sets of nonlinear operators, the SFP was named the split fixed point problem in [9]. More precisely, the split fixed point problem (shortly, SFPP) is to find an element $x^\dagger \in H$ satisfying

$$x^\dagger \in Fix(\psi) \quad \text{and} \quad Ax^\dagger \in Fix(\varphi), \tag{4}$$

where $Fix(\psi)$ and $Fix(\varphi)$ denote the fixed point sets of two nonlinear operators $\psi : H_1 \to H_1$ and $\varphi : H_2 \to H_2$. The solution set of the SFPP is denoted by Ω_2.

In order to solve the SFPP, Censor and Segal ([9]) suggested the following iterative algorithm: for given $x_0 \in H_1$, let the sequence $\{x_n\}$ defined by

$$x_{n+1} = \psi(x_n - \tau A^*(I-\varphi)Ax_n), \tag{5}$$

where ψ and φ are two directed operators.

Iterative algorithms for solving the split problems have been further studied and developed ([39]-[44]). Moudafi ([22]) extended the SFPP to demicontractive operators. Cholamjiak and Shehu ([11]) investigated the SFPP regarding an asymptotically nonexpansive semigroup and a total asymptotically strict pseudocontractve operator. Liu, Chen and Liu ([19]) considered the SFPP for strict quasi-phi-pseudocontractive operators in Banach spaces. Reich and Tuyen ([25]) suggested two projection algorithms for solving the SFPP. Taiwo, Alakoya and Mewomo ([30]) proposed a Halpern-type iterative process for solving the

SFPP and monotone variational inclusion problem between Banach spaces. Some related results, please refer to [10, 14, 52].

It is our main purpose of this paper that we devote to investigate the iterative methods for solving the VI and the SFPP in Hilbert spaces. We propose an iterative algorithm for finding a common solution of the VI and the SFPP with the help of the resolvent operator technique. We show that the suggested algorithm has strong convergence.

2. **Preliminaries**

Let H be a real Hilbert space. In what follows, we shall write $\to$ and $\rightharpoonup$ to denote respectively, the strong norm convergence and the weak convergence on H.

Definition 2.1. *An operator $S : H \to H$ is said to be*

(i) *L-Lipschitz if there exists a constant $L \geq 0$ such that*

$$\|S(x) - S(y)\| \leq L\|x - y\|, \forall x, y \in H.$$

If $L = 1$, then S is called nonexpansive. If $L < 1$, then S is called L-contractive.

(ii) *Directed if*

$$\langle S(x) - x^\dagger, S(x) - x\rangle \leq 0$$

for all $x \in H$ and $x^\dagger \in Fix(S)$;

(iii) *Demicontractive if*

$$\|S(x) - x^\dagger\|^2 \leq \|x - x^\dagger\|^2 + k\|S(x) - x\|^2$$

for all $x \in H$ and $x^\dagger \in Fix(S)$, where $k \in [0, 1)$.

(iv) *Strictly pseudocontractive if*

$$\|S(x) - S(y)\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2$$

for all $x, y \in H$, where $k \in [0, 1)$.

Definition 2.2. *An operator $S : H \to H$ is said to be pseudocontractive if $\forall x, y \in H$,*

$$\langle S(x) - S(y), x - y\rangle \leq \|x - y\|^2,$$

which equals

$$\|S(x) - S(y)\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2.$$

It is obvious that the pseudocontractive operator S with $Fix(S) \neq \emptyset$ includes the nonexpansive operator, the directed operator, the demicontractive operator and the strictly pseudocontractive operator as special cases ([2]).

Definition 2.3. *Let $C \subset H$ be a nonempty closed convex set. The orthogonal projection $proj_C : H \to C$ is defined by*

$$proj_C(x^\dagger) := \arg\min_{x\in C}\{\|x - x^\dagger\|\}, x^\dagger \in H.$$

It is obvious that $proj_C$ is the nearest point projection from H onto C and satisfies the characteristic inequality ([50]*)*

$$\langle x^\dagger - proj_C(x^\dagger), x - proj_C(x^\dagger)\rangle \leq 0,\ \forall x^\dagger \in H, x \in C.$$

Let a multi-valued operator $\psi : H \to 2^H$ be maximal monotone. Let $\zeta >$ be a constant. Define the resolvent operator $(I + \zeta\psi)^{-1}$, which is single-valued and firmly nonexpansive, i.e.,

$$\|(I + \zeta\psi)^{-1}(x) - (I + \zeta\psi)^{-1}(x^\dagger)\|^2 \leq \langle (I + \zeta\psi)^{-1}(x) - (I + \zeta\psi)^{-1}(x^\dagger), x - x^\dagger\rangle$$

for all $x, x^\dagger \in H$.

For all $x, y \in H$, we have

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ t \in \mathbb{R}, \tag{6}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{7}$$

and

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle. \tag{8}$$

Lemma 2.1 ([24])**.** *Let H be a real Hilbert space. Let $f : H \to 2^H$ be a maximal monotone operator and $g : H \to H$ be an ω-inverse strongly monotone operator. Let $\sigma > 0$ be a constant. Then, we have*

(i) *If $\sigma \in (0, 2\omega)$, then $(I + \sigma f)^{-1}(I - \sigma g)$ is an averaged operator.*

(ii) *$0 \in f(x^\dagger) + g(x^\dagger)$ if and only if $x^\dagger \in \mathrm{Fix}((I + \sigma f)^{-1}(I - \sigma g))$.*

Lemma 2.2 ([42])**.** *Let H be a real Hilbert space. Let $S : H \to H$ be an L-Lipschitz pseudocontractive operator with $Fix(S) \ne \emptyset$. Then, for all $x \in H$ and $x^\dagger \in Fix(S)$, we have*

$$\|S((1-\eta)x + \eta S(x)) - x^\dagger\|^2 \le \|x - x^\dagger\|^2 + (1-\eta)\|x - S((1-\eta)x + \eta S(x))\|^2,$$

where η is a constant in $(0, \frac{1}{\sqrt{1+L^2}+1})$.

Lemma 2.3 ([16])**.** *Let H be a real Hilbert space. Let $S : H \to H$ be a nonexpansive operator. Then $I - S$ is demi-closed at zero.*

Lemma 2.4 ([57])**.** *Let H be a real Hilbert space. Let $S : H \to H$ be a continuous pseudo-contractive operator. Then S is demi-closed.*

Lemma 2.5 ([36])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \mu_n, n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\mu_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\mu_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\mu_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([20])**.** *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \le w_{n_k+1}$ for all $k \ge 0$. For every $n \ge N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \le n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge N_0$

$$\max\{w_{\tau(n)}, w_n\} \le w_{\tau(n)+1}.$$

3. **Main results**

In this section, we present our main results. Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to 2^{H_1}$ be a maximal monotone operator and $g : H_1 \to H_1$ be an ω-inverse strongly monotone operator. Let $A : H_1 \to H_2$ be a bounded linear operator and $B : H_1 \to H_1$ be a ϖ-strongly positive bounded linear operator. Let $\phi : H_1 \to H_1$ be a κ-contractive operator. Let $\varphi : H_2 \to H_2$ be an L_1-Lipschitz pseudocontractive operator and $\psi : H_1 \to H_1$ be an L_2-Lipschitz pseudocontractive operator.

Now, we present our iterative procedure for solving the split fixed point problem and the variational inclusion problem. First, we given several iterative parameters. Let $\{\lambda_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\eta_n\}$ be five real number sequences in $(0,1)$. Let σ and μ be two positive constants.

Algorithm 3.1. *For any given initial point $x_0 \in H_1$, define an iterative sequence $\{x_n\}$ by the following form*

$$\begin{cases} y_n = (I+\sigma f)^{-1}(x_n - \sigma g(x_n)), & (9)\\ w_n = (1-\eta_n)Ay_n + \eta_n \varphi(Ay_n), & (10)\\ z_n = (1-\alpha_n)Ay_n + \alpha_n \varphi(w_n), & (11)\\ v_n = y_n - \mu A^*(Ay_n - z_n), & (12)\\ u_n = \lambda_n \phi(x_n) + (I - \lambda_n B)v_n, & (13)\\ r_n = (1-\gamma_n)u_n + \gamma_n \psi(u_n), & (14)\\ x_{n+1} = (1-\beta_n)u_n + \beta_n \psi(r_n), n \geq 0. & (15) \end{cases}$$

Theorem 3.1. *Suppose that $\Gamma := \Omega_1 \cap \Omega_2 \neq \emptyset$. Assume that the following conditions are satisfied:*
(i) $L_1 > 1$, $L_2 > 1$, $\varpi > \kappa$, $\sigma \in (0, 2\omega)$ *and* $\mu \in (0, \frac{1}{\|A\|^2})$;
(ii) $0 < a_1 < \alpha_n < a_2 < \eta_n < a_3 < \frac{1}{\sqrt{1+L_1^2}+1}$;
(iii) $0 < b_1 < \beta_n < b_2 < \gamma_n < b_3 < \frac{1}{\sqrt{1+L_2^2}+1}$;
(iv) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$.
Then the sequence $\{x_n\}$ defined by (15) *converges strongly to $q^\dagger = proj_\Gamma(I+\phi-B)(q^\dagger)$.*

Proof. Owing to the operator $proj_\Gamma(I+\phi-B)$ ([21]) is contractive, denote its unique fixed point by $q^\dagger$, i.e., $q^\dagger = proj_\Gamma(I+\phi-B)(q^\dagger)$. By Lemma 2.1, the operator $(I+\sigma f)^{-1}(I-\sigma g)$ is averaged. Thus, $(I+\sigma f)^{-1}(I-\sigma g)$ can be written as $(I+\sigma f)^{-1}(I-\sigma g) = (1-\varsigma)I + \varsigma S$ in which $\varsigma \in (0,1)$ is a constant and $S : H_1 \to H_1$ is a nonexpansive operator. From (9), we have

$$y_n = (I+\sigma f)^{-1}(x_n - \sigma g(x_n)) = (1-\varsigma)x_n + \varsigma S(x_n). \tag{16}$$

It follows from (6) and (16) that

$$\begin{aligned} \|y_n - q^\dagger\|^2 &= \|\varsigma(x_n - q^\dagger) + (1-\varsigma)(S(x_n) - q^\dagger)\|^2 \\ &= (1-\varsigma)\|x_n - q^\dagger\|^2 + \varsigma\|S(x_n) - q^\dagger\|^2 - \varsigma(1-\varsigma)\|x_n - S(x_n)\|^2 \\ &\leq \|x_n - q^\dagger\|^2 - \frac{1-\varsigma}{\varsigma}\|x_n - y_n\|^2 \\ &\leq \|x_n - q^\dagger\|^2. \end{aligned} \tag{17}$$

Using Lemma 2.2, we deduce

$$\|\varphi(w_n) - Aq^\dagger\|^2 \leq \|Ay_n - Aq^\dagger\|^2 + (1-\eta_n)\|\varphi(w_n) - Ay_n\|^2, \tag{18}$$

and

$$\|\psi(r_n) - q^\dagger\|^2 \leq \|u_n - q^\dagger\|^2 + (1-\gamma_n)\|u_n - \psi(r_n)\|^2. \tag{19}$$

By (6) and (11), we obtain

$$\begin{aligned} \|z_n - Aq^\dagger\|^2 &= \|(1-\alpha_n)(Ay_n - Aq^\dagger) + \alpha_n(\varphi(w_n) - Aq^\dagger)\|^2 \\ &= (1-\alpha_n)\|Ay_n - Aq^\dagger\|^2 + \alpha_n\|\varphi(w_n) - Aq^\dagger\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|\varphi(w_n) - Ay_n\|^2. \end{aligned} \tag{20}$$

Combining (18) and (20), we obtain

$$\begin{aligned} \|z_n - Aq^\dagger\|^2 &\leq \|Ay_n - Aq^\dagger\|^2 - \alpha_n(\eta_n - \alpha_n)\|\varphi(w_n) - Ay_n\|^2 \\ &\leq \|Ay_n - Aq^\dagger\|^2. \end{aligned} \tag{21}$$

From (6), (15) and (19), we have

$$\begin{aligned} \|x_{n+1} - q^\dagger\|^2 &= \|(1-\beta_n)(u_n - q^\dagger) + \beta_n(\psi(r_n) - q^\dagger)\|^2 \\ &= (1-\beta_n)\|u_n - q^\dagger\|^2 + \beta_n\|\psi(r_n) - q^\dagger\|^2 - \beta_n(1-\beta_n)\|u_n - \psi(r_n)\|^2 \\ &\le \|u_n - q^\dagger\|^2 - \beta_n(\gamma_n - \beta_n)\|u_n - \psi(r_n)\|^2 \\ &\le \|u_n - q^\dagger\|^2. \end{aligned} \tag{22}$$

Since the linear operator B is ϖ-strongly positive, $\|I - \lambda_n B\| \le (1 - \lambda_n\varpi)$. By virtue of (13), we get

$$\begin{aligned} \|u_n - q^\dagger\| &= \|\lambda_n(\phi(x_n) - Bq^\dagger) + (I - \lambda_n B)(v_n - q^\dagger)\| \\ &\le \lambda_n\|\phi(x_n) - Bq^\dagger\| + \|I - \lambda_n B\|\|v_n - q^\dagger\| \\ &\le \lambda_n\|\phi(x_n) - \phi(q^\dagger)\| + \lambda_n\|\phi(q^\dagger) - Bq^\dagger\| + (1 - \lambda_n\varpi)\|v_n - q^\dagger\| \\ &\le \lambda_n\kappa\|x_n - q^\dagger\| + \lambda_n\|\phi(q^\dagger) - Bq^\dagger\| + (1 - \lambda_n\varpi)\|v_n - q^\dagger\|. \end{aligned} \tag{23}$$

According to (12), we derive

$$\begin{aligned} \|v_n - q^\dagger\| &= \|y_n - q^\dagger - \mu A^*(Ay_n - z_n)\|^2 \\ &= \|y_n - q^\dagger\|^2 + \mu^2\|A^*(Ay_n - z_n)\|^2 - 2\mu\langle y_n - q^\dagger, A^*(Ay_n - z_n)\rangle. \end{aligned} \tag{24}$$

Note that

$$\begin{aligned} \langle y_n - q^\dagger, A^*(Ay_n - z_n)\rangle &= \langle Ay_n - Aq^\dagger, Ay_n - z_n\rangle \\ &= \|Ay_n - z_n\|^2 + \langle z_n - Aq^\dagger, Ay_n - z_n\rangle. \end{aligned} \tag{25}$$

Applying (7), we obtain

$$\langle z_n - Aq^\dagger, Ay_n - z_n\rangle = \frac{1}{2}(\|Ay_n - Aq^\dagger\|^2 - \|z_n - Aq^\dagger\|^2 - \|Ay_n - z_n\|^2). \tag{26}$$

Combining (21), (25) and (26), we deduce

$$\begin{aligned} \langle y_n - q^\dagger, A^*(Ay_n - z_n)\rangle &= \frac{1}{2}(\|Ay_n - Aq^\dagger\|^2 + \|Ay_n - z_n\|^2 - \|z_n - Aq^\dagger\|^2) \\ &\ge \frac{1}{2}(\|Ay_n - Aq^\dagger\|^2 + \|Ay_n - z_n\|^2 - \|Ay_n - Aq^\dagger\|^2) \\ &= \frac{1}{2}\|Ay_n - z_n\|^2. \end{aligned}$$

This together with (17) and (24) implies that

$$\begin{aligned} \|v_n - q^\dagger\|^2 &\le \|y_n - q^\dagger\|^2 + \mu^2\|A^*(Ay_n - z_n)\|^2 - \mu\|Ay_n - z_n\|^2 \\ &\le \|y_n - q^\dagger\|^2 - \mu(1 - \mu\|A\|^2)\|Ay_n - z_n\|^2 \\ &\le \|y_n - q^\dagger\|^2 \\ &\le \|x_n - q^\dagger\|^2. \end{aligned} \tag{27}$$

Substituting (27) into (23) to deduce

$$\|u_n - q^\dagger\| \le \lambda_n\|\phi(q^\dagger) - Bq^\dagger\| + [1 - (\varpi - \kappa)\lambda_n]\|x_n - q^\dagger\|. \tag{28}$$

In the light of (22) and (28), we get

$$\begin{aligned} \|x_{n+1} - q^\dagger\| &\le \lambda_n\|\phi(q^\dagger) - Bq^\dagger\| + [1 - (\varpi - \kappa)\lambda_n]\|x_n - q^\dagger\| \\ &\le \max\{\|x_n - q^\dagger\|, \frac{\|\phi(q^\dagger) - Bq^\dagger\|}{\varpi - \kappa}\}, \end{aligned}$$

which implies that $\|x_n - q^\dagger\| \le \max\{\|x_0 - q^\dagger\|, \frac{\|\phi(q^\dagger)-Bq^\dagger\|}{\varpi-\kappa}\}$ by induction. Therefore, the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded.

Next, we consider two possibilities. Possibility 1. There exists some n_0 such that the sequence $\{\|x_n - q^\dagger\|\}$ is decreasing when $n \ge n_0$. Possibility 2. For any n_0, there exists at least an integer $m \ge n_0$ such that $\|x_m - q^\dagger\| \le \|x_{m+1} - q^\dagger\|$.

For Possibility 1, we deduce that $\lim_{n\to\infty} \|x_n - q^\dagger\|$ exists. Based on (22) and (23), we obtain

$$\begin{aligned}\|x_{n+1} - q^\dagger\|^2 &\le \big(\lambda_n(\|x_n - q^\dagger\| + \|\phi(q^\dagger) - Bq^\dagger\|) + (1-\lambda_n\varpi)\|v_n - q^\dagger\|\big)^2 \\ &\le \lambda_n(\|x_n - q^\dagger\| + \|\phi(q^\dagger) - Bq^\dagger\|)^2/\varpi + (1-\lambda_n\varpi)\|v_n - q^\dagger\|^2.\end{aligned} \tag{29}$$

Moreover, it follows from (17) and (27) that

$$\|v_n - q^\dagger\|^2 \le \|x_n - q^\dagger\|^2 - \frac{1-\varsigma}{\varsigma}\|x_n - y_n\|^2 - \mu(1-\mu\|A\|^2)\|Ay_n - z_n\|^2. \tag{30}$$

Substituting (30) into (29), we have

$$\begin{aligned}\|x_{n+1} - q^\dagger\|^2 &\le (1-\lambda_n\varpi)\|x_n - q^\dagger\|^2 - \frac{\varsigma(1-\lambda_n\varpi)}{1-\varsigma}\|x_n - y_n\|^2 \\ &\quad - (1-\lambda_n\varpi)(\mu - \mu^2\|A\|^2)\|Ay_n - z_n\|^2 + M\lambda_n \\ &\le M\lambda_n + \|x_n - q^\dagger\|^2,\end{aligned} \tag{31}$$

where $M > 0$ is a constant such that $\sup_n\{(\|x_n - q^\dagger\| + \|\phi(q^\dagger) - Bq^\dagger\|)^2/\varpi + \|x_n - q^\dagger\|^2\} \le M$.

Hence,

$$\begin{aligned}(1-\lambda_n\varpi)(\mu - \mu^2\|A\|^2)\|Ay_n - z_n\|^2 &+ \frac{\varsigma(1-\lambda_n\varpi)}{1-\varsigma}\|x_n - y_n\|^2 \\ &\le (1-\lambda_n\varpi)\|x_n - q^\dagger\|^2 - \|x_{n+1} - q^\dagger\|^2 + M\lambda_n \\ &\to 0.\end{aligned}$$

It results in that

$$\lim_{n\to\infty} \|Ay_n - z_n\| = 0, \tag{32}$$

and

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - (I+\sigma f)^{-1}(I-\sigma g)x_n\| = 0. \tag{33}$$

By (11), we have $z_n - Ay_n = \alpha_n(\varphi(w_n) - Ay_n)$. This together with (32) implies that

$$\lim_{n\to\infty} \|Ay_n - \varphi(w_n)\| = 0. \tag{34}$$

Since φ is L_1-Lipschitz, from (10), we conclude

$$\begin{aligned}\|Ay_n - \varphi(Ay_n)\| &\le \|Ay_n - \varphi(w_n)\| + \|\varphi(w_n) - \varphi(Ay_n)\| \\ &\le \|Ay_n - \varphi(w_n)\| + L_1\eta_n\|Ay_n - \varphi(Ay_n)\|.\end{aligned}$$

It yields that

$$\|Ay_n - \varphi(Ay_n)\| \le \frac{1}{1 - L_1\eta_n}\|Ay_n - \varphi(w_n)\|,$$

which together with (34) implies that

$$\lim_{n\to\infty} \|Ay_n - \varphi(Ay_n)\| = 0. \tag{35}$$

Taking into account (12) and (13), we have

$$\begin{aligned}\|u_n - y_n\| &= \|\mu A^*(z_n - Ay_n) + \lambda_n(\phi(x_n) - Bv_n)\| \\ &\le \mu\|A\|\|z_n - Ay_n\| + \lambda_n\|\phi(x_n) - Bv_n\|.\end{aligned}$$

This together with (32) implies that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{36}$$

On account of (22) and (28), we deduce

$$\|x_{n+1} - q^\dagger\|^2 \leq \|x_n - q^\dagger\|^2 + \lambda_n M - \beta_n(\gamma_n - \beta_n)\|u_n - \psi(r_n)\|^2.$$

It leads to that

$$\begin{aligned}\beta_n(\gamma_n - \beta_n)\|u_n - \psi(r_n)\|^2 &\leq \|x_n - q^\dagger\|^2 - \|x_{n+1} - q^\dagger\|^2 + \lambda_n M \\ &\to 0.\end{aligned}$$

So,

$$\lim_{n\to\infty} \|u_n - \psi(r_n)\| = 0. \tag{37}$$

Owing to ψ is L_2-Lipschitz, by (14), we get

$$\begin{aligned}\|u_n - \psi(u_n)\| &\leq \|u_n - \psi(r_n)\| + \|\psi(r_n) - \psi(u_n)\| \\ &\leq \|u_n - \psi(r_n)\| + L_2\gamma_n\|u_n - \psi(u_n)\|.\end{aligned}$$

It follows that

$$\|u_n - \psi(u_n)\| \leq \frac{1}{1 - L_2\gamma_n}\|u_n - \psi(r_n)\|.$$

With the help of (37), we have

$$\lim_{n\to\infty} \|u_n - \psi(u_n)\| = 0. \tag{38}$$

Now, we show that

$$\limsup_{n\to\infty}\langle\phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle \leq 0.$$

Thanks to the boundedness of the sequence of $\{u_n\}$, we can choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup z$ and

$$\limsup_{n\to\infty}\langle\phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle = \lim_{i\to\infty}\langle\phi(q^\dagger) - Bq^\dagger, u_{n_i} - q^\dagger\rangle \tag{39}$$

From (33) and (36), we deduce $\|x_n - u_n\| \to 0$. Then,

$$x_{n_i} \rightharpoonup z \text{ and } y_{n_i} \rightharpoonup z. \tag{40}$$

Since the operator $(I + \sigma f)^{-1}(I - \sigma g)$ is averaged, it is also nonexpansive. Using Lemma 2.3 and (40), we conclude that $z \in Fix((I + \sigma f)^{-1}(I - \sigma g)) = \Omega_1$. By Lemma 2.4 and (38), we have $z \in Fix(\psi)$. Noting that $Ay_{n_i} \rightharpoonup Az$, by Lemma 2.4 and (35), we conclude that $Az \in Fix(\varphi)$. Thus, $z \in \Omega_1 \cap \Omega_2 = \Gamma$. Thanks to $q^\dagger = proj_\Gamma(I + \phi - B)(q^\dagger)$, we deduce

$$\begin{aligned}\limsup_{n\to\infty}\langle\phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle &= \lim_{i\to\infty}\langle\phi(q^\dagger) - Bq^\dagger, u_{n_i} - q^\dagger\rangle \\ &= \lim_{i\to\infty}\langle\phi(q^\dagger) - Bq^\dagger, z - q^\dagger\rangle \\ &\leq 0.\end{aligned} \tag{41}$$

Applying (8) to (13) to get

$$\begin{aligned}\|u_n - q^\dagger\|^2 &= \|(I-\lambda_n B)(v_n - q^\dagger) + \lambda_n(\phi(x_n) - Bq^\dagger)\|^2\\ &\le (1-\varpi\lambda_n)^2\|v_n - q^\dagger\|^2 + 2\lambda_n\langle \phi(x_n) - Bq^\dagger, u_n - q^\dagger\rangle\\ &\le (1-\varpi\lambda_n)^2\|x_n - q^\dagger\|^2 + 2\lambda_n\langle \phi(x_n) - \phi(q^\dagger), u_n - q^\dagger\rangle\\ &\quad + 2\lambda_n\langle \phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle\\ &\le (1-\varpi\lambda_n)^2\|x_n - q^\dagger\|^2 + 2\kappa\lambda_n\|x_n - q^\dagger\|\|u_n - q^\dagger\|\\ &\quad + 2\lambda_n\langle \phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle\\ &\le (1-\varpi\lambda_n)^2\|x_n - q^\dagger\|^2 + \lambda_n\kappa\|x_n - q^\dagger\|^2 + \kappa\lambda_n\|u_n - q^\dagger\|^2\\ &\quad + 2\lambda_n\langle \phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|u_n - q^\dagger\|^2 \le& \left[1 - \frac{2(\varpi-\kappa)\lambda_n}{1-\kappa\lambda_n}\right]\|x_n - q^\dagger\|^2 + \frac{\varpi^2\lambda_n^2}{1-\kappa\lambda_n}\|x_n - q^\dagger\|^2\\ &+ \frac{2\lambda_n}{1-\kappa\lambda_n}\langle \phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle.\end{aligned} \tag{42}$$

Combining (22) with (42), we have

$$\begin{aligned}\|x_{n+1} - q^\dagger\|^2 \le& \left[1 - \frac{2(\varpi-\kappa)\lambda_n}{1-\kappa\lambda_n}\right]\|x_n - q^\dagger\|^2 + \frac{2(\varpi-\kappa)\lambda_n}{1-\kappa\lambda_n}\\ &\times\left\{\frac{\varpi^2 M\lambda_n}{2(\varpi-\kappa)} + \frac{1}{\varpi-\kappa}\langle \phi(q^\dagger) - Bq^\dagger, u_n - q^\dagger\rangle\right\}.\end{aligned} \tag{43}$$

According to Lemma 2.5, (41) and (43), we deduce $x_n \to q^\dagger$.

Now, we consider Possibility 2. In this case, for any integer m, there exists integer $k \ge m$ such that $\|x_k - q^\dagger\| \le \|x_{k+1} - q^\dagger\|$. Set $\vartheta_n = \|x_n - q^\dagger\|^2$. Then, $\vartheta_m \le \vartheta_{m+1}$. Define an integer sequence $\{\tau_n\}$ for all $n \ge m$ as follows:

$$\tau(n) = \max\{i \in \mathbb{N} | m \le i \le n, \vartheta_i \le \vartheta_{i+1}\}.$$

It is obviously that the sequence $\tau(n)$ is non-decreasing and satisfies $\lim_{n\to\infty}\tau(n) = \infty$ and $\vartheta_{\tau(n)} \le \vartheta_{\tau(n)+1}$ for all $n \ge m$.

By the similar proof as that of Possibility 1, we can deduce

$$\limsup_{n\to\infty}\langle \phi(q^\dagger) - Bq^\dagger, u_{\tau(n)} - q^\dagger\rangle \le 0, \tag{44}$$

and

$$\begin{aligned}\vartheta_{\tau(n)+1} \le& \left[1 - \frac{2(\varpi-\kappa)\lambda_{\tau(n)}}{1-\kappa\lambda_{\tau(n)}}\right]\vartheta_{\tau(n)} + \frac{2(\varpi-\kappa)\lambda_{\tau(n)}}{1-\kappa\lambda_{\tau(n)}}\\ &\times\left\{\frac{\varpi^2 M\lambda_{\tau(n)}}{2(\varpi-\kappa)} + \frac{1}{\varpi-\kappa}\langle \phi(q^\dagger) - Bq^\dagger, u_{\tau(n)} - q^\dagger\rangle\right\}.\end{aligned} \tag{45}$$

Since $\vartheta_{\tau(n)} \le \vartheta_{\tau(n)+1}$, it follows from (45) that

$$\vartheta_{\tau(n)} \le \frac{\varpi^2 M\lambda_{\tau(n)}}{2(\varpi-\kappa)} + \frac{1}{\varpi-\kappa}\langle \phi(q^\dagger) - Bq^\dagger, u_{\tau(n)} - q^\dagger\rangle. \tag{46}$$

Combining (44) and (45), we have $\limsup_{n\to\infty}\vartheta_{\tau(n)} \le 0$ and hence

$$\lim_{n\to\infty}\vartheta_{\tau(n)} = 0. \tag{47}$$

Moreover, by (45), we obtain

$$\limsup_{n\to\infty} \vartheta_{\tau(n)+1} \leq \limsup_{n\to\infty} \vartheta_{\tau(n)}.$$

This together with (47) implies that

$$\lim_{n\to\infty} \vartheta_{\tau(n)+1} = 0.$$

Applying Lemma 2.6 to deduce

$$0 \leq \vartheta_n \leq \max\{\vartheta_{\tau(n)}, \vartheta_{\tau(n)+1}\}.$$

Therefore, $\vartheta_n \to 0$. That is, $x_n \to q^\dagger$. This completes the proof. □

Remark 3.1. *If the operators φ and ψ are quasi-nonexpansive operators or directed operators or demicontractive operators, Theorem 3.1 is still correct.*

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