ASPECTS OF RUIN PROBABILITY IN INSURANCE

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Se studiază probabilitatea de ruinare a unei companii de asigurări ţinând cont de frecvenţa si amploarea cererilor de despăgubire. O primă evaluare se obţine cu inegalitatea lui Lundberg iar apoi se stabileşte o ecuatie integro-diferentială verificată de aceasta. Se va aplica pe cazul particular al cererilor cu repartiţie exponenţială.

Study of ruin probability for an insurance company taking into account the frequency and size of the claims. A first evaluation is obtained using the Lundberg inequality and then an integro-differential equation satisfied by the ruin probability is established. This will be applied to the particular case of exponentially distributed claims.

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Introduction

This paper presents the classical model for the risk process in insurance and exposes results regarding the infinite time ruin probability. It is firstly recalled the adjustment coefficient and then established the Lundberg inequality (see [1], [2]). Furthermore, the article presents an integro-differential equation verified by the ruin probability for arbitrary claims, and determines its solution for the case of exponentially distributed claims.

1. The classical risk model

The standard mathematical model for insurance risk consists of the following: the occurrence of the claims is described by a Poisson process with intensity λ. It is denoted by N_t the number of claims within the interval $(0,t]$. The amounts of money to be paid by the company at each claim is considered to be a sequence of random variables, denoted $\{X_1, X_2, ...\}$ which are independent identically distributed (i.i.d.) positive variables with distribution function F and finite mean μ. It is denoted by S_t the total amount of claims

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within interval $(0,t]$, thus we have $S_t = \sum_{j=1}^{N_t} X_j$. Therefore the process $\{S_t, t \geq 0\}$ is a composed Poisson process and for each $t > 0$ it results: $E(S_t) = E(N_t)E(X_j) = (\lambda t)(\mu) = \mu\lambda t$.

Furthermore, it is assumed that the insurance premiums are paid continuously, with a premium income rate c, the total income within $(0,t]$ being ct.

In order for the company to cover its liabilities, the premium rate must verify the following inequality: $ct > E(S_t)$, equivalent to $c > \mu\lambda$.

Thus, there exists $\theta > 0$ called the safety loading so that:

$$c = (1+\theta)\lambda\mu \tag{1.1}$$

The capital at time t is: $U_t = U_0 + ct - S_t, t \geq 0$, where $U_0 = u$ is the initial capital.

The survival probability in infinite time, with initial capital u is defined by: $\phi(u) = \Pr(U_t \geq 0, \forall t \geq 0 | U_0 = u)$. Thus, the ruin probability in infinite time will be: $\psi(u) = 1 - \phi(u)$.

2. Lundberg inequality

In this section it is demonstrated an inequality satisfied by the ruin probability $\psi(u)$. For this, the following concept is introduced:

Definition: It is called an adjustment coefficient for a positive random variable X of mean μ, the unique positive solution of the k variable equation:

$$1 + (1+\theta)\mu k = E(e^{kX}). \tag{1.2}$$

We present here a justification for the existence and singularity of a strict positive solution for the equation (1.2).

It is obvious that the equation admits $k = 0$ as a solution.

Then, let: $y_1(t) = 1 + (1+\theta)\mu t$ and $y_2(t) = E(e^{tX})$ be the functions which will be graphically represented. The graphic of y_1 is a line of slope $(1+\theta)\mu$, which passes through (0, 1), as shown in Figure 1. Concerning y_2, we notice that it is an ascending function, since $y_2'(t) = E(Xe^{tX}) > 0$ and it is also convex because $y_2''(t) = E(X^2 e^{tX}) > 0$.

The two curves cross each other in (0, 1), as we previously noticed. Since $y_2'(0) = \mu < (1+\theta)\mu = y_1'(0)$, the graphical representation of y_2 descends below the graphical representation of y_1, and because of its profile, it will intersect the graphic of y_1 once more. The x-value of the point of intersection is the strict positive value of k that we are looking for.

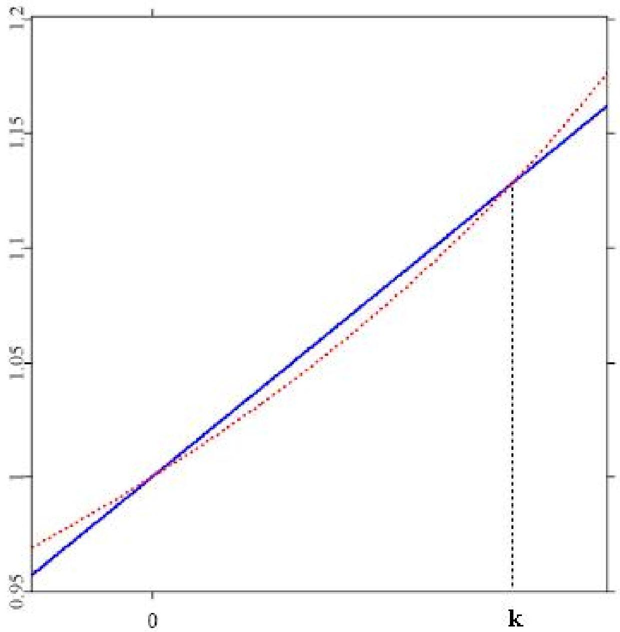

Fig. 1: Illustration of the existence of the adjustment coefficient

We will add an alternative definition for the adjustment coefficient, which will be later used.

By integrating the expression of $E(e^{kX})$ it results:

$$E(e^{kX}) = \int_0^\infty e^{kx} dF(x) = -e^{kx}[1 - F(x)]\Big|_0^\infty + k\int_0^\infty e^{kx}[1 - F(x)]dx .$$

The first term of this sum is analyzed. In order to do this, the following inequalities are established:

$$0 \le e^{kx}[1 - F(x)] = e^{kx}\int_x^\infty dF(y) \le \int_x^\infty e^{ky} dF(y)$$

and because $\lim_{x\to\infty}\int_x^\infty e^{ky} dF(y) = 0,$ we obtain: $\lim_{x\to\infty} e^{kx}[1 - F(x)] = 0$.

Thus, for $x \to \infty$, the expression from the first term tends to 0, and for $x \to 0$ the expression tends to 1.

This results in: $E(e^{kX}) = \int_0^\infty e^{kx} dF(x) = 1 + k\int_0^\infty e^{kx}[1 - F(x)]dx$.

Because $1 + (1+\theta)\mu k = E(e^{kX})$, it obtains:

$$1 + \theta = \int_0^\infty e^{kx} f_e(x)dx \tag{1.3}$$

where:

$$f_e(x) = \frac{1 - F(x)}{\mu},\ x > 0 \tag{1.4}$$

The choice for the denotation is justified by the fact that f_e is in fact a distribution density: $\int_0^\infty f_e(x)dx = \frac{1}{\mu}\int_0^\infty [1 - F(x)]dx = 1$

<u>Theorem</u>. The ruin probability satisfies the Lundberg inequality:

$$\psi(u) \le e^{-ku}, u \ge 0. \tag{1.5}$$

Demonstration. It is denoted by $\psi_n(u)$ the probability that the ruin occurs at the n-th claim or before. Then $\psi(u) = \lim_{n\to\infty} \psi_n(u)$.

It will be proven that $\psi_n(u) \le e^{-ku}, u \ge 0$ for any $n \ge 0$

For $n = 0$, $\psi_0(u) = 0 \le e^{-ku}, u \ge 0$.

Supposing that $\psi_n(u) \le e^{-ku}, u \ge 0$, it will be shown that $\psi_{n+1}(u) \le e^{-ku}, u \ge 0$.

First, the moment when the first claim appears is analyzed. The time elapsed until the appearance follows an exponential repartition with a repartition density of $\lambda e^{-\lambda t}$ because the process $\{N_t; t \ge 0\}$ is assumed to be a Poisson process of intensity λ. It is denoted by $t > 0$ the moment of the first claim appearance. The company's surplus at that time is $u + ct$. There are two possible situations: either the claim quantum surpasses the value of the present capital, in which case ruin occurs, this event having the probability $\Pr(X_1 > u + ct) = 1 - F(u + ct)$, or the claim quantum is inferior to the present capital, $0 \le x \le u + ct$, in which case the claim will be paid and the process will be retaken with a capital of $u + ct - x$. If it hasn't occurred because of the first claim, ruin can occur at the next n claims. Because the process has stationary and independent increments, the probability of ruin occurring in the interval between the second claim and the n-th claim starting with a surplus of $u + ct$ at moment t is the same as the probability of ruin at the first n claims if the starting surplus is $u + ct - x$.

Using the total probability formula, it results that:

$$\psi_{n+1}(u) = \int_0^\infty [1 - F(u+ct) + \int_0^{u+ct} \psi_n(u+ct-x)dF(x)]\lambda e^{-\lambda t}dt$$

$$= \int_0^\infty [\int_{u+ct}^\infty dF(x) + \int_0^{u+ct} \psi_n(u+ct-x)dF(x)]\lambda e^{-\lambda t}dt$$

Using the inductive hypothesis, it results:

$$\psi_{n+1}(u) \le \int_0^\infty [\int_{u+ct}^\infty e^{-k(u+ct-x)} dF(x) + \int_0^{u+ct} e^{-k(u+ct-x)}dF(x)]\lambda e^{-\lambda t}dt,$$

Then: $\psi_{n+1}(u) \le \int_0^\infty [\int_0^\infty e^{-k(u+ct-x)}dF(x)]\lambda e^{-\lambda t}dt$

$$= \lambda e^{-ku}\int_0^\infty e^{-kct}[\int_0^\infty e^{kx}dF(x)]e^{-\lambda t}dt = \lambda e^{-ku}\int_0^\infty e^{-(\lambda+kc)t}[E(e^{kX})]dt$$

.

$$= \lambda E(e^{kX})e^{-ku}\int_0^\infty e^{-(\lambda+kc)t}dt = \frac{\lambda E(e^{kX})}{\lambda+kc}e^{-ku}$$

According to expressions (1.2) and (1.1), it results:

$\lambda E(e^{kX}) = \lambda[1+(1+\theta)k\mu] = \lambda + k(1+\theta)\lambda\mu = \lambda + kc$

Thus: $\psi_{n+1}(u) \le e^{-ku}$, which completes the induction.

Because $\psi(u) = \lim_{n\to\infty}\psi_n(u)$ and $\psi_n(u) \le e^{-ku}, \forall n,$

then $\psi(u) \le e^{-ku}$.

The importance of the result consists in the fact that it establishes a connection between the initial surplus u and the value of the safety coefficient θ.

For example, if the insurer wants to take a maximum risk of $\alpha = 0.01$ and he has a fixed initial capital u, the safety coefficient which he has to take into account is: $$\theta = \frac{u\{E[\exp(-\frac{\log\alpha}{u}X)]-1\}}{-\mu\log\alpha} - 1,$$

where $k = (-\log\alpha)/u$.

By applying the above theorem, it is obtained: $\psi(u) \le e^{-ku} = e^{\log\alpha} = \alpha.$

Otherwise if one wants to have a certain predetermined value of the safety coefficient, and the initial capital may vary, one will consider: $u = (-\log\alpha)/k$. Furthermore, expression (1.5) confirms a fact which is intuitively obvious, namely that the ruin probability, for infinite initial capital, is 0.

$$\psi(\infty) = \lim_{u \to \infty} \psi(u) = 0 \tag{1.6}$$

In order to justify this, the next argument is given: In accordance with the theorem:

$$0 \le \psi(u) \le e^{-ku}. \tag{1.7}$$

Tending $u \to \infty$, it results: $0 \le \lim\limits_{u \to \infty} \psi(u) \le \lim\limits_{u \to \infty} e^{-ku} = 0$.

From equality (1.6) it results that the survival probability satisfies the condition:

$$\phi(\infty) = 1 \tag{1.8}$$

2. Integro-differential equations satisfied by the survival and ruin probabilities

At this point, the article presents an explicit formula for the ruin probability $\psi(u)$ or an equivalent for the survival probability $\phi(u)$.

Definition. For $u \ge 0$ and $y \ge 0$, it is denoted by $G(u, y)$ the probability of ruin when starting with an initial capital u, with a produced deficit not greater then y monetary units.

Thus:

$$\psi(u) = \lim_{y \to \infty} G(u, y), u > 0. \tag{2.1}$$

Theorem. The function $G(u, y)$ satisfies the equation:

$$\frac{\partial}{\partial u} G(u, y) = \frac{\lambda}{c} G(u, y) - \frac{\lambda}{c} \int_0^u G(u-x, y)dF(x) - \frac{\lambda}{c}[F(u+y) - F(u)], u > 0 \tag{2.2}$$

Demonstration. It will be studied what happens in the first h time units, for very small values of h. Because the number of claims is a Poisson process, during this time frame there can be 0 or 1 claims. The probability of having more then one claim is negligible.

If no claims occur, which happens with a probability of $1 - \lambda h$, the probability of ruin with a deficit not greater then y is $G(u + ch, y)$. If a claim occurs, which happens with a probability of λh, we have the following possible situations: either its quantum x satisfies the inequality $0 \le x \le u + ch$ which means that the ruin has not yet occurred, but can occur from now on with the probability of $G(u + ch - x, y)$ or it satisfies the inequality $u + ch \le x \le u + ch + y$, at which point ruin has already occurred with a deficit not greater then y. For the case of $x \ge u + ch + y$, the deficit will surpass y. Using the total probability formula, it is obtained:

$$G(u,y) = (1-\lambda h)G(u+ch,y) + \lambda h \int_0^{u+ch} G(u+ch-x,y)dF(x) +$$
$$+\lambda h[F(u+ch+y) - F(u+ch)] + o(h)$$

This is equivalent to: $c\dfrac{G(u+ch,y)-G(u,y)}{ch} = \lambda G(u+ch,y) -$

$$-\lambda \int_0^{u+ch} G(u+ch-x,y)dF(x) -$$

$$-\lambda[F(u+ch+y) - F(u+ch)] + \frac{o(h)}{h}$$

Dividing it by c and tending $h \to 0$, the desired expression is obtained:

$$\frac{\partial}{\partial u}G(u,y) = \frac{\lambda}{c}G(u,y) - \frac{\lambda}{c}\int_0^u G(u-x,y)dF(x) - \frac{\lambda}{c}[F(u+y)-F(u)]$$

<u>Theorem</u>. The function G(0,y) is provided by the equality:

$$G(0,y) = \frac{\lambda}{c}\int_0^y [1-F(x)]dx\,,\ \ y \ge 0 \quad (6.16)$$

Demonstration. Because: $0 \le G(u,y) \le \psi(u) \le e^{-ku}$,

then: $0 \le G(\infty,y) = \lim\limits_{u\to\infty} G(u,y) \le \lim\limits_{u\to\infty} e^{-ku} = 0$, which gives: $G(\infty,y) = 0$.

Moreover, $\int_0^\infty G(u,y)du \le \int_0^\infty e^{-ku}du = k^{-1} < \infty$.

The following function is introduced: $\tau(y) = \int_0^\infty G(u,y)du$.

From the above results, this function verifies the inequality: $0 < \tau(y) < \infty, \forall y$.

By integrating the relation (2.2) in respect to u from 0 to ∞, it results:

$$-G(0,y) = \frac{\lambda}{c}\tau(y) - \frac{\lambda}{c}\int_0^\infty\int_0^u G(u-x,y)dF(x)du - \frac{\lambda}{c}\int_0^\infty [F(u+y)-F(u)]du.$$

By changing the order of integration, it is obtained:

$$G(0,y) = -\frac{\lambda}{c}\tau(y) + \frac{\lambda}{c}\int_0^\infty\int_x^\infty G(u-x,y)dudF(x) + \frac{\lambda}{c}\int_0^\infty [F(u+y)-F(u)]du$$

Inside the double integral, changing the variable from u into $v = u - x$ leads to:

$$G(0,y) = -\frac{\lambda}{c}\tau(y) + \frac{\lambda}{c}\int_0^\infty\int_0^\infty G(v,y)dvdF(x) + \frac{\lambda}{c}\int_0^\infty [F(u+y)-F(u)]du$$

$$= -\frac{\lambda}{c}\tau(y) + \frac{\lambda}{c}\int_0^\infty \tau(y)dF(x) + \frac{\lambda}{c}\int_0^\infty [F(u+y) - F(u)]du$$

Because $\int_0^\infty dF(x) = 1$, the following equality is obtained:

$$G(0,y) = \frac{\lambda}{c}\int_0^\infty [F(u+y) - F(u)]du = \frac{\lambda}{c}\int_0^\infty [1 - F(u)]du - \frac{\lambda}{c}\int_0^\infty [1 - F(u+y)]du$$

In the second integral the variable is changed from u into $x = u + y$

$$G(0,y) = \frac{\lambda}{c}\int_0^\infty [1 - F(u)]du - \frac{\lambda}{c}\int_y^\infty [1 - F(x)]dx = \frac{\lambda}{c}\int_0^y [1 - F(x)]dx .$$

The next theorem provides an important result regarding the survival probability with initial capital zero.

<u>Theorem.</u> The probability of survival with zero initial capital is given by the expression:

$$\phi(0) = \frac{\theta}{1+\theta}. \tag{2.3}$$

Demonstration. Inside the equality: $G(0,y) = \frac{\lambda}{c}\int_0^y [1 - F(x)]dx$,

tending $y \to \infty$ leads to:

$$\psi(0) = \lim_{y\to\infty} G(0,y) = \frac{\lambda}{c}\int_0^\infty [1 - F(x)]dx = \frac{\lambda\mu}{c} = \frac{1}{1+\theta}$$

For this, expression (1.1) and $\int_0^\infty [1 - F(x)]dx = \mu$ are used.

Thus:

$$\phi(0) = 1 - \psi(0) = \frac{\theta}{1+\theta} .$$

Further, it is established the integro-differential equation satisfied by $\phi(u)$, with initial condition (2.3).

<u>Theorem</u>. The survival probability $\phi(u)$ satisfies the equation:

$$\phi^{'}(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)dF(x), u \geq 0 . \tag{2.4}$$

Demonstration. From equality (2.1) and $\psi^{'}(u) = \lim_{y\to\infty}\frac{\partial}{\partial u}G(u,y),$

using (2.2) it results:

$$\psi'(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c}\int_0^u \psi(u-x)dF(x) - \frac{\lambda}{c}[1-F(u)], u \ge 0. \quad (2.5)$$

From this the next expression in $\phi'(u)$ is obtained:

$$-\phi'(u) = \frac{\lambda}{c}[1-\phi(u)] - \frac{\lambda}{c}\int_0^u [1-\phi(u-x)]dF(x) - \frac{\lambda}{c}[1-F(u)]$$

$$-\phi'(u) = -\frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u dF(x) + \frac{\lambda}{c}\int_0^u \phi(u-x)dF(x) + \frac{\lambda}{c}F(u)$$

$$= -\frac{\lambda}{c}\phi(u) + \frac{\lambda}{c}\int_0^u \phi(u-x)dF(x)$$

This proves the desired result.
As an application of this result, we will determine the solution to equation (2.4) for a particular distribution function F .

3. Determining the survival probability for exponentially distributed claims

Supposing that X is an exponential distribution, having the repartition function:

$F(x) = 1 - e^{-x/\mu}, x > 0$

We want to calculate $\phi(u)$.

In this case, equation (2.4) becomes:

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{\mu c}\int_0^u \phi(u-x)e^{-x/\mu}dx$$

By changing the variable x into $y = u - x$, it results:

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{\mu c}e^{-u/\mu}\int_0^u \phi(y)e^{y/\mu}dy \; . \; (6.20)$$

By differentiating expression (6.20) with respect to u, it is obtiained:

$$\phi''(u) = \frac{\lambda}{c}\phi'(u) + \frac{\lambda}{\mu^2 c}e^{-u/\mu}\int_0^u \phi(y)e^{y/\mu}dy - \frac{\lambda}{\mu c}\phi(u)$$

The integral can be calculated from expression (6.20), and then by replacing it inside the expression we obtain the following differential equation:

$$\phi''(u) = \frac{\lambda}{c}\phi'(u) - \frac{\lambda}{\mu c}\phi(u) + \frac{1}{\mu}[\frac{\lambda}{c}\phi(u) - \phi'(u)].$$

or, yet: $\phi''(u) = (\frac{\lambda}{c} - \frac{1}{\mu})\phi'(u)$

Using the method of separation of variables and the expression (1.1), it is obtained:

$$\frac{\phi''(u)}{\phi'(u)} = -\frac{\theta}{\mu(1+\theta)}.$$

Integrating in respect to u, the general solution is obtained:

$$\log\phi'(u) = -\frac{\theta u}{\mu(1+\theta)} + K_1.$$

In order to evaluate K_1, let $u = 0$ in (6.20), and then using (2.3) it is obtained: $\phi'(0) = \frac{\lambda}{c}\frac{\theta}{1+\theta} = \frac{\lambda}{\lambda\mu(1+\theta)}\frac{\theta}{1+\theta} = \frac{\theta}{\mu(1+\theta)^2}$

From the previous expression the exact value of K_1 is obtained: $K_1 = \log[\frac{\theta}{\mu(1+\theta)^2}]$

Getting back to $\phi(u)$: $\phi'(u) = \frac{\theta}{\mu(1+\theta)^2}\exp[-\frac{\theta u}{\mu(1+\theta)}]$.

Integrating, the general solution is obtained:

$$\phi(u) = -\frac{1}{1+\theta}\exp[-\frac{\theta u}{\mu(1+\theta)}] + K_2$$

In order to obtain K_2, let $u = 0$ and expression (2.3) will lead to $K_2 = 1$.

In the end, the desired formula for the survival probability is obtained:

$$\phi(u) = 1 - \frac{1}{1+\theta}\exp[-\frac{\theta u}{\mu(1+\theta)}].$$

Remark: Even though the mathematical model presented in this paper is an idealized one, the results it provides correspond to real world situations and can be useful in the development of a financial long term plan for an insurance company.

R E F E R E N C E S

1. *Grandell, J.* „Aspects of Ruin Theory”, Springer, New York (1992).
2. *Klugman, S.,Panjer, H, Wilmot, G.* „Loss Models” J. Willey (2000)
3. *Albrecher, H.,* “Dependent Risks and Ruin Probabilities in Insurance” IIASA Interim Report, IR-98-072.
4 *Keller, G., Warrack, B*. “Essentials of Business Statistics” Wadsworth Publishing Company (1991)