U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 2, 2015 ISSN 1223-7027

# PARALLEL TRANSPORT AND MULTI-TEMPORAL CONTROLLABILITY

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*Our idea was to make a bridge between the parallel transport in differential geometry and the controllability of evolutionary linear PDE systems. The purpose is threefold: (i) to highlight the origin of some linear homogeneous PDE systems; (ii) to formulate new controllability theorems regarding the bilinear systems; (iii) to prove that the problem of the Riemannian metric can be thought of as a problem of multitime controllability. The results are concise and confirmed by examples and counterexamples. In particular, the control of metric phases in Riemannian geometry is a new subject in our research group.*

**Keywords:** multitime controllability, parallel transport, controls in differential geometry.

**MSC2010:** 93B 05, 53C 05, 35B 37.

## 1. Controlled multitime linear homogeneous PDE systems

### 1.1. Multitime linear homogeneous PDE systems

Our aim is to study homogeneous PDE systems of the type

$$\frac{\partial x}{\partial t^\alpha}(t) = U_\alpha(t)x(t), \quad \forall \alpha = \overline{1,m}, \tag{1}$$

where $t = (t^1, \dots, t^m) \in \mathbb{R}^m$, $x\colon \mathbb{R}^m \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$, and $U_\alpha : \mathbb{R}^m \to \mathcal{M}_n(\mathbb{R})$ are $\mathcal{C}^1$ matrix functions indexed after $\alpha = \overline{1,m}$.

**Proposition 1.1.** *The PDE system* (1) *is completely integrable iff, for any* $\alpha, \beta = \overline{1,m}$ *and any* $t \in \mathbb{R}^m$, *we have*

$$\frac{\partial U_\alpha}{\partial t^\beta}(t) + U_\alpha(t)U_\beta(t) = \frac{\partial U_\beta}{\partial t^\alpha}(t) + U_\beta(t)U_\alpha(t). \tag{2}$$

**Specification.** A matrix-valued function $F(t) \in \mathcal{M}_n(\mathbb{R})$, $t \in I$ is said to be (i) *proper* on the interval $I$ if $F(t) = f(t, A), t \in I$, and a fixed constant matrix $A \in \mathcal{M}_n(\mathbb{R})$, where $f$ is a scalar function, and (ii) $F(t)$ is said to be *semiproper* on $I$ if $F(t)F(\tau) = F(\tau)F(t)$, $\forall t, \tau \in I$.

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The PDE system (1) is called *semiproper* if

$$U_\alpha(t)U_\beta(\tau) = U_\beta(\tau)U_\alpha(t),\ t,\tau \in \mathbb{R}^m, \forall \alpha,\beta \in \overline{1,m}$$

(*functional commutativity* of the matrices $U_\alpha$). A semiproper PDE system has a closed form fundamental matrix

$$\chi(t,t_0) = exp \int_{\gamma_{t_0 t}} U_\alpha(s)\, ds^\alpha.$$

Consequently, the problem of solving a semiproper system amounts to that of finding a finite form expression for the exponential matrix.

If the complete integrability conditions are taken simultaneously with semiproper conditions, then the matrices $U_\alpha$ are the components of gradient of a matrix $U$.

Using Theorem 5.5 and Proposition 5.2 of the paper [5], we obtain

**Proposition 1.2.** *Let $t_0 \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$. If the matrix functions $U_\alpha(\cdot)$ verify the relations* (2)*, then the Cauchy problem* $\{(1),\ x(t_0) = x_0\}$ *has a unique solution*

$$x : \mathbb{R}^m \to \mathbb{R}^n, \quad x(t) = \chi(t,t_0)x_0, \quad \forall t \in \mathbb{R}^m, \tag{3}$$

*where $\chi(\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathcal{M}_n(\mathbb{R})$ is the fundamental matrix associated to the PDE system* (1)*. The solution $x(\cdot)$ is a function of class $\mathcal{C}^2$.*

A source for PDE systems of type (1) is the parallel transport in differential geometry (on a manifold or on jet bundle of order one), modulo the notations of coordinates and functions.

For example, on an $n$-dimensional manifold $M$, with local coordinates $x = (x^i)$, we define a *covariant derivative operator* $\nabla$, and its components $\Gamma^i_{jk}(x)$ (*connection coefficients*), to perform the components of covariant derivative, in a way independent of coordinates.

(i) A vector field $X = (X^i)$ on $M$ is called *parallel* if

$$\nabla_j X^i(x) = \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(x)X^k(x) = 0,\ i,j,k \in \overline{1,n}.$$

(ii) A second order tensor $g = (g_{ij})$ on $M$ is called *parallel* if

$$\nabla_k g_{ij}(x) = \frac{\partial g_{ij}}{\partial x^k}(x) - \Gamma^h_{ki}(x)g_{hj}(x) - \Gamma^h_{kj}(x)g_{hi}(x)$$

$$= \frac{\partial g_{ij}}{\partial x^k}(x) - \left(\Gamma^h_{ki}(x)\delta^l_j + \Gamma^h_{kj}(x)\delta^l_i\right)\ g_{hl}(x) = 0,\ i,j,h,k,l \in \overline{1,n}.$$

For identifying with the PDE system (1), we need to identify: (i) the ranges $\overline{1,m}$, $\overline{1,n}$; (ii) $t$ with $x$; and (iii) $X^i$, resp. $g_{ij}$ with the unknowns $x^i$.

**Remark 1.1.** In differential geometry, the parallel transport is a way of transporting geometrical data along smooth curves in a manifold. If the manifold is equipped with an affine connection (a covariant derivative or connection on the tangent bundle), then this connection allows one to transport tensors of the manifold along curves so that they stay parallel with respect to the connection.

The papers [1]–[4], [6]–[13], [15]–[17] include information and related techniques, the relevant sources for original ideas in this article.

### 1.2. Controlled multitime linear homogeneous PDE systems

Linear PDE systems on Lie groups are a natural generalization of linear PDE systems on Euclidean spaces. Indeed, a homogeneous linear system on a Lie group is a controlled system

$$\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha^\beta(t) X_\beta(x(t)),$$

where $X_\alpha(x)$ are linear vector fields. Of course, these PDE systems can be written in the form (1) and conversely.

We reconsider the PDE system (1) as a *controlled bilinear system*, where the matrix functions

$$(U_\alpha(\cdot))_{\alpha=\overline{1,m}},\ \ U_\alpha : \mathbb{R}^m \to \mathcal{M}_n(\mathbb{R}),$$

are controls of class $\mathcal{C}^1$, $\forall \alpha = \overline{1,m}$, which verify the relations (2) on $\mathbb{R}^m$, for any $\alpha, \beta = \overline{1,m}$. If in addition, for any $\alpha$, the functions $U_\alpha(\cdot)$ are constant, then $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$ is called *constant* control.

**Definition 1.1.** *The family of matrix functions $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$, of class $\mathcal{C}^1$, which verify the relations* (2) *on $\mathbb{R}^m$, is called control for the PDE system* (1)*.*

*The set of controls, resp. the set of constant controls, is denoted by*

$$\mathcal{U}_1 := \Big\{ (U_\alpha)_{\alpha=\overline{1,m}} \,\Big|\, (U_\alpha)_{\alpha=\overline{1,m}} \text{ is control for the PDE system (1)} \Big\},$$

$$\mathcal{U}_2 := \Big\{ (U_\alpha)_{\alpha=\overline{1,m}} \,\Big|\, (U_\alpha)_{\alpha=\overline{1,m}} \text{ is constant control for the PDE system (1)} \Big\}.$$

Hence, the system (1) is completely integrable if and only if $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$ is a control function.

It is obvious that $\mathcal{U}_1$ and $\mathcal{U}_2$ are real vector spaces and $\mathcal{U}_2 \subseteq \mathcal{U}_1$.

**Definition 1.2.** *Let $(t_0, x_0), (t_1, x_1) \in \mathbb{R}^m \times \mathbb{R}^n$ be two phases. We say that the phase $(t_0, x_0)$ is transferred to phase $(t_1, x_1)$, if there exists a control $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$, in the PDE system* (1)*, such that the solution $x(\cdot)$ of the Cauchy problem $\{(1),\ x(t_0) = x_0\}$, verifies also the relations $x(t_1) = x_1$. In other words, for the same control $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$, the Cauchy problems $\{(1),\ x(t_0) = x_0\}$ and $\{(1),\ x(t_1) = x_1\}$ have the same solution. We will say that the control $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$ transfers the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$.*

**Remark 1.2.** Let $(t_0, x_0)$, $(t_1, x_1) \in \mathbb{R}^m \times \mathbb{R}^n$. The control $(U_\alpha(\cdot))_{\alpha=\overline{1,m}} \in \mathcal{U}_1$, transfers the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$, iff $\chi(t_1, t_0)x_0 = x_1$ (here $\chi(\cdot, \cdot)$ is the fundamental matrix of the PDE system (1) in which the family $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$ is as above, which makes the transfer of phases). This follows immediately from Proposition 1.2.

**Remark 1.3.** Let $x(\cdot)$ be a solution of the PDE system (1). If there exists $s \in \mathbb{R}^m$ such that $x(s) = 0$, then $x(t) = 0$, $\forall t \in \mathbb{R}^m$. Suppose that the phase $(t_0, x_0)$ transfers to the phase $(t_1, x_1)$. Then $x_0 = 0$ iff $x_1 = 0$. It is noticed immediately that any control transfers the phase $(t_0, 0)$ into the phase $(t_1, 0)$ ($\forall t_0, t_1 \in \mathbb{R}^m$).

For $x_0 \neq 0$ and $x_1 \neq 0$, we further study the transfer of the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$. In the case $n \geq 2$, we prove that for any $t_0$, $t_1$, different, the phase $(t_0, x_0)$ transfers in the phase $(t_1, x_1)$.

**Lemma 1.1.** *If $a, b \in \mathbb{R}$ and $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, then $e^A = e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}$.*

*Proof.* For an original proof, we show that $e^{\tau A} = e^{a\tau} \begin{pmatrix} \cos(\tau b) & \sin(\tau b) \\ -\sin(\tau b) & \cos(\tau b) \end{pmatrix}$, $\forall \tau \in \mathbb{R}$; then we set $\tau = 1$ and we obtain the conclusion.

We denote $F(\tau) := e^{a\tau} \begin{pmatrix} \cos(\tau b) & \sin(\tau b) \\ -\sin(\tau b) & \cos(\tau b) \end{pmatrix}$, $\tau \in \mathbb{R}$. It suffices to show that $F'(\tau) = AF(\tau)$, $\forall \tau \in \mathbb{R}$, and $F(0) = I_2$. The equality $F(0) = I_2$ is obvious. By computation, we find

$$F'(\tau) = ae^{a\tau} \begin{pmatrix} \cos(\tau b) & \sin(\tau b) \\ -\sin(\tau b) & \cos(\tau b) \end{pmatrix} + e^{a\tau} \begin{pmatrix} -b\sin(\tau b) & b\cos(\tau b) \\ -b\cos(\tau b) & -b\sin(\tau b) \end{pmatrix}.$$

$$AF(\tau) = \left(aI_2 + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot e^{a\tau} \begin{pmatrix} \cos(\tau b) & \sin(\tau b) \\ -\sin(\tau b) & \cos(\tau b) \end{pmatrix}$$

$$= ae^{a\tau} \begin{pmatrix} \cos(\tau b) & \sin(\tau b) \\ -\sin(\tau b) & \cos(\tau b) \end{pmatrix} + b\, e^{a\tau} \begin{pmatrix} -\sin(\tau b) & \cos(\tau b) \\ -\cos(\tau b) & -\sin(\tau b) \end{pmatrix} = F'(\tau).$$

□

**Lemma 1.2.** *Let $n \geq 3$. For any $x_0, x_1 \in \mathbb{R}^n \setminus \{0\}$, there exists $Y_0, Y_1 \in \mathbb{R}^2 \setminus \{0\}$, and there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$, such that*

$$Px_0 = \begin{pmatrix} Y_0 \\ O_{n-2,1} \end{pmatrix} \quad \text{and} \quad Px_1 = \begin{pmatrix} Y_1 \\ O_{n-2,1} \end{pmatrix}. \tag{4}$$

*Proof.* First, do the proof for the case in which vectors $x_0$ and $x_1$ are linearly independent.

Let $S = Sp\{x_0, x_1\}$. The set $\{x_0, x_1\}$ is a basis of the vector subspace $S$ and hence $\dim S = 2$. Then $\dim S^\perp = n - 2 \geq 1$. Let $\{c_3, \ldots, c_n\}$ be a basis of the vector subspace $S^\perp$. The set $\{x_0, x_1, c_3, \ldots, c_n\}$ is a basis of $\mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$, since $S \oplus S^\perp = \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$. It follows that the matrix $P := \begin{pmatrix} x_0^\top \\ x_1^\top \\ c_3^\top \\ \vdots \\ c_n^\top \end{pmatrix}$ is invertible. For $j = \overline{3, n}$, we have $c_j^\top x_0 = x_0^\top c_j = \langle c_j, x_0 \rangle = 0$ and $c_j^\top x_1 = x_1^\top c_j = \langle c_j, x_1 \rangle = 0$, due to the fact $c_j \in S^\perp$. Hence

$$Px_0 = \begin{pmatrix} \|x_0\|^2 \\ \langle x_1, x_0 \rangle \\ O_{n-2,1} \end{pmatrix} \quad \text{and} \quad Px_1 = \begin{pmatrix} \langle x_0, x_1 \rangle \\ \|x_1\|^2 \\ O_{n-2,1} \end{pmatrix}.$$

It is enough to choose $Y_0 = \begin{pmatrix} \|x_0\|^2 \\ \langle x_1, x_0\rangle \end{pmatrix}$ and $Y_1 = \begin{pmatrix} \langle x_0, x_1\rangle \\ \|x_1\|^2 \end{pmatrix}$. The vectors $Y_0$ and $Y_1$ are nonzero, since $\|x_0\| \neq 0$ and $\|x_1\| \neq 0$.

Let us suppose now that $x_0$ and $x_1$ are linearly dependent. There exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$, such that $x_1 = \lambda x_0$. Let $S_0 = Sp\{x_0\}$. The set $\{x_0\}$ is a basis of $S_0$ and hence $\dim S_0 = 1$. Then $\dim S_0^{\perp} = n - 1 \geq 2$.

Let $\{c_2, \ldots, c_n\}$ be a basis of $S_0^{\perp}$. The set $\{x_0, c_2, \ldots, c_n\}$ is a basis of $\mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$. It follows that the matrix $P = \begin{pmatrix} x_0^{\top} \\ c_2^{\top} \\ \vdots \\ c_n^{\top} \end{pmatrix}$ is invertible.

For $j = \overline{2, n}$, we have $c_j^{\top} x_0 = x_0^{\top} c_j = \langle c_j, x_0\rangle = 0$, since $c_j \in S^{\perp}$. Consequently $Px_0 = \begin{pmatrix} \|x_0\|^2 \\ O_{n-1,1} \end{pmatrix}$ and $Px_1 = \lambda P x_0 = \begin{pmatrix} \lambda\|x_0\|^2 \\ O_{n-1,1} \end{pmatrix}$. We choose $Y_0 = \begin{pmatrix} \|x_0\|^2 \\ 0 \end{pmatrix}$ and $Y_1 = \lambda Y_0$. The vectors $Y_0$ and $Y_1$ are non-zero, since $x_0 \neq 0$ and $\lambda \neq 0$. $\square$

**Proposition 1.3.** *Let $n \geq 2$. For any $x_0, x_1 \in \mathbb{R}^n \setminus \{0\}$, there exists $A \in \mathcal{M}_n(\mathbb{R})$ such that $e^A x_0 = x_1$. If $x_0 = x_1$, we can choose $A = 0$. If $x_0 \neq x_1$, then $A$ can be chosen with* $\operatorname{rank} A = 2$.

*Proof.* First, we refer to the case $n = 2$. Let $x_0 = (x_0^1, x_0^2)^{\top}$, $x_1 = (x_1^1, x_1^2)^{\top}$. We consider the complex numbers $z_0 := x_0^2 + i x_0^1$ and $z_1 := x_1^2 + i x_1^1$. The numbers $z_0$ and $z_1$ are non-zero since $x_0$ and $x_1$ are non-zero.

Write $z_0$ and $z_1$ in trigonometric form. There exists $r_0, r_1 \in (0, \infty)$ and there exists $\theta_0, \theta_1 \in [0, 2\pi)$, such that $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$ and $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$. Let $z = \dfrac{z_1}{z_0}$. Obviously $z = \dfrac{r_1}{r_0}\big(\cos(\theta_1 - \theta_0) + i\sin(\theta_1 - \theta_0)\big)$.

Let $G := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \,\middle|\, a, b \in \mathbb{R} \right\}$. We consider the function

$$f : \mathbb{C} \to G, \quad f(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (a, b \in \mathbb{R}).$$

It is easily proved that $G$, with operations of addition and multiplication of matrices, is a field and the function $f$ is an isomorphism of fields.

Consequently we have $z z_0 = z_1 \Longrightarrow f(z) f(z_0) = f(z_1)$, i.e.,

$$\frac{r_1}{r_0}\begin{pmatrix} \cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0) \\ -\sin(\theta_1 - \theta_0) & \cos(\theta_1 - \theta_0) \end{pmatrix}\begin{pmatrix} x_0^2 & x_0^1 \\ -x_0^1 & x_0^2 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1^1 \\ -x_1^1 & x_1^2 \end{pmatrix}. \tag{5}$$

Let $a := \ln\left(\dfrac{r_1}{r_0}\right)$, $b := \theta_1 - \theta_0$ and $A := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. We use Lemma 1.1 and it follows

$$e^A = e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} = \frac{r_1}{r_0}\begin{pmatrix} \cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0) \\ -\sin(\theta_1 - \theta_0) & \cos(\theta_1 - \theta_0) \end{pmatrix}. \tag{6}$$

From (5) and (6) we deduce the equality

$$e^A \begin{pmatrix} x_0^2 & x_0^1 \\ -x_0^1 & x_0^2 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1^1 \\ -x_1^1 & x_1^2 \end{pmatrix}.$$

Equalizing the second column in the left hand side with second column from the right, and we get

$$e^A \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} = \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix} \iff e^A x_0 = x_1.$$

The matrix $A$, chosen above, is void if and only if $a = b = 0 \iff r_1 = r_0$ and $\theta_1 = \theta_0$ $\iff z_1 = z_0 \iff x_1 = x_0$. Hence $A = 0$ iff $x_0 = x_1$.

It follows that if $x_0 \neq x_1$, then $A$ is non-zero, hence $a \neq 0$ or $b \neq 0$, and from $\det(A) = a^2 + b^2$ we obtain $\operatorname{rank} A = 2$.

We remark that in the case $n = 2$, we can choose $A \in G$ ($A = O_2$ or $\operatorname{rank} A = 2$).

So we have proved the Proposition for $n = 2$.

Let $n \geq 3$.

If $x_0 = x_1$, we choose $A = 0$ and it is obvious that $e^A x_0 = I_2 x_0 = x_1$.

Now treat the case where $x_0 \neq x_1$.

According Lemma 1.2, there exists $Y_0, Y_1 \in \mathbb{R}^2 \setminus \{0\}$, and there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$, such that the equalities (4) hold.

We apply the case $n = 2$ to the vectors $Y_0$, $Y_1$. We saw that one can choose $A_1 \in G$, such that $e^{A_1} Y_0 = Y_1$.

Let $A_2 := \begin{pmatrix} A_1 & O_{2,n-2} \\ O_{n-2,2} & O_{n-2,n-2} \end{pmatrix}$. Using the relations (4), we obtain

$$e^{A_2} P x_0 = \begin{pmatrix} e^{A_1} & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix} \begin{pmatrix} Y_0 \\ O_{n-2,1} \end{pmatrix}$$

$$= \begin{pmatrix} e^{A_1} Y_0 \\ O_{n-2,1} \end{pmatrix} = \begin{pmatrix} Y_1 \\ O_{n-2,1} \end{pmatrix} = P x_1.$$

Hence $e^{A_2} P x_0 = P x_1$. Since $P$ is invertible, it follows $P^{-1} e^{A_2} P x_0 = x_1$. We choose $A = P^{-1} A_2 P$. We get

$$e^A x_0 = e^{P^{-1} A_2 P} x_0 = (P^{-1} e^{A_2} P) x_0 = x_1.$$

Obviously, $\operatorname{rank} A = \operatorname{rank} A_2 = \operatorname{rank} A_1$.

If we had $\operatorname{rank} A_1 \neq 2$, then $\det(A_1) = 0$. Since $A_1 \in G$ and $\det(A_1) = 0$, it follows that $A_1 = O_2$, hence also $A_2 = O_n$. From $A = P^{-1} A_2 P$, we obtain $A = O_n$, and from $e^A x_0 = x_1$, it follows $x_0 = x_1$, which is false.

Hence $\operatorname{rank} A_1 = 2$. It follows $\operatorname{rank} A = 2$. □

In general, the matrix $A$, appearing in the conclusion of Proposition 1.3 can not have the rank 1. We show this in the following example.

**Example 1.1.** *Let $n \geq 2$, $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ and $x_1 = -\xi x_0$, with $\xi \in (0, \infty)$. If the matrix $A \in \mathcal{M}_n(\mathbb{R})$ verifies the relations $e^A x_0 = x_1$, then* $\operatorname{rank} A \neq 1$.

Let us suppose that $\operatorname{rank} A = 1$. Let $J \in \mathcal{M}_n(\mathbb{C})$, be the canonical Jordan form o the matrix $A$. $\operatorname{Rank} J = \operatorname{rank} A = 1$. If on the principal diagonal of $J$ should exists at least two nonzero elements, then $\operatorname{rank} J \geq 2$, which is impossible.

It follows that at least $n-1$ from the eigenvalues of the matrix $A$ are zero, and the last eigenvalue, $\lambda$, is real, since $\lambda = Tr(A) \in \mathbb{R}$.

Then the matrix $e^A$ has $n-1$ eigenvalues equal to 1, an the $n$-th eigenvalue is $e^\lambda$. It follows that $e^A + \xi I_n$ has $n-1$ eigenvalues equal to $1+\xi$, and the $n$-th eigenvalue is $e^\lambda + \xi$. Hence $\det(e^A + \xi I_n) = (1+\xi)^{n-1}(e^\lambda + \xi) > 0$. We deduce that the matrix $e^A + \xi I_n$ is invertible.

The equality $e^A x_0 = x_1$ is equivalent to $e^A x_0 + \xi x_0 = 0$ or $(e^A + \xi I_n)x_0 = 0$. Since the matrix $e^A + \xi I_n$ is invertible, it follows $x_0 = 0$, that is false. □

We denote by $\mathcal{U}_3$ the set of families of functions $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$,
$U_\alpha(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$, with the property

– $U_\alpha(\cdot)$ are all identically zero,

or

– there exists $\alpha_0 = \overline{1,m}$ and there exists a matrix $M \in \mathcal{M}_n(\mathbb{R})$, with $\operatorname{rank} M = 2$, such that

$$U_\alpha(t) = \begin{cases} M, \ \forall t \in \mathbb{R}^m; & \text{if } \alpha = \alpha_0 \\ O_n, \ \forall t \in \mathbb{R}^m; & \text{if } \alpha \neq \alpha_0. \end{cases}$$

Note that for such families, the relations (2) are true. It follows that the elements of the set $\mathcal{U}_3$ are constant controls for the PDE system (1).

Hence $\mathcal{U}_3 \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$.

**Theorem 1.1.** *Let $n \geq 2$. For any $(t_0, x_0)$, $(t_1, x_1) \in \mathbb{R}^m \times \mathbb{R}^n$, with $x_0 \neq 0$, $x_1 \neq 0$ and $t_0 \neq t_1$, there exists a control in $\mathcal{U}_3$ which transfers the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$.*

*Proof.* Since $t_0 \neq t_1$, there exists $\alpha_0 = \overline{1,m}$, such that $t_0^{\alpha_0} \neq t_1^{\alpha_0}$.

According the Proposition 1.3, there exists $A \in \mathcal{M}_n(\mathbb{R})$, with $A = 0$ or $\operatorname{rank} A = 2$, such that $e^A x_0 = x_1$. Let $M = \dfrac{1}{t_1^{\alpha_0} - t_0^{\alpha_0}} A$. We choose

$$U_\alpha(t) = \begin{cases} M, \ \forall t \in \mathbb{R}^m; & \text{if } \alpha = \alpha_0 \\ O_n, \ \forall t \in \mathbb{R}^m; & \text{if } \alpha \neq \alpha_0. \end{cases}$$

Obviously, $(U_\alpha(\cdot))_{\alpha=\overline{1,m}} \in \mathcal{U}_3$.

Taking into account the Proposition 1.2 and the formula

$$\chi(t, t_0) = e^{M_\alpha (t^\alpha - t_0^\alpha)}, \quad \forall (t, t_0) \in \mathbb{R}^m \times \mathbb{R}^m, \tag{7}$$

it follows that the solution of the Cauchy problem $\{(1),\ x(t_0) = x_0\}$ is

$$x : \mathbb{R}^m \to \mathbb{R}^n, \quad x(t) = e^{(t^{\alpha_0} - t_0^{\alpha_0})M} x_0, \quad \forall t = (t^1, \ldots, t^m) \in \mathbb{R}^m.$$

We obtain $x(t_1) = e^{(t_1^{\alpha_0} - t_0^{\alpha_0})M} x_0 = e^A x_0 = x_1$, therefore chosen control transfers the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$. □

The Theorem 1.1 says that for $n \geq 2$, in the case of the PDE system (1), it is sufficient to consider as controls only elements of the set $\mathcal{U}_3$.

In the case $n = 1$, we have the next result

**Theorem 1.2.** *Let $n = 1$, $(t_0, x_0)$, $(t_1, x_1) \in \mathbb{R}^m \times \mathbb{R}$, with $x_0 \neq 0$ and $x_1 \neq 0$.*
*a) If the phase $(t_0, x_0)$ transfers into the phase $(t_1, x_1)$, then $sgn(x_0) = sgn(x_1)$.*
*b) Let us suppose that $sgn(x_0) = sgn(x_1)$ and $t_0 \neq t_1$. Let $\alpha_0 = \overline{1, m}$, such that $t_0^{\alpha_0} \neq t_1^{\alpha_0}$. We consider the functions*

$$U_\alpha : \mathbb{R}^m \to \mathbb{R}, \quad U_\alpha(t) = \begin{cases} \dfrac{1}{t_1^{\alpha_0} - t_0^{\alpha_0}} \ln\left(\dfrac{x_1}{x_0}\right), \forall t \in \mathbb{R}^m; \text{ if } \alpha = \alpha_0 \\ 0 \qquad\qquad , \ \forall t \in \mathbb{R}^m; \text{ if } \alpha \neq \alpha_0. \end{cases} \tag{8}$$

*Then $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$ is a constant control for the PDE system (1), and transfers the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$.*

*Proof.* *a)* The function $\chi(\cdot, t_0) : \mathbb{R}^m \to \mathbb{R}$ is continuous. Since for any $t \in \mathbb{R}^m$, the matrix $\chi(t, t_0)$ is invertible, i.e., $\chi(t, t_0) \neq 0$, it follows that the function $\chi(\cdot, t_0)$ has constant sign. But $\chi(t_0, t_0) = 1 > 0$; hence $\chi(t, t_0) > 0$, $\forall t \in \mathbb{R}^m$.

According the Remark 1.2, we have $\chi(t_1, t_0)x_0 = x_1$. Since $\chi(t_1, t_0) > 0$, it follows that $x_0$ and $x_1$ have the same sign.

*b)* We have $\dfrac{x_1}{x_0} > 0$, since $sgn(x_0) = sgn(x_1)$. Hence $U_\alpha$ is well definite. Because the relations (2) are obviously true, it follows that $(U_\alpha(\cdot))_{\alpha=\overline{1,m}} \in \mathcal{U}_2$.

Using the formula (7), we obtain,

$$\chi(t_1, t_0)x_0 = e^{\ln\left(\frac{x_1}{x_0}\right)} x_0 = \frac{x_1}{x_0} \cdot x_0 = x_1.$$

Consequently the control $(U_\alpha(\cdot))_{\alpha=\overline{1,m}}$ transfers the phase $(t_0, x_0)$ into the phase $(t_1, x_1)$, according the Remark 1.2. □

The Theorem 1.2 says that, for $n = 1$, in the case of the PDE system (1), it is sufficient to consider as controls only constant functions form (8).

For the PDE systems of the type (1) we may consider other sets of controls, how was it $\mathcal{U}_3$, for example. Also, it may impose conditions on solutions $x(\cdot)$; for example, require that $x(\cdot)$ take values in a given set, included in $\mathbb{R}^n$.

## 2. Metric parallelism (metric phase transfer)

In the paper [14] one solves the following problem: giving a second order differential equation, under what conditions the graphs of solutions of this equations are geodesics for a Riemannian manifold $\left(D \subseteq \mathbb{R}^2, g_{ij}\right)$? Obviously $g = (g_{ij})_{i,j \in \{1,2\}}$, is a Riemannian metric that must be determined. It shows that it is necessary that the given equation to be a Riccati second order equation and $\Gamma^1_{ij} = 0$. In these conditions, the coefficients of the Levi-Civita connection, $\Gamma^2_{ij}$, are expressed with respect to the functions which defines the Riccati equation and conversely.

Equivalently, the Riccati equation is known if and only if the coefficients $\Gamma^2_{ij}$, $\forall i,j\in\{1,2\}$ are known. Let us show that $(g_{11},g_{12},g_{22})^\top$ is the solution of the linear homogeneous PDE system (9), written below.

We consider the linear homogeneous PDE system

$$\frac{\partial x}{\partial t^1}=U_1(t)x,\quad \frac{\partial x}{\partial t^2}=U_2(t)x, \tag{9}$$

where $t=(t^1,t^2)\in\mathbb{R}^2$, $x=(x^1,x^2,x^3)^\top:\mathbb{R}^2\to\mathbb{R}\times\mathbb{R}\times(0,\infty)$,
and $U_1,U_2:\mathbb{R}^2\to\mathcal{M}_3(\mathbb{R})$ are $\mathcal{C}^1$ matrix functions of the form

$$U_1(t)=\begin{pmatrix}0 & 2\Gamma^2_{11}(t) & 0\\ 0 & \Gamma^2_{12}(t) & \Gamma^2_{11}(t)\\ 0 & 0 & 2\Gamma^2_{12}(t)\end{pmatrix},\quad U_2(t)=\begin{pmatrix}0 & 2\Gamma^2_{12}(t) & 0\\ 0 & \Gamma^2_{22}(t) & \Gamma^2_{12}(t)\\ 0 & 0 & 2\Gamma^2_{22}(t)\end{pmatrix}. \tag{10}$$

Note that systems (9) are a special cases of the PDE systems (1), with $m=2$, $n=3$.

In the paper [14] was proved the following result

**Theorem 2.1.** *We consider given $\mathcal{C}^1$ functions $\Gamma^2_{11},\Gamma^2_{12},\Gamma^2_{22}:\mathbb{R}^2\to\mathbb{R}$.*
*a) The following statements are equivalent:*

*i) On $\mathbb{R}^2$ there are true the relations*

$$\frac{\partial\Gamma^2_{12}}{\partial t^2}=\frac{\partial\Gamma^2_{22}}{\partial t^1},\qquad \frac{\partial\Gamma^2_{12}}{\partial t^1}+\left(\Gamma^2_{12}\right)^2=\frac{\partial\Gamma^2_{11}}{\partial t^2}+\Gamma^2_{11}\Gamma^2_{22}. \tag{11}$$

*ii) There exists the $\mathcal{C}^3$ function $W:\mathbb{R}^2\to\mathbb{R}$, such that on $\mathbb{R}^2$ we have the relations*

$$\frac{\partial W}{\partial t^2}>0,\quad \Gamma^2_{11}=\frac{\dfrac{\partial^2 W}{\partial (t^1)^2}}{\dfrac{\partial W}{\partial t^2}},\quad \Gamma^2_{12}=\frac{\dfrac{\partial^2 W}{\partial t^1\partial t^2}}{\dfrac{\partial W}{\partial t^2}},\quad \Gamma^2_{22}=\frac{\dfrac{\partial^2 W}{\partial (t^2)^2}}{\dfrac{\partial W}{\partial t^2}}. \tag{12}$$

*iii) The PDE system* (9) *has at least a solution $x:\mathbb{R}^2\to\mathbb{R}\times\mathbb{R}\times(0,\infty)$, of class $\mathcal{C}^1$.*
*b) Suppose the equivalent statements i), ii), iii) are true. Let $W$ be a function as in the point ii).*

*Then all the solutions, $x=(x^1,x^2,x^3)^\top:\mathbb{R}^2\to\mathbb{R}\times\mathbb{R}\times(0,\infty)$, of the PDE system* (9)*, are of the form*

$$x^1=a\left(\frac{\partial W}{\partial t^1}+c\right)^2+b;\ x^2=a\left(\frac{\partial W}{\partial t^1}+c\right)\cdot\left(\frac{\partial W}{\partial t^2}\right);\ x^3=a\left(\frac{\partial W}{\partial t^2}\right)^2 \tag{13}$$

*with $a>0$, $b$, $c$ arbitrarily real constants.*

*In these conditions, the functions $x^1(\cdot)$, $x^2(\cdot)$, $x^3(\cdot)$ are of class $\mathcal{C}^2$.*

Proposition next check immediately.

**Proposition 2.1.** *Suppose that we are considering Theorem 2.1 and that the equivalent statements i), ii), iii) are true. Let $W$ as in the statement ii) of the Theorem 2.1.*

*Then* $\forall t_0 \in \mathbb{R}^2$, $\forall x_0 \in \mathbb{R}^2 \times (0,\infty)$, *the Cauchy problem* $\{(9),\ x(t_0) = x_0\}$ *has a unique solution. This is given by the formulas* (13)*, where*

$$a = \frac{x_0^3}{\left(\dfrac{\partial W}{\partial t^2}(t_0)\right)^2}, \quad b = \frac{1}{x_0^3}\begin{vmatrix} x_0^1 & x_0^2 \\ x_0^2 & x_0^3 \end{vmatrix} = x_0^1 - \frac{(x_0^2)^2}{x_0^3}, \tag{14}$$

$$c = \frac{1}{x_0^3}\begin{vmatrix} x_0^2 & x_0^3 \\ \dfrac{\partial W}{\partial t^1}(t_0) & \dfrac{\partial W}{\partial t^2}(t_0) \end{vmatrix} = \frac{x_0^2}{x_0^3}\cdot\frac{\partial W}{\partial t^2}(t_0) - \frac{\partial W}{\partial t^1}(t_0). \tag{15}$$

**Definition 2.1.** *The pair of matrix functions* $\big(U_1(\cdot), U_2(\cdot)\big)$ *is called control for the PDE system* (9) *if there exists a function* $W : \mathbb{R}^2 \to \mathbb{R}$*, of class* $\mathcal{C}^3$*, with* $\dfrac{\partial W}{\partial t^2}(t) > 0$, $\forall t \in \mathbb{R}^2$*, such that,* $U_1$*,* $U_2$ *are defined by the relations* (10)*, for* $\Gamma^2_{ij}$ *given by the formulas* (12)*. The function* $W$ *defines completely the control. We denote by* $U_W$*, the control defined the function* $W$*.*

We observe that the pair $\big(U_1, U_2\big)$ is control for the PDE system (9) iff the functions $\Gamma^2_{ij}$, which define $U_1$ and $U_2$, verify the equivalent statements *i*), *ii*), of the Theorem 2.1.

For the PDE system (9), the phase transfer is similar to the exposed in Section 1, for the PDE system (1). It differs only choice controls; they are now the ones presented in Definition 2.1.

**Proposition 2.2.** *For any control* $\big(U_1, U_2\big)$ *and any solution* $x(\cdot)$ *of the PDE system* (9)*, there exists a constant* $k \in \mathbb{R}$*, such that*

$$x^1(t) - \frac{(x^2(t))^2}{x^3(t)} = k, \quad \forall t \in \mathbb{R}^2, \tag{16}$$

*i.e., the function* $F(x^1, x^2, x^3) = x^1 - \dfrac{(x^2)^2}{x^3}$ *is a first integral for the PDE system* (9)*.*

*Proof.* Any solution is given by the formulas (13). We have

$$x^1(t)x^3(t) = a^2\left(\frac{\partial W}{\partial t^1}(t) + c\right)^2 \cdot \left(\frac{\partial W}{\partial t^2}(t)\right)^2 + bx^3(t) = (x^2(t))^2 + bx^3(t).$$

Hence $x^1(t)x^3(t) = (x^2(t))^2 + bx^3(t)$, equivalent to $x^1(t) - \dfrac{(x^2(t))^2}{x^3(t)} = b$. We choose $k = b$. □

**Proposition 2.3.** *Let* $x_0, x_1 \in \mathbb{R}^2 \times (0,\infty)$ *and* $t_0, t_1 \in \mathbb{R}^2$*.*

*a) If the phase* $(t_0, x_0)$ *transfers to the phase* $(t_1, x_1)$*, then*

$$x_0^1 - \frac{(x_0^2)^2}{x_0^3} = x_1^1 - \frac{(x_1^2)^2}{x_1^3}. \tag{17}$$

*b) Let us suppose that the equality (17) is true.*

*Let* $W:\mathbb{R}^2\to\mathbb{R}$, *of class* $\mathcal{C}^3$, *with* $\dfrac{\partial W}{\partial t^2}(t)>0$, $\forall t\in\mathbb{R}^2$. *Then the control* $U_W$ *transfers the phase* $(t_0,x_0)$ *into the phase* $(t_1,x_1)$ *iff*

$$\frac{x_0^3}{\left(\dfrac{\partial W}{\partial t^2}(t_0)\right)^2}=\frac{x_1^3}{\left(\dfrac{\partial W}{\partial t^2}(t_1)\right)^2}\quad \textit{and} \tag{18}$$

$$\frac{x_0^2}{x_0^3}\cdot\frac{\partial W}{\partial t^2}(t_0)-\frac{\partial W}{\partial t^1}(t_0)=\frac{x_1^2}{x_1^3}\cdot\frac{\partial W}{\partial t^2}(t_1)-\frac{\partial W}{\partial t^1}(t_1). \tag{19}$$

*Proof.* *a*) If the phase $(t_0,x_0)$ transfers to the phase $(t_1,x_1)$, then there exists a control, such that the Cauchy problems $\{(9),\, x(t_0)=x_0\}$ and $\{(9),\, x(t_1)=x_1\}$ have the same solution $x(\cdot)$. According the Proposition 2.2, there exists $k\in\mathbb{R}$, such that the relation (16) holds true, $\forall t\in\mathbb{R}^2$. Hence we have

$$x^1(t_0)-\frac{(x^2(t_0))^2}{x^3(t_0)}=x^1(t_1)-\frac{(x^2(t_1))^2}{x^3(t_1)}=k\Longleftrightarrow x_0^1-\frac{(x_0^2)^2}{x_0^3}=x_1^1-\frac{(x_1^2)^2}{x_1^3}=k.$$

*b*) The control $U_W$ transfers the phase $(t_0,x_0)$ into the phase $(t_1,x_1)$ iff the Cauchy problems $\{(9),\, x(t_0)=x_0\}$ and $\{(9),\, x(t_1)=x_1\}$ have the same solution $x(\cdot)$. This is equivalent, according the Proposition 2.1, to the fact that the numbers $a$, $b$, $c$ are

$$a=\frac{x_0^3}{\left(\dfrac{\partial W}{\partial t^2}(t_0)\right)^2}=\frac{x_1^3}{\left(\dfrac{\partial W}{\partial t^2}(t_1)\right)^2},\quad b=x_0^1-\frac{(x_0^2)^2}{x_0^3}=x_1^1-\frac{(x_1^2)^2}{x_1^3}\quad\text{and}$$

$$c=\frac{x_0^2}{x_0^3}\cdot\frac{\partial W}{\partial t^2}(t_0)-\frac{\partial W}{\partial t^1}(t_0)=\frac{x_1^2}{x_1^3}\cdot\frac{\partial W}{\partial t^2}(t_1)-\frac{\partial W}{\partial t^1}(t_1).$$

□

**Theorem 2.2.** *Let* $x_0,x_1\in\mathbb{R}^2\times(0,\infty)$, *such that the relation* (17) *is true. Then, for any* $t_0,t_1\in\mathbb{R}^2$, $t_0\neq t_1$, *the phase* $(t_0,x_0)$ *transfers to the phase* $(t_1,x_1)$.

*a*) *If* $t_0^1\neq t_1^1$, *we consider*

$$W:\mathbb{R}^2\to\mathbb{R},\ W(t^1,t^2)=A\frac{(t^1-t_0^1)^2}{2}+e^{B(t^1-t_0^1)}\cdot(t^2-t_0^2),\quad\forall(t^1,t^2)\in\mathbb{R}^2,$$

$$\textit{where}\quad B=\frac{\ln x_1^3-\ln x_0^3}{2(t_1^1-t_0^1)},\quad A=\frac{1}{t_1^1-t_0^1}\cdot\left(\frac{x_1^2}{\sqrt{x_1^3x_0^3}}-\frac{x_0^2}{x_0^3}-B(t_1^2-t_0^2)\frac{\sqrt{x_1^3}}{\sqrt{x_0^3}}\right).$$

*Then the control* $U_W$ *transfers the phase* $(t_0,x_0)$ *into the phase* $(t_1,x_1)$.

*b*) *If* $t_0^1=t_1^1$, $t_0^2\neq t_1^2$, $x_0^3\neq x_1^3$, *we consider*

$$W:\mathbb{R}^2\to\mathbb{R},\ W(t^1,t^2)=\frac{1}{B}\cdot e^{AB(t^1-t_0^1)}\cdot e^{B(t^2-t_0^2)},\quad\forall(t^1,t^2)\in\mathbb{R}^2,$$

$$\textit{where}\quad A=\frac{1}{\sqrt{x_1^3}-\sqrt{x_0^3}}\cdot\left(\frac{x_1^2}{\sqrt{x_1^3}}-\frac{x_0^2}{\sqrt{x_0^3}}\right),\quad B=\frac{\ln x_1^3-\ln x_0^3}{2(t_1^2-t_0^2)}.$$

*Then the control* $U_W$ *transfers the phase* $(t_0,x_0)$ *into the phase* $(t_1,x_1)$.

*c) If* $t_0^1 = t_1^1$, $t_0^2 \neq t_1^2$, $x_0^3 = x_1^3$, *we consider*

$$W : \mathbb{R}^2 \to \mathbb{R},\ \ W(t^1,t^2) = e^{A(t^1-t_0^1)} \cdot (t^2 - t_0^2), \quad \forall (t^1,t^2) \in \mathbb{R}^2,$$

*where* $A = \dfrac{x_1^2 - x_0^2}{x_0^3}$.

*Then the control* $U_W$ *transfers the phase* $(t_0, x_0)$ *into the phase* $(t_1, x_1)$.

*Proof.* One applies the Proposition 2.3.

We have to show that $\dfrac{\partial W}{\partial t^2} > 0$ and that the equalities (18), (19) are true.

*a)* $\dfrac{\partial W}{\partial t^2}(t^1,t^2) = e^{B(t^1-t_0^1)} > 0$, $\dfrac{\partial W}{\partial t^1}(t^1,t^2) = A(t^1 - t_0^1) + Be^{B(t^1-t_0^1)} \cdot (t^2 - t_0^2)$.

$$\frac{\partial W}{\partial t^2}(t_0) = 1, \quad \frac{\partial W}{\partial t^1}(t_0) = 0,$$

$$\frac{\partial W}{\partial t^2}(t_1) = e^{B(t_1^1-t_0^1)}, \quad \frac{\partial W}{\partial t^1}(t_1) = A(t_1^1 - t_0^1) + Be^{B(t_1^1-t_0^1)} \cdot (t_1^2 - t_0^2).$$

The equality (18) is equivalent to

$$\frac{x_0^3}{1} = \frac{x_1^3}{e^{2B(t_1^1-t_0^1)}} \Longleftrightarrow e^{B(t_1^1-t_0^1)} = \frac{\sqrt{x_1^3}}{\sqrt{x_0^3}} \Longleftrightarrow B = \frac{\ln x_1^3 - \ln x_0^3}{2(t_1^1 - t_0^1)},$$

the last equality being true.

The equality (19) is equivalent to

$$\frac{x_0^2}{x_0^3} = \frac{x_1^2}{x_1^3} \cdot e^{B(t_1^1-t_0^1)} - A(t_1^1 - t_0^1) - Be^{B(t_1^1-t_0^1)} \cdot (t_1^2 - t_0^2). \tag{20}$$

Since $e^{B(t_1^1-t_0^1)} = \dfrac{\sqrt{x_1^3}}{\sqrt{x_0^3}}$, the equality (20) is equivalent to

$$\frac{x_0^2}{x_0^3} = \frac{x_1^2}{x_1^3} \cdot \frac{\sqrt{x_1^3}}{\sqrt{x_0^3}} - A(t_1^1 - t_0^1) - B \cdot \frac{\sqrt{x_1^3}}{\sqrt{x_0^3}} \cdot (t_1^2 - t_0^2),$$

$$\text{or} \quad A = \frac{1}{t_1^1 - t_0^1} \cdot \left( \frac{x_1^2}{\sqrt{x_1^3 x_0^3}} - \frac{x_0^2}{x_0^3} - B(t_1^2 - t_0^2)\frac{\sqrt{x_1^3}}{\sqrt{x_0^3}} \right),$$

which is true.

Analogously, we treat the cases *b*) and *c*). □

**Acknowledgments**

The work has been funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Ministry of European Funds through the Financial Agreement POSDRU/159/1.5/S/132395.

Partially supported by University Politehnica of Bucharest, and by Academy of Romanian Scientists.

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